Soft gluon evolution beyond leading colour

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- Ultimate goal: systematically to go beyond leading colour in an event generator.

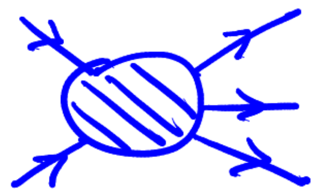
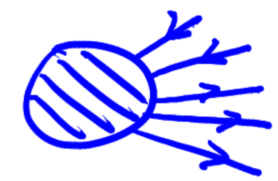
- So far: soft gluons only and for the simplest initial colour flow = electron-positron annihilation.

Plan of this talk

1. The algorithm we use and other preliminaries for Simon's talk on the implementation.

2. A remarkable result concerning the loop corrections.

Soft gluon evolution

$|H\rangle =$  $=$ 

$|H\rangle\langle H| = H =$ 

$$D_i = \sum_j T_j \, E_i \, \frac{p_j}{p_j \cdot q_i}$$

$$V_{a,b} = \exp\left[\int_a^b \frac{dE}{E}\,\Gamma\right]$$

$$\Gamma = \frac{\alpha_s}{\pi} \sum_{i<j} \left(-T_i \cdot T_j\right) \left\{ \int \frac{d\Omega_k}{4\pi}\, w_{ij}(\hat{k}) - i\pi\, \tilde{\delta}_{ij} \right\}$$

$$w_{ij}(\hat{k}) = E_k^2 \, \frac{p_i \cdot p_j}{p_i \cdot k \;\, p_j \cdot k}$$

e.g. $A_1 =$

$$d\sigma_n = \mathrm{Tr}\, A_n(\mu)\, d\Pi_n$$

$\mu = 0$ always
& $\mu = Q_0$ if inclusive
for $E < Q_0$

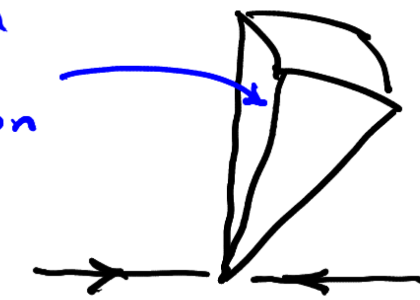
$$\Sigma = \sum_n \int d\sigma_n \; u_n(q_1, \dots, q_n)$$

measurement function

$$A_n(E) = V_{E,E_n} D_n A_{n-1}(E_n) D_n^\dagger V_{E,E_n}^\dagger \, \Theta(E \leq E_n)$$

Ángeles Martínez, de Angelis, JF, Plätzer, Seymour
JHEP 05(2018) 044
arXiv:1802.08531

An IR finite formulation



e.g. $$u_m = \prod_{i=1}^{m} \underbrace{\left[\Theta_{out}(q_i) + \Theta_{in}(q_i)\,\Theta(E_i < Q_0)\right]}_{= u(q_i)}$$

$\mu = Q_0$

Put $\Gamma = \Gamma_u + \bar{\Gamma}_u$

$$\bar{\Gamma}_u \sim \int d\Omega_k \,(1 - u(k))\, w_{ij}(\hat{k}) \sim \int_{in} d\Omega_k$$

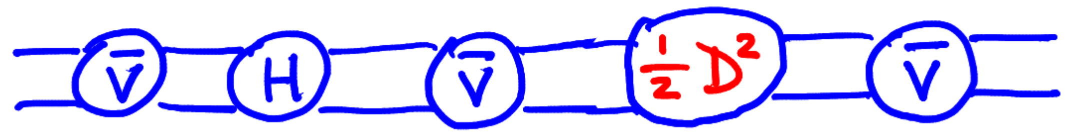
$$\Gamma_u \sim \int d\Omega_k \, u(k)\, w_{ij}(\hat{k}) \sim \int_{out} d\Omega_k$$

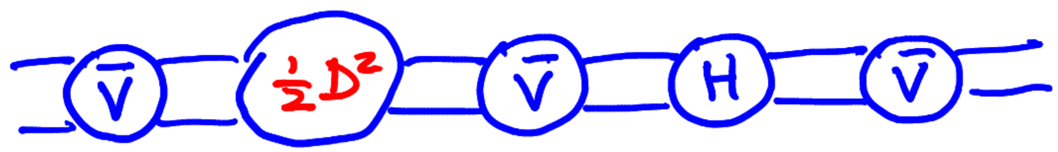
Sum $\bar{\Gamma}_u$ and expand in Γ_u

$\Rightarrow$ IR poles cancel between real & virtual order-by-order

$\Sigma_0 \sim$  = "global" part

$\Sigma_1 \sim \int_{out}$ [ = "non-global" part

$-$ 

$-$ ]

etc.

