Some Features of Scattering Amplitudes in String Theory and Quantum Field Theory

Michael B. Green

University of Cambridge & Queen Mary University London

SAGEX Kickoff Workshop, QMUL, September 6 2018

UNIVERSITY OF CAMBRIDGE

Queen Mary University of London

The SAGEX collaboration covers a range of exciting ideas that focus largely on perturbative properties of quantum field theories.

However, conventional quantum field theory is almost certainly not the right language for describing quantum gravity.

String theory is an extension of quantum field theory that reduces at low energy to particular quantum field theories interacting with gravity in a manner that is consistent with quantum mechanics (ALTHOUGH THE PRECISE DEFINITION OF STRING THEORY IS STILL LACKING!).

This talk will describe some aspects of the relationship between string theory and quantum field theories.

- **CONTRASTING FEATURES OF QUANTUM FIELD THEORY AND STRING THEORY**

- **PERTURBATION EXPANSIONS IN QFT AND STRING THEORY**

  Automorphic features of superstring amplitudes

  Rich dependence of string theory on scalar fields - interconnections with number theory

  SUPERSYMMETRY is an important mathematical tool – whether or not it is a symmetry of nature

- **ULTRA-VIOLET PROPERTIES OF STRING THEORY AND SUPERGRAVITY**

  How do field theory UV divergences arise in low energy limit of string theory?

STRING THEORY - FROM EXPERIMENT TO THEORY

STRING THEORY EMERGED DIRECTLY OUT OF THE EXPERIMENTS ON THE STRONG FORCE !!

1967

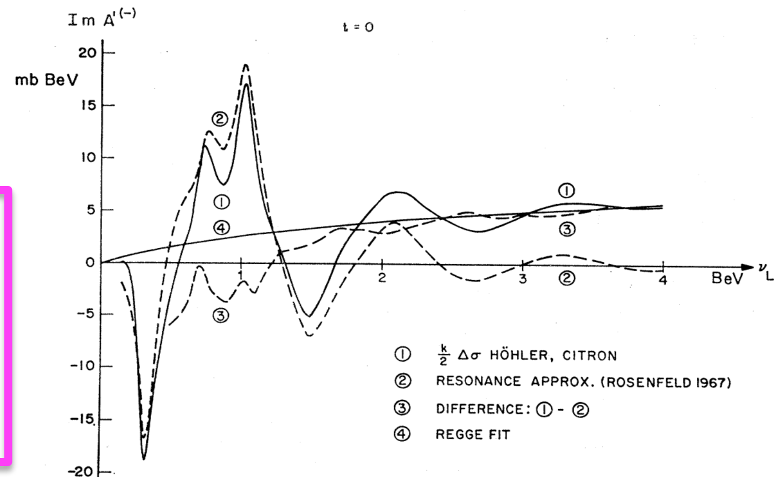
K. Igi and S. Matsuda, Phys. Rev. Lett. 18 (1967) 625
R. Dolen, D. Horn and C. Schmid, Phys. Rev. Lett. 19 (1967) 402
M. Ademollo, H.R. Rubinstein, G. Veneziano, M.A. Virasoro Phys.Rev.Lett. 19 (1967) 1402

πN charge exchange

As a first application we consider k [σ_T (π^- p) - σ_T (π^+ p)] which should be dominated by the ρ Regge pole. There exist very good data for low energies, and a smooth fit by the Regge pole can be observed from 4 GeV onwards. The sum rule holds within experimental accuracy.

Igi and Matsuda, PRL 1967.
Horn and Schmid, 1967.
Logunov et al PL 1967
Dolen Horn Schmid PRL 1967 PR 1968

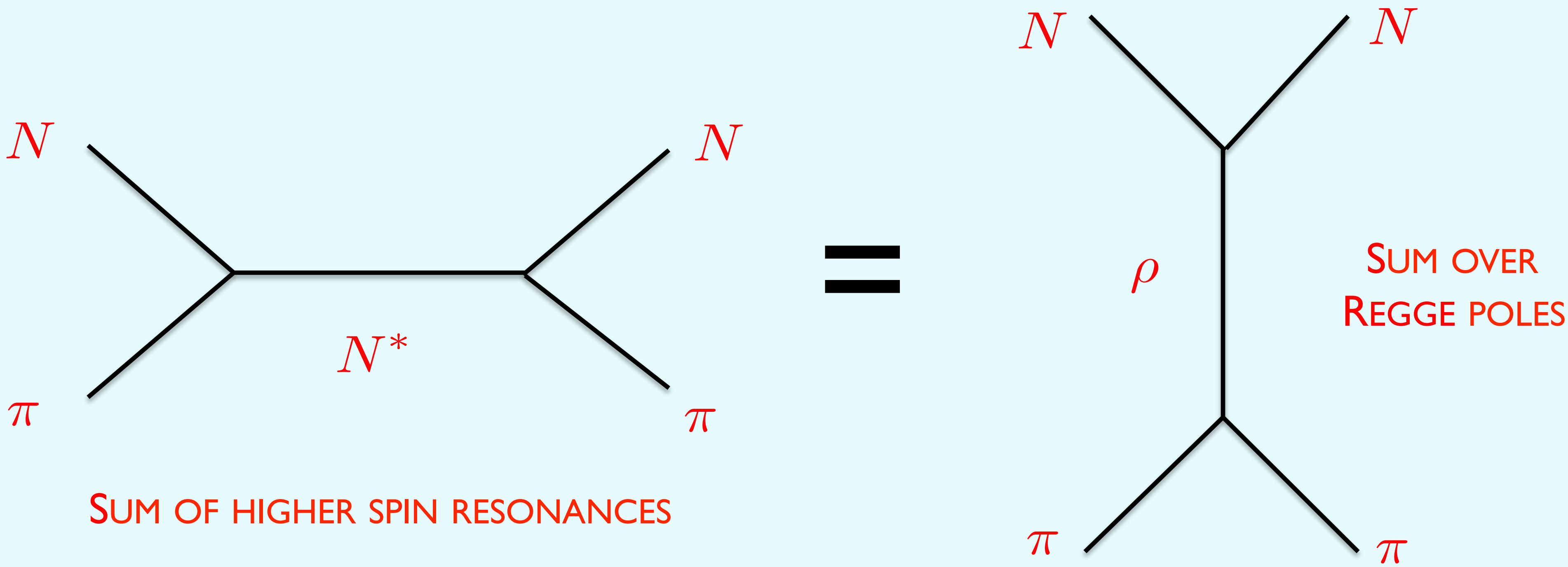
This figure compares $\Delta\sigma_T$ (1) with the Regge fit (4). It shows also that sum of resonances (2) approximates quite well $\Delta\sigma_T$ at low energies.



