Transcendental Functions & Integrability

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Based on work with M. Isachenkov

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

DESY.

I.1 Feynman Integrals & Transcendentals

[De Calan et al.]

Step 1: Mellin-Barnes representation of Feynman integral

$$\frac{1}{(X+Y)^\mu} = \frac{1}{\Gamma(\mu)}\frac{1}{2\pi i} \times \int_{-i\infty}^{i\infty} ds \frac{Y^s}{X^{\mu+s}}\Gamma(\mu+s)\Gamma(-s)$$

Step 2: Evaluate Mellin-Barnes integral with Cauchy's theorem

Pick poles of Γ functions

Sum over poles →

$$\sum_{k_1,\ldots,k_{n+m}} \prod_{a,b} \frac{\Gamma(\sum_{i=1}^m A_{ai}k_i + B_a)}{\Gamma(\sum_{j=1}^n C_{bj}k_j + D_b)} x_1^{k_1} \cdots x_{m+n}^{k_{n+m}}$$

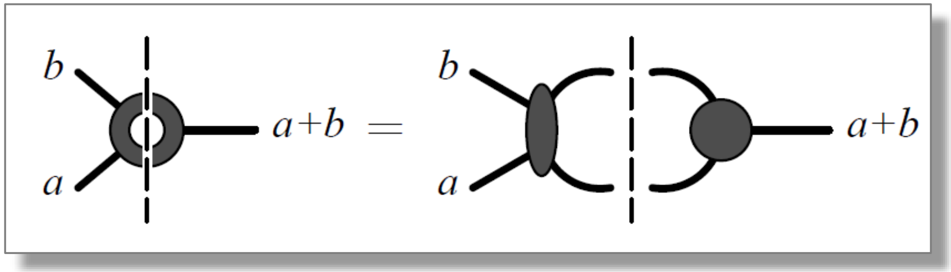
← kinematic invariants

Step 3: expand in regulator ϵ

$ABCD = ABCD(\mu,\epsilon)$ linear

I.2 Transcendentals and Polylogarithms

Example: one-loop splitting amplitude.



[Bern et al] [Kosower, Uwer]

involves $\int \frac{d^{4-2\epsilon}p}{p\cdot q} \frac{1}{p^2(p-k_a)^2(p-k_a-k_b)^2} \sim \frac{1}{\epsilon^2 z}\, {}_2F_1\left({1,-\epsilon \atop 1-\epsilon}; 1-\frac{1}{z}\right)$

Expand in regulator ϵ using

$${}_2F_1\left({1,-\epsilon \atop 1-\epsilon}; x\right) = 1 + \ln(1-x)\epsilon - \mathrm{Li}_2(x)\epsilon^2 - \mathrm{Li}_3(x)\epsilon^3 + \ldots$$

more generally use [Moch,Uwer,Weinzierl]..

I.3 Higher Transcendental Functions

Lauricella functions $F_C^{(N)}$ appear for *N-1* loop massive scalar self-energy diagrams with *N* propagators.

[Berends, Böhm, Buza, Scharf]