From Dolen, Horn, Schmid
Phys. Rev.166.1768 (1967)

FINITE ENERGY SUM RULES: Unitarity, Analyticity, Crossing Symmetry

DUALITY: **Sum of Resonances = Sum of Regge Poles**



Radicaly different from conventional QFT

Embodied in the **Veneziano model** (Euler beta function) **(1968)**

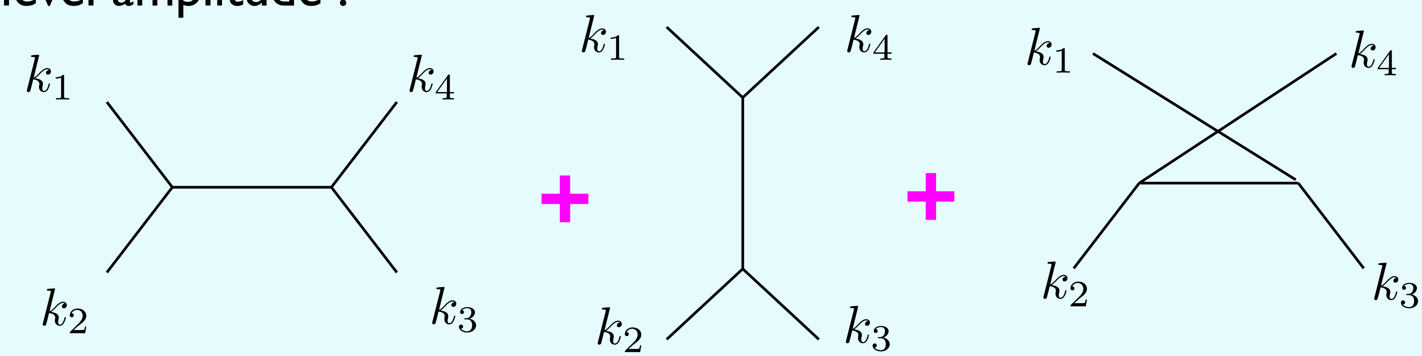
50th Anniversary of the origin of String Theory

Duality: A relation between two descriptions of the same physical or mathematical system

DISTINCTIONS BETWEEN FIELD THEORY AND STRING THEORY

Consider 4-particle tree-level amplitude :

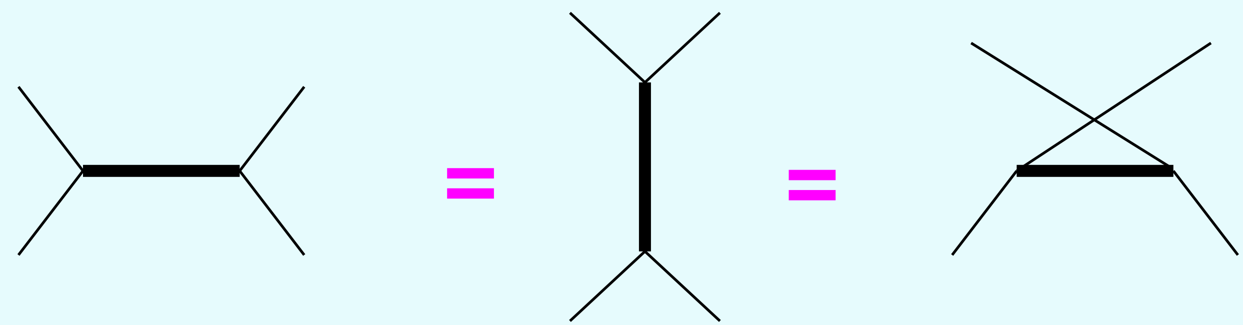
QUANTUM FIELD THEORY



ADD FEYNMAN DIAGRAMS with poles in the s-channel, t-channel and u-channel propagators

$$s = (-k_1 + k_2)^2\,,\ \ t = -(k_1 + k_4)^2\,,\ \ u = -(k_1 + k_3)^2$$

STRING THEORY



Sum of INFINITE NUMBER of massive propagating string states – excited modes of string

DO NOT ADD DIAGRAMS with poles in the s-channel, t-channel and u-channel propagators

PROPERTY OF LARGE-N QCD; CONFORMAL BOOTSTRAP: S-MATRIX BOOTSTRAP

LINKS WITH MANY OTHER MODERN DEVELOPMENTS; …………..

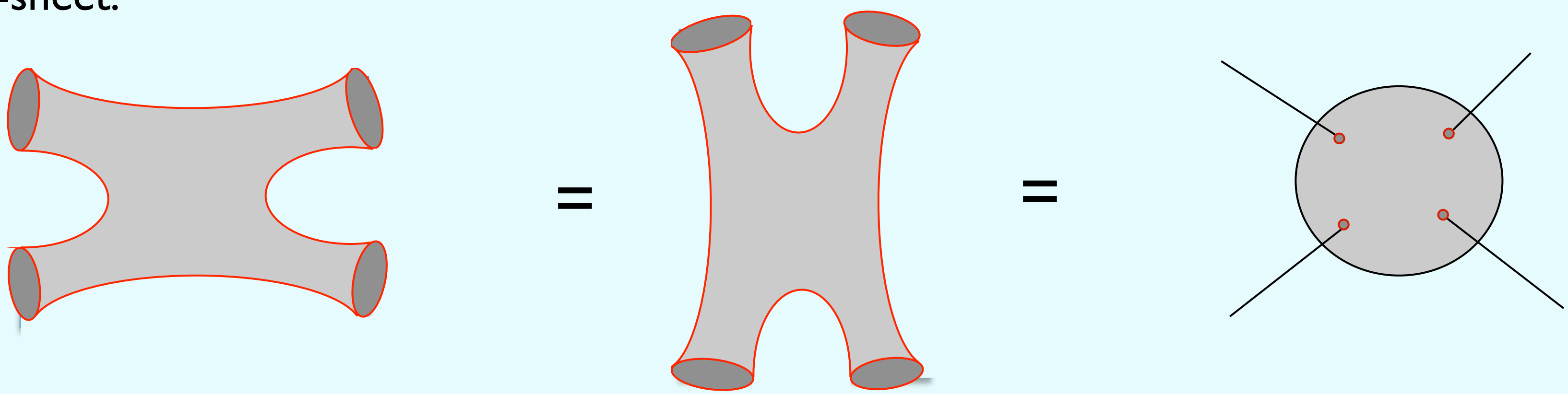
An iconic feature of string theory and the bootstrap

Requiring all pole residues to be positive (zeroth order UNITARITY) leads to very strong constraints, which imply:

- There must be an infinite number of massive states with spins that increase with mass;
- Asymptotically linear Regge trajectories as $\text{Spin} \sim \alpha'(\text{Mass})^2$ when $\alpha'(\text{Mass})^2 \gg 1$

These are characteristic features of string theory.

World-sheet:



Conformal invariance for tree-level amplitude implies:: Spherical world-sheet

- These results apply to large-N QCD and strongly suggest a string description

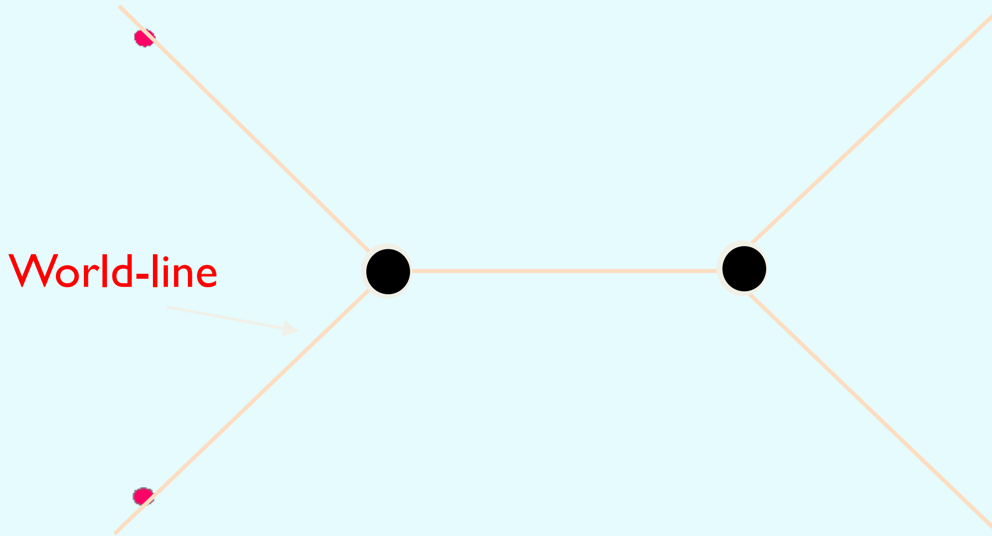
c.f. Meson or Glueball scattering in large-N SU(N) QCD

Viz.

Scattering of point particles in QFT

(scattering of small plane wave excitations of a field)

Feynman Diagram



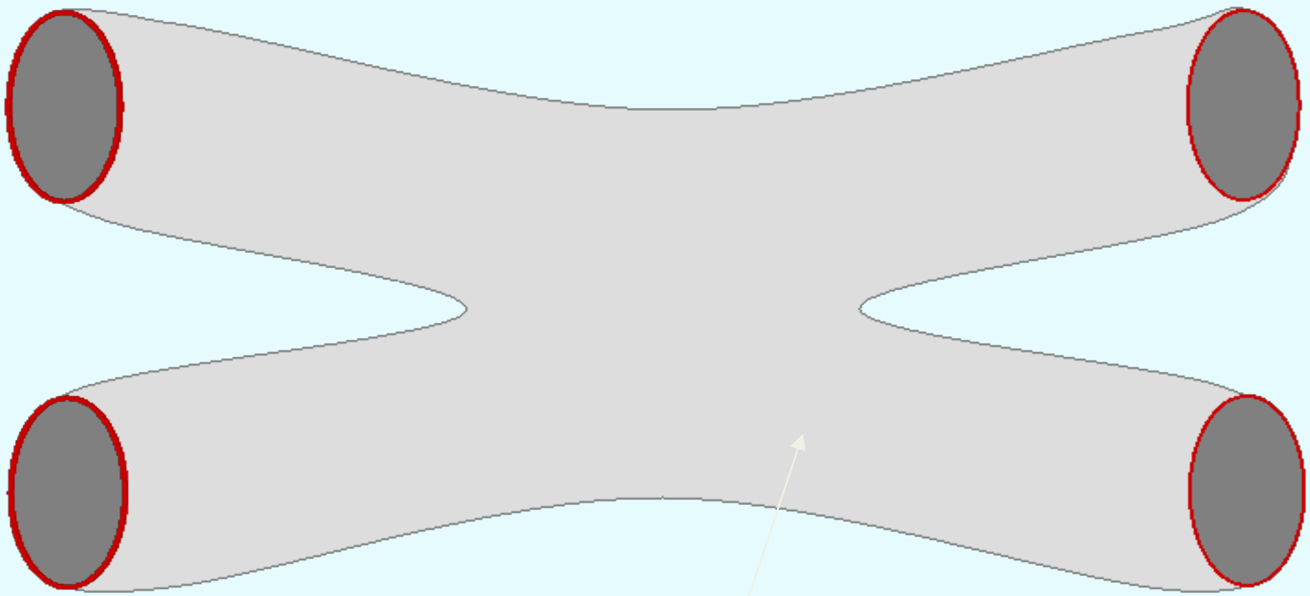
Point-like Nodes ⟶ Ultraviolet divergences of loops