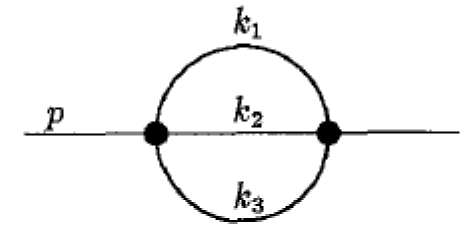
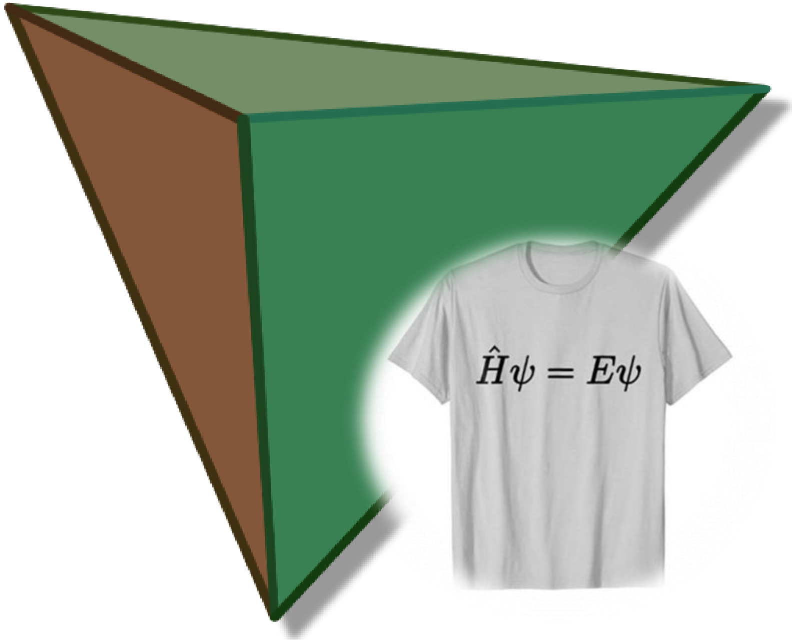


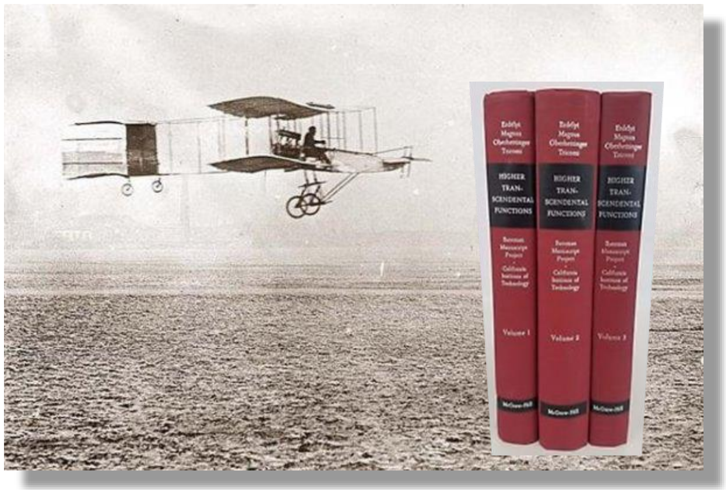
Fig. 1. The London transport diagram

Lauricella functions are wave functions of a particle in semi-infinite tetrahedron with expo-nential wall A_N Calogero-Sutherland

[Shimeno, Tamaoka 2018]



I.4 Outline



Modern theory of hypergeometric functions [1987....] is intimately linked to integrable quantum mechanics

1. **Revisiting Gaussian Hypergeometric Function**
2. **Multivariate Extensions and KZ-type Equations**

Gaussian Hypergeometric Function

2.1 Hypergeometrics & Poeschl-Teller

Hamiltonian of single Poeschl-Teller particle on half-line

$$H_{\mathrm{PT}}^{(a,b)} = -\frac{d^2}{du^2} - \frac{ab}{\sinh^2 \frac{u}{2}} + \frac{(a+b)^2 - \frac{1}{4}}{\sinh^2 u}$$

Wave functions with energy $E = -\lambda^2$ given by Gaussian hypergeometric function as

$$\psi(\lambda; u) \sim {}_2F_1\left({a+1/2+\lambda, a+1/2-\lambda \atop 1+a-b} ; -\sinh^2 \frac{u}{2} \right) \sim {}_2F_1\left({\alpha, \beta \atop \gamma} ; z \right)$$

Note: QM couplings, eigenvalues ~ QFT exponents, regulator

QM particle coordinates ~ QFT kinematic invariants

2.2 Integrability of Poeschl-Teller

On reflection symmetric functions Poeschl-Teller H_{PT} is a square $H_{PT} = \nabla^2$,

$$R\psi(u) = \psi(-u)$$

$$\nabla = \partial_u + \left[\frac{2b}{1-e^{-u}} - \frac{2a+2b+1}{1-e^{-2u}}\right] R$$