Joining and splitting of world-lines

- Consistent interpretation in renormalisable quantum field theories.
  e.g. Standard Model – Yang—Mills theories

- Ultraviolet incompleteness – no understanding of physics at ultra-short distances.
  Inconsistency of perturbative quantum gravity – ultraviolet divergences of loop amplitudes.
  – quantum gravity is not well-defined in perturbation theory.

Stringy "Feynman Diagram"



smooth world-sheet – NO NODES

- Superstrings join and split "smoothly" (euclidean signature).
- No ultra-violet divergences in loop amplitudes.
- **Finite** perturbative theory of quantum gravity.
- Sign of ultraviolet completeness – requires presence of non-perturbative objects.

D-branes, p-branes, instantons, ….

Connection between quantum fields and strings

Closed (super) string:

Massless spin-2 **graviton** mode + massless scalar (dilaton φ) .

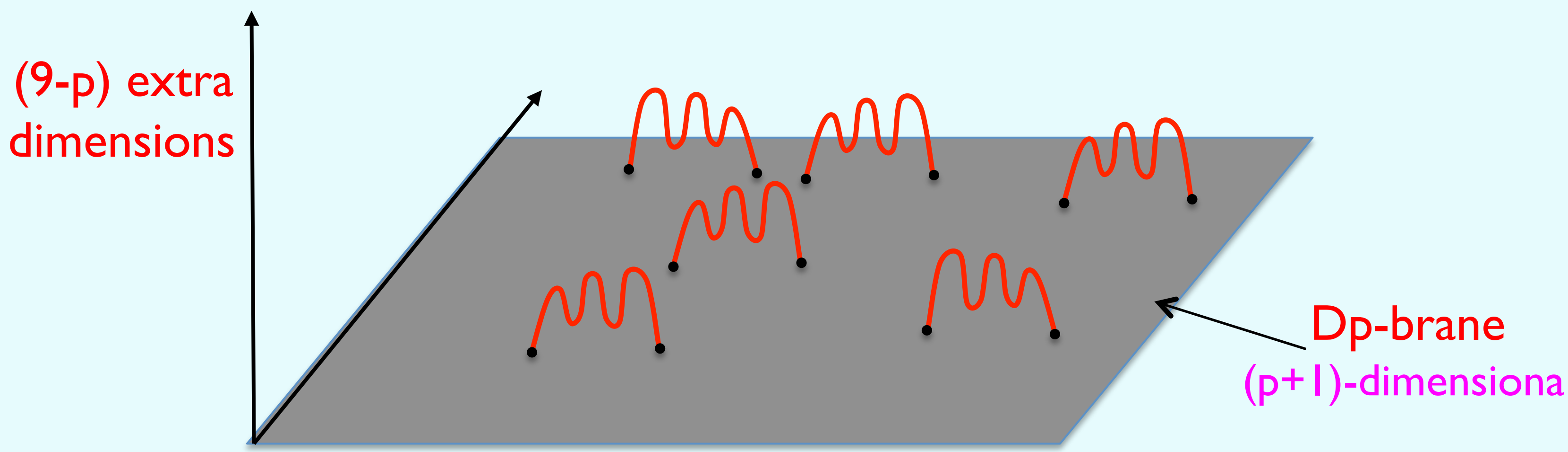
Infinite set of massive modes with masses spaced in units of $1/\ell_s$.

Open (super) string: Endpoints tethered to a (p+1) dimensional hyperplane

This defines a solitonic object – a **Dp-brane** non-perturbative – mass $\sim 1/g_s$

Massless spin-1 SU(N) **Yang-Mills** mode.

e.g. Dp-Brane Dynamics



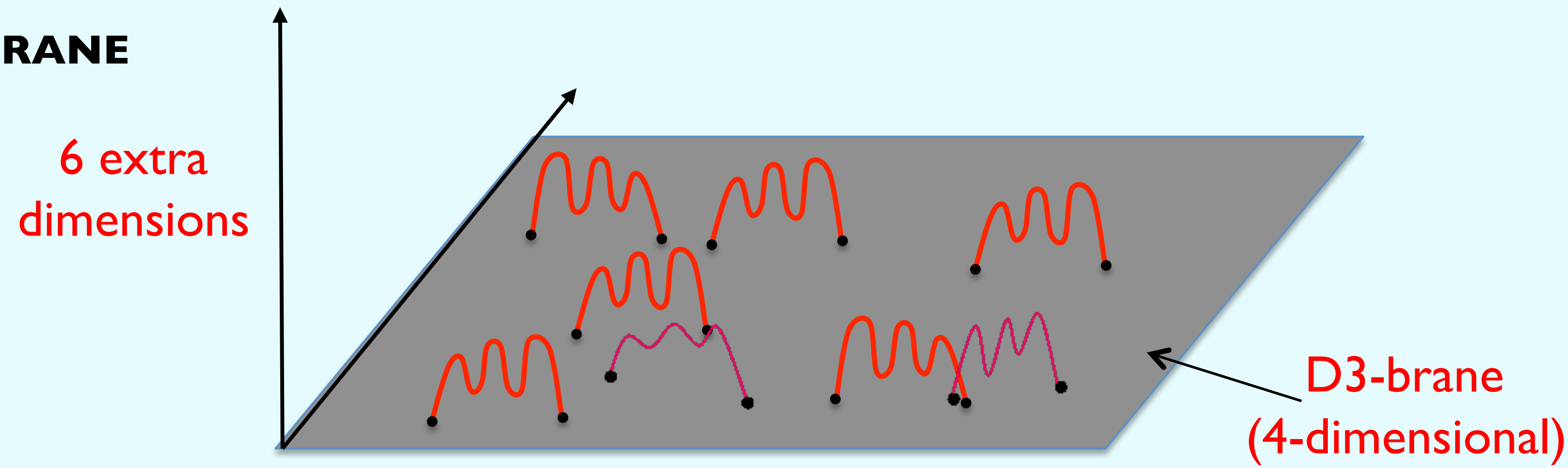
Open Strings move in four dimensional D3-brane

CONNECTION BETWEEN QUANTUM FIELDS AND STRINGS

UNIFICATION OF YANG-MILLS AND GRAVITY

graviton may be viewed as a "BOUND STATE" of Yang-Mills

e.g. D3-BRANE



UNIFIED ORIGIN OF GAUGE THEORY AND GRAVITY.

- Explicit in the KAWAI-LLEWELLYN-TYE (KLT) construction of closed-string tree amplitudes from open-string trees.

- Related to gauge/kinematic duality of BERN-CARRASCO-JOHANSSON (BCJ) in QFT.

$$[\text{Gravity}] = [\text{Yang}-\text{Mills}] \otimes [\text{Yang}-\text{Mills}]$$
$$\mathcal{N}=8 \qquad (\mathcal{N}=4) \otimes (\mathcal{N}=4)$$

- Leads to HOLOGRAPHIC EQUIVALENCE of closed (type IIB) superstring theory in 5-dimensional anti de-Sitter space and large-N maximally supersymmetric SU(N) Yang-Mills conformal field theory in four dimensions. Many extensions and generalizations.

Comments on non-perturbative objects in string theory

Wrapping a Dp-brane around p compact dimensions gives a massive state in (10-p) dimensions of mass $1/g_D$

$g_D^2 = \frac{g_s^2}{r^p}$ (10-p)-dimensional string coupling

- Construct **Black holes** as superpositions of D-branes wrapped around compact cycles.
- In perturbative gravitational field theory the effect of black holes and instantons on scattering amplitudes is generally ignored but this is not possible in string theory – constraints of non-perturbative symmetries require their inclusion.

N=8 supergravity does not arise as a limit of string theory

even though perturbative Type II string theory reduces to perturbative supergravity at low energies

PROOF

In order to get N=8 supergravity field theory as a limit of supergravity we need to compactify on a 6-torus of radius r and take the limit

$\ell_s \to 0\,,\quad r \to 0\,,$ with fixed Planck scale $\ell_4 = g_4\,\ell_s$

$g_4 = \frac{g_s}{r^3}$ 4-dim. string coupling

This is the STRONG COUPLING limit, $g_4 \to \infty\,,$ in which infinite towers of "non-perturbative" states become massless

N=8 supergravity is in the "Swampland".

Non-perturbative Dualities

Beyond perturbation theory superstring theory possesses a rich set of dualities:

Discrete transformations between descriptions of the same theory, in which the fundamental degrees of freedom may be very different.

Example: Ten-dimensional Type IIB closed string theory is self-dual – invariant under the modular group, $SL(2,\mathbb{Z})$, which is an arithmetic subgroup of $SL(2,\mathbb{R})$.

- Compactifying on a d-torus to D=10-d dimensions, the duality group is enlarged to a group of rank-$(d+1)$ in the series known as $E_{d+1}(\mathbb{Z})$. These dualities are discrete gauge symmetries.
- This contrasts with supergravity field theories, which are invariant (in perturbation theory) under the corresponding continuous global symmetries:

  e.g. D=10 type IIB supergravity is invariant under $SL(2,\mathbb{R}) \equiv E_1(\mathbb{R})$
- Four dimensional (D=4, d=6) maximal supergravity, $\mathcal{N}=8$ SUGRA, is invariant under $E_{7(7)}(\mathbb{R})$.

Consequently:

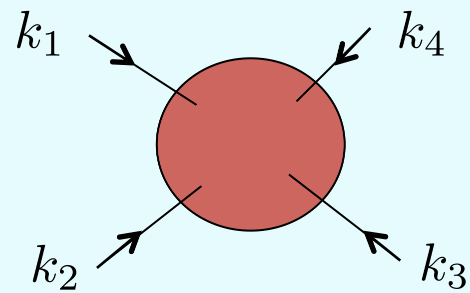
Amplitudes in perturbative supergravity are independent of moduli

massless scalar fields that transform by shifts under the action of the continuous duality symmetry

Perturbative superstring amplitudes have a rich dependence on moduli

since there is no continuous duality symmetry – Automorphic functions, **Langland's Eisenstein Series**, ..

CLOSED TYPE II (SUPER)STRING SCATTERING AMPLITUDES



e.g. four-graviton amplitude

AMPLITUDE $A^{(4)}(\epsilon_r, k_r;\, m)$

Moduli m – specify geometry of target space (space-time)

polarisation

TWO KINDS OF EXPANSION:

I) "LOW ENERGY" EXPANSION - approximation for $k_r \ll 1/\ell_s$

ℓ_s- STRING LENGTH SCALE

- LOWEST ORDER TERM reproduces classical (super)gravity – **EINSTEIN-HILBERT LAGRANGIAN**

  Proportional to $e^{-2\phi} R$ where R is the curvature scalar and ϕ is a scalar field

  $e^{-\phi} = \frac{1}{g_s}$

  STRING COUPLING CONSTANT

- INFINITE SET OF HIGHER ORDER TERMS ENCODE DISTINCTLY STRINGY EFFECTS.