Dunkl operator

<u>Rem</u>: $\nabla, R, x = e^u$ generate dDAHA = Heisenberg/Weyl algebra for system with reflection symmetry $u \to -u$

<u>Rem</u>: Obtain wave functions from solution of KZ eqn:

$$\frac{d}{dx}F = \left[\frac{1}{x}\begin{pmatrix}\lambda & 2a-1\\ 0 & -\lambda\end{pmatrix} + \frac{(a-b+\frac{1}{2})}{1-x}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{(a+b+\frac{1}{2})}{1+x}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] F$$

2.3 Formal Knizhnik-Zamolodchikov Eqs.

Consider following Knizhnik-Zamolodchikov type eqn.

matrices

$$\frac{dG}{dz} = \left[\frac{\varrho(X)}{z} + \frac{\varrho(Y)}{1-z}\right] G$$

It possesses fundamental system of solutions of form

$$G = \sum_W Li(W;z)\varrho(W)$$

$$\frac{d}{dz} Li(XW;z) = \frac{\varrho(X)}{z} Li(W;z)$$

$$\frac{d}{dz} Li(YW;z) = \frac{\varrho(Y)}{1-z} Li(W;z)$$

In application to Poeschl-Teller wave functions choose

$$\varrho(X) = \begin{pmatrix} 0 & \beta \\ 0 & 1-\gamma \end{pmatrix} \qquad \varrho(Y) = \begin{pmatrix} 0 & 0 \\ \alpha & \alpha+\beta+1-\gamma \end{pmatrix}$$

2.4 Harmonic Polylogs from Integrability

$_2F_1$ admits convergent expansion in regime $|t_X| < 1/2$

$$ {}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};w\right) = 1 + \alpha\beta \sum_{k,n,s>0} G_0(k,n,s;w)\, t^{k-n-s} t_{\alpha\beta}^{n-s} t_\alpha^s t_\beta^s $$

$t = 1-\gamma \quad t_{\alpha\beta} = \alpha+\beta+1-\gamma$

$t_\alpha = \alpha+1-\gamma \qquad t_\beta = \beta+1-\gamma$

[Shu Oi, 2009]

$$ G_0(k,n,s;w) = \sum_{\substack{k_1+\cdots+k_n=k\\ k_1\geq 2, k_i\geq 1\\ \#\{i|k_i\geq 2\}=s}} Li_{k_1,\ldots,k_n}(z) $$

k = |w| = # of letters = weight

n = d(w) = # of letters y = depth

s = h(w) = # of yx + 1 = height

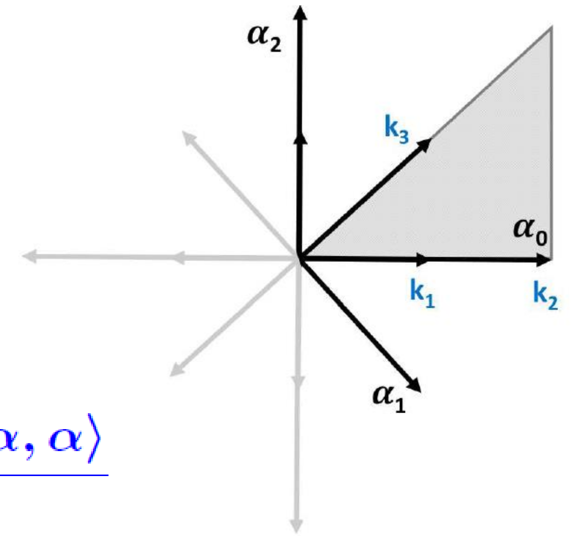
→ Formulas (75) of Moch et al.

Multivariate Extensions

3.1 Calogero-Sutherland Models

[Calogero 71] [Sutherland 72]

Integrable multi-particle generalization of Poeschl-Teller model



Associated with root systems Σ

$$H_{\rm CS} = -\sum_{i=1}^{N} \frac{d^2}{du_i^2} + \sum_{\alpha\in\Sigma^+} \frac{k_\alpha(k_\alpha + 2k_{2\alpha} - 1)\langle\alpha,\alpha\rangle}{4\sinh^2\frac{\langle\alpha,u\rangle}{2}}$$

Poeschl-Teller appears as special case $\Sigma^+_{\rm PT} = \{e_1, 2e_1\}$

$$k_1 = -2b \quad k_2 = a + b + \frac{1}{2}$$

Scattering states $\psi(\lambda, k; u)$ with momentum λ given by Heckmann-Opdam hypergeometrics [Heckman,Opdam]...

3.2 Integrability and KZ-Equations

On Weyl symmetric functions, Calogero-Sutherland H_{CS}

takes form $H_{CS} = \sum_j \nabla_j^2$ with commuting ∇_j, $i.e.$ $[\nabla_i, \nabla_j] = 0$

$$\nabla_j = \partial_j - \sum_{\alpha \in \Sigma +} \frac{k_\alpha \langle \alpha, e_j \rangle}{1 - e^{-\langle \alpha, u \rangle}} R_\alpha$$

Dunkl operators

Weyl reflection

HO Hypergeometrics obtained from solution of system of matrix Knizhnik-Zamolodchikov equations

N variables, $|\Sigma^+| + N$ singularities, |W| components

From ∇_j one can also build differential shift operators

shift parameters k_a

$\rightarrow$ Differential reduction

3.3 Bispectral-Duality and Recurrence

Rem: $\nabla_j, R_\alpha, x_i = e^{u_i}$ generate dDAHA = Heisenberg/Weyl algebra for system with Weyl reflection symmetry

Free particle: Heisenberg algebra is symmetric x $\leftrightarrow \partial$

Heckman-Opdam hypergeometrics satisfy higher order integrable difference eqs in the eigenvalues λ, e.g.

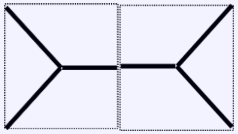
$$\sum_{j=1,2} \left[v_j(\lambda)(\psi_{\lambda+e_j} - \psi_\lambda) + v_j(-\lambda)(\psi_{\lambda-e_j} - \psi_\lambda)\right] = 4 \sum_{j=1,2} \sinh^2 u_i/2 \ \psi_\lambda$$

$$v_j(\lambda) = \frac{(\lambda_j + a + 1/2)(\lambda_j - b + 1/2)}{\lambda_j(\lambda_j + 1/2)} \frac{(\lambda_1 + \lambda_2 + \epsilon/2)(\lambda_1 - \lambda_2 + \epsilon/2)}{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)}$$

and further recurrence equations involving shifts of k_a

contiguous relations

4.1 Outlook: Hypergeometrics and CFT



$$= \quad G_{\Delta,l}(z,\bar{z}) \sim \qquad \mu = \frac{x_{10}}{x_{10}^2} - \frac{x_{20}}{x_{20}^2} \qquad \tilde{\mu} = \frac{x_{40}}{x_{40}^2} - \frac{x_{30}}{x_{30}^2}$$

$$\sim \int d^d x_0 \, x_{10}^{l+a-\Delta} x_{20}^{l-a-\Delta} x_{30}^{l-b+\Delta-d} x_{40}^{l+b+\Delta-d} \left(|\mu||\tilde{\mu}|\right)^l Y_l^d \left(\frac{\mu \cdot \tilde{\mu}}{|\mu||\tilde{\mu}|} \right)$$

Geodesic Witten diagram Zonal spherical functions

Dolan-Osborn Casimir differential equation for CPW

is eigenvalue equation for CS Hamiltonian (for BC_2)

[Isachenkov,VS]

Through connection with CS we can construct CPWs

for external long multiplets 4D supersymmetric CFTs

4.2 Conclusions & Open Questions

Modern mathematical theory of HO Hypergeometrics provides rich class functions satisfying KZ-equations & recurrence relations ….

much is known

Does this beautiful theory possess non-trivial applications in perturbative quantum field theory ?