  Effective action $$\frac{1}{\alpha'} \int d^{10}x \sqrt{-\det G}\, \mathcal{E}(\phi \ldots)\, R^4 + \ldots$$

  $\alpha' = \ell_s^2$

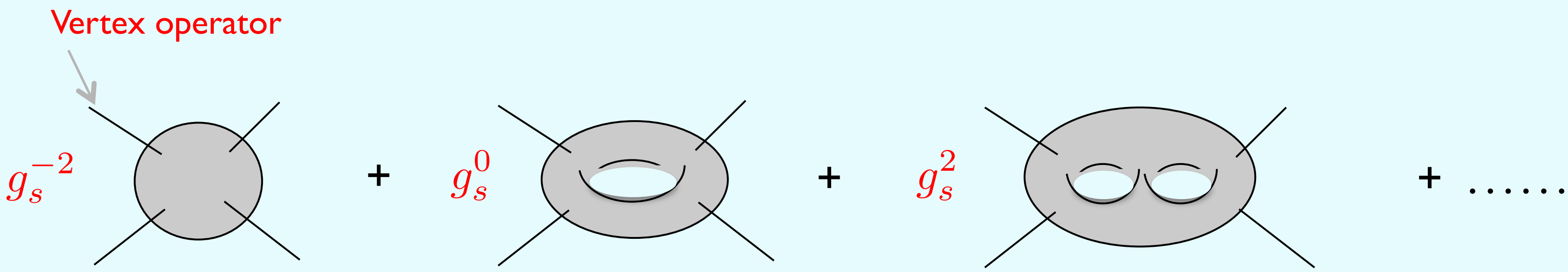
  Coefficients depend on MODULI (SCALAR FIELDS)

II. String Perturbation Expansion

Expand around a boundary of moduli space (large values of a scalar field).

e.g. In powers of scalar (dilaton) field $g_s = e^{-\phi}$ as $g_s \to 0$

Expansion in series of genus-h WORLD-SHEETS - in powers of string coupling g_s^{-2+2h}.



- Each term in the low energy expansion can be expanded in string perturbation theory. Interesting constraints imposed by non-perturbative dualities.

Some Features of Closed Superstring and Supergravity Perturbation Theory

- A unique string diagram contributes to the N-graviton amplitude at order g_s^{-2+2h}. This has the geometry associated with a genus-h orientable Riemann surface, which is a sphere with h handles and N punctures.

- The genus-h amplitude is given by an INTEGRAL OVER THE (SUPER) MODULI that parameterise inequivalent surfaces.

- The immense number of individual Feynman diagrams of supergravity arise in degeneration limits, which correspond to points on the boundary of moduli space at which world-sheet handle(s) become infinitely long.

- These field theory Feynman diagrams become dominant at low energies and in low enough dimensions.

- String theory has no ultraviolet divergences - and superstring theory has no divergences at any genus when expanded around consistent backgrounds. The low energy "field theory" limit defines a particular effective field theory of (non-renormalisable) perturbative supergravity.

Four-Graviton Scattering Amplitude

Maximally supersymmetric supergravity:

Maximally supersymmetric supergravity compactified on a (d+1)-torus from 11 dimensions to D=10-d dimensions.

$$\mathcal{A}^{(4)}(\epsilon_r, k_r; \Lambda, D) = \mathcal{R}^4\, T^{(4)}(s,t,u;\Lambda,D)$$

$s = -2\,k_1\cdot k_2 \qquad t = -2\,k_1\cdot k_4 \qquad u = -2\,k_1\cdot k_3$

$s+t+u=0$

$\mathcal{R}$ linearized curvature $\sim k_\mu\, k_\nu\, \epsilon_{\rho\sigma}$

momentum cut-off Λ

Tree-level maximal supergravity:

Sum of tree diagrams

$$\mathcal{A}_0^{(4)} = \mathcal{R}^4 \frac{1}{s\,t\,u} = \mathcal{R}^4 \frac{3}{\sigma_3}$$

$\sigma_3 = s^3 + t^3 + u^3 = 3s\,t\,u$

$\sigma_n = s^n + t^n + u^n$

s, t, u measured in units of the D-dimensional Planck length ℓ_D

Superstring Theory

Moduli in (10-D) dimensions

$$\mathcal{A}^{(4)}(\epsilon_r, k_r; \mu_D) = \mathcal{R}^4\, T^{(4)}(s,t,u;\mu_D)$$

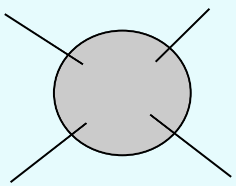
All coupling constants are expectation values of scalar fields

e.g. In 10 dimensions: One complex modulus $\mu_D \to \Omega = \Omega_1 + \Omega_2$

$\Omega_2 = e^{-\varphi} = \frac{1}{g_s}$ inverse string coupling constant determined by scalar dilaton φ

$\mathcal{A}^{(4)}(\epsilon_r, k_r; \Omega)$ is a $SL(2,\mathbb{Z})$-invariant function of scalar field: $\Omega \to \frac{a\Omega + b}{c\Omega + d}$ $\quad a,\ b,\ c,\ d \in \mathbb{Z}$, $\quad ad - bc = 1$

Tree-level closed superstring four-graviton amplitude:



$$\mathcal{A}_0^{(4)} = \mathcal{R}^4 \frac{1}{g_s^2} \frac{1}{stu} \frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+t\alpha' u)}$$

Virasoro amplitude

s, t, u measured in units of string length ℓ_s $\quad \alpha' = \ell_s^2$

Infinite number of massive and higher spin poles at $s,t,u = 0,1,2,3,\ldots$

(property of Euler Γ function)

LOW ENERGY EXPANSION:

Riemann zeta values $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$

$\sigma_n = s^n + t^n + u^n$

$$T_0^{(4)} = \frac{1}{stu} \frac{\Gamma(1-\alpha' s)\,\Gamma(1-\alpha' t)\,\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\,\Gamma(1+\alpha' t)\,\Gamma(1+\alpha' u)} \qquad = \frac{3}{\sigma_3} \exp\left[\sum_{n=1}^{\infty} \frac{2\zeta(2n+1)}{2n+1} {\alpha'}^{2n+1} \sigma_{2n+1}\right]$$

Tree-level SUPERGRAVITY

R^4 $\quad d^4 R^4$ $\quad d^6 R^4$ $\quad d^8 R^4$

$$= \frac{3}{\sigma_3} + 2\zeta(3)\,{\alpha'}^3 + \zeta(5)\,{\alpha'}^5 \sigma_2 + \frac{2\,\zeta(3)^2}{3}\,{\alpha'}^6\,\sigma_3 + \frac{\zeta(7)}{2}\,{\alpha'}^7\,\sigma_2^2$$

$$+ \frac{2\,\zeta(3)\,\zeta(5)}{3}\,{\alpha'}^8\,\sigma_2\,\sigma_3 + \frac{\zeta(9)}{4}\,{\alpha'}^8\,\sigma_2^3 + \frac{2}{27}(2\,\zeta(3)^2 + \zeta(9))\,{\alpha'}^9\,\sigma_3^2 + \ldots$$

$s^k R^4 \sim d^{2k} R^4$

$\sigma_2 = s^2 + t^2 + u^2$

$\sigma_3 = s^3 + t^3 + u^3$

$d^{10} R^4$ $\quad d^{12} R^4$

INFINITE SERIES of $d^{2k} R^4$ terms. COEFFICIENTS ARE POWERS OF **ODD RIEMANN ζ VALUES** WITH RATIONAL COEFFICIENTS

Coefficients of low energy expansion of tree-level N-PARTICLE OPEN-STRING scattering are

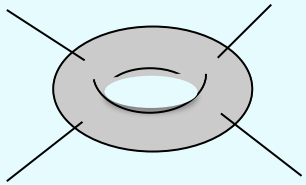
MULTIPLE-ZETA VALUES. $\quad \zeta(s_1, \ldots, s_r) = \sum_{0<k_1<\cdots<k_r} \prod_{\ell=1}^{r} k_\ell^{-s_\ell}$

Coefficients of low energy expansion of tree-level N-PARTICLE CLOSED-STRING scattering are

SINGLE-VALUED MULTIPLE-ZETA VALUES. (no even ζ values)

- which also arise in various higher loop amplitudes in quantum field theory

Genus one superstring Theory



$$A_1^{(4)}(\epsilon_r) = \frac{\pi}{16}\mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d\tau^2}{\tau_2^2} \mathcal{B}_1(s,t,u;\tau)$$

Integral over complex structure $\tau = \tau_1 + i\tau_2$

Integral over positions of vertices on torus with complex structure τ - invariant under modular transformations – reparameterisations of the torus.

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

$a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1$

Low energy expansion: $$\mathcal{B}_1(s,t,u;\tau) = \sum_{p,q} \sigma_2^p\, \sigma_3^q\, \mathcal{B}_w(\tau)$$

$\sim s^{2p+3q}$ $\quad w = 2p + 3q$

$\mathcal{B}_w(\tau)$ = sum of Feynman diagrams on toroidal world-sheet - Modular invariants for surface

elliptic generalisations of single-valued MZV

"Modular Graph functions"

Expansion near $\tau_2 \to \infty$ $$\mathcal{B}_w(\tau) = \sum_{k=-w+1}^{w} b_k\, \tau_2^k + O(e^{-\pi\tau_2})$$

- Laurent polynomial has $(2w-1)$ terms,
- Coefficients b_k are single-valued multiple zeta values (that arose in tree-level expansion)
- Modular graph functions satisfy remarkable polynomial identities

c.f. correlations between MZVs

Genus-one low-energy expansion of the amplitude

Integrating over τ gives the one-loop expansion coefficients (in D=10 dimensions):

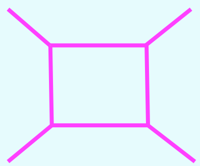
$$\mathcal{R}^4 \quad d^4\,\mathcal{R}^4 \quad d^6\,\mathcal{R}^4 \quad d^8\,\mathcal{R}^4 \quad d^{10}\,\mathcal{R}^4$$

Local terms

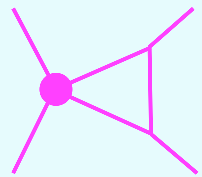
$$A_1^{(4)} = \frac{\pi}{3}\left(1 + 0\,\sigma_2 + \frac{\zeta(3)}{3}\,\sigma_3 + 0\,\sigma_2^2 + \frac{116\zeta(5)}{5}\sigma_2\,\sigma_3 + O(s^6)\right)\,\mathcal{R}^4$$

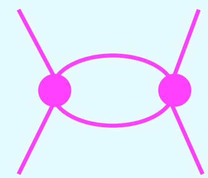
Threshold terms (schematic)

$$+\left(a\,s\log(-\alpha' s) + b\,\zeta(3)\,s^4\,\log(-\alpha' s/\rho_1) + c\,\zeta(3)^2\,s^6\,\log(-\alpha' s/\rho_2) + O(s^7) +\right)\,\mathcal{R}^4$$



A specific Ultraviolet finite value of supergravity amplitude





Higher-derivative effective interactions

$\zeta(3)\,\mathcal{R}^4\,, \zeta(5)\,d^4\mathcal{R}^4\,,\ldots$ $\qquad$ $\zeta(3)^2\,\mathcal{R}^4\,,\ldots$

- Coefficients are consistent with the conditions imposed by non-perturbative dualities.

- Compactify to D=8

$$\mathcal{A}_1^{(4)} \sim \left(d\,\log(-\alpha' s/\rho_3) + e\,\zeta(3)\,s^2\,\log(-\alpha' s/\rho_4) + f\,\zeta(3)^2\,s^4\,\log(-\alpha' s/\rho_5) + O(s^5)\right)\,\mathcal{R}^4$$

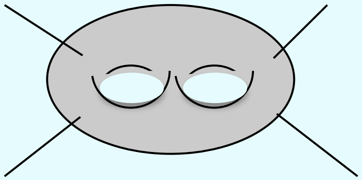
Same coefficient as $d\,\log(-s/\Lambda^2)$ term in D=8 supergravity

"stringy" thresholds

The scales ρ_i are uniquely specified numbers, whereas supergravity has arbitrary scale $\sim \Lambda^2$

GENUS TWO

Amplitude is explicit



$$\int_{\mathcal{M}_2} \frac{|d^3\Omega|^2}{\det(\mathrm{Im}\,\Omega)^3}\,\mathcal{B}_2(s,t,u;\Omega)$$

2X2 period matrix $\Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix}$

GENUS-TWO $Sp(4,\mathbb{Z})$-INVARIANT

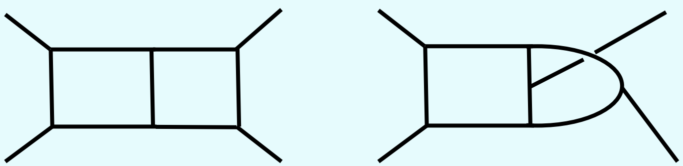
Low energy expansion has the form $\mathcal{B}_2(s,t,u;\Omega) = \sum_{p,q} \sigma_2^p\,\sigma_3^q\,\mathcal{B}^{(2)}_{(p,q)}(\Omega) \leftarrow$ $Sp(4,\mathbb{Z})$ INVARIANTS,

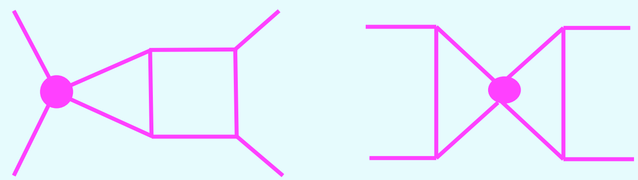
HYPER-ELLIPTIC GENERALISATIONS OF SINGLE-VALUED MZV

Very interesting connections with algebraic geometry

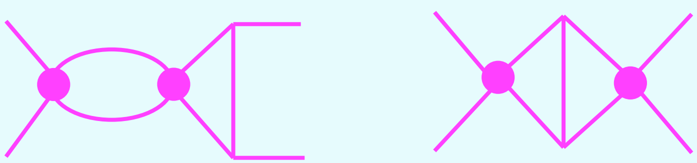
FIELD THEORY LIMIT:

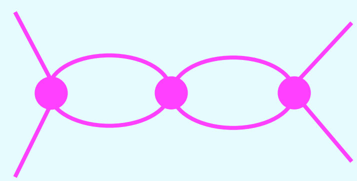
A variety of degenerations of the surface





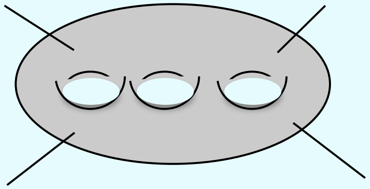
A specific renormalised value of **TWO-LOOP MAXIMAL SUPERGRAVITY**





Higher derivative effective interactions lead to new “stringy” thresholds

Genus Three: Technical difficulties analysing 3-loops beyond the leading low energy behaviour



$$A_3^{(4)} = g_s^4 \left(\frac{4}{27} \zeta(6)\, \sigma_3 + \dots \right) \mathcal{R}^4$$

$d^6 \, R^4$

Higher Genera: **No UV divergences – but no explicit expressions**

- Non-perturbative duality (U-duality) relates different orders in the perturbation expansion

  Crucial rôle played by instanton contributions

  - This is explicitly demonstrated at low orders in the low-energy expansion.

  - The coefficients of the log terms corresponding to the ultraviolet divergences in multi-loop supergravity are determined by imposing U-duality

Connection Between String Duality and SUGRA UV divergences

Maximal SUGRA has $\log\ \Lambda$ UV divergences in "Critical" dimensions $D = D_c$

These are reflected by $\log g$ terms in the corresponding string theory amplitudes (in Planck units).

"Protected" terms:

One loop in D=8	$\mathcal{R}^4$	
Two loops in D=7	$d^4\,\mathcal{R}^4$	
Three loops in D=6	$d^6\,\mathcal{R}^4$	$D_L = 4 + \frac{6}{L}$
Four loops in D=11/2	$d^8\,\mathcal{R}^4$	

Five loops in D=24/5 $\quad d^8\,\mathcal{R}^4$

$d^8\,\mathcal{R}^4$ Is not protected !!!

Might be a signal for a seven-loop divergence in N=8 supergravity in D=4 dimensions

But might not!

UV properties of supergravity

- Supersymmetry plays a key rôle in constraining the topology of the Feynman diagrams Contribute to higher-loop supergravity – this is manifest in the pure spinor approach to Quantum field theory (which is inspired by Berkovits' formulation of superstring theory).

- To what extent do string theory dualities constrain the structure of perturbative supergravity? – ultraviolet divergences??

- Superstring perturbation theory is free of UV divergences. Can we understand the origin of UV divergences of N=8 supergravity by careful consideration of the low energy limit of perturbative superstring theory?