$$\Sigma_1 = -\frac{C_A C_F}{2}\zeta(2)\left(\frac{\alpha_s}{\pi}\right)^2 \log^2(Q/Q_0)$$

("out" = hemisphere)

$$+ 2 C_A^2 C_F \zeta(3)\left(\frac{\alpha_s}{\pi}\right)^3 \frac{1}{3!}\log^3(Q/Q_0) + \cdots\cdots$$

Leading N_c : Σ_n is iterative solution to BMS

$$\frac{\partial g_{ab}(t)}{\partial t} = \int_{\text{out}} \frac{d\Omega_k}{4\pi} w_{ab}(\hat{k}) \left[\frac{V_{ak} \; V_{kb}}{V_{ab}} g_{ak}(t)\, g_{kb}(t) - g_{ab}(t) \right]$$

$t = \frac{C_A \alpha_s}{\pi} \log\left(E/Q_0 \right)$

$V_{ij} = \exp\left[-t \int_{\text{in}} \frac{d\Omega_k}{4\pi} w_{ij}(\hat{k}) \right]$

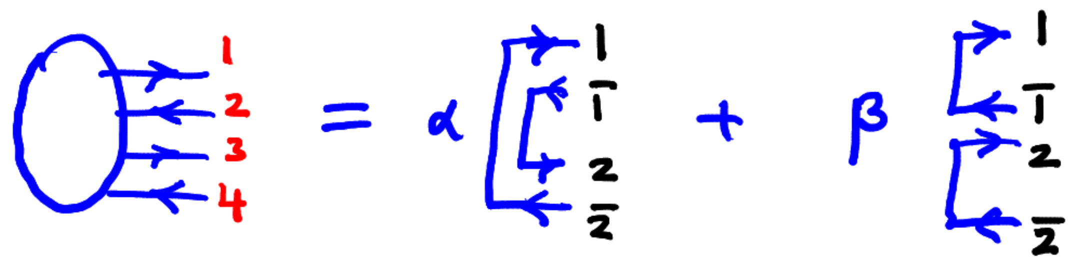
i.e. $\Sigma_n(E) = V_{ab}(t)\, g^{(n)}_{ab}(t)$

$g^{(0)}_{ab}(t) = 1$

Case of single $q\bar{q}$ pair

Colour evolution

Work in colour flow basis



i	c_i	$\bar{c}_i$
1	1	0
2	0	$\bar{1}$
3	2	0
4	0	$\bar{2}$

$$|M\rangle = \alpha \left| \begin{matrix} 1 & 2 \\ \bar{2} & \bar{1} \end{matrix} \right\rangle + \beta \left| \begin{matrix} 1 & 2 \\ \bar{1} & \bar{2} \end{matrix} \right\rangle$$

$$= \alpha\,|2\,1\rangle + \beta\,|1\,2\rangle$$

length of σ, τ

$$\langle \sigma | \tau \rangle = N_c^{\,n - \#(\sigma,\tau)}$$

\# transpositions by which σ & τ differ

e.g. $\langle 1 2 | 1 2 \rangle = N_c^2$

$\langle 2 1 | 1 2 \rangle = N_c$

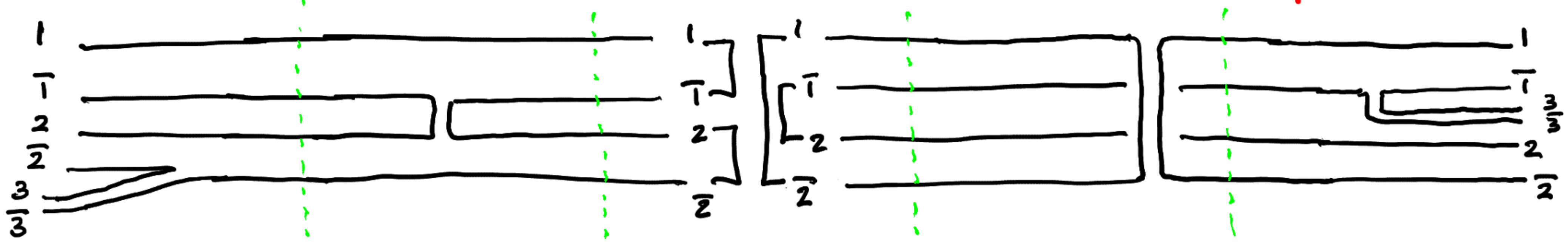
$$\sum_{\sigma} |\sigma\rangle[\sigma| = \mathbb{1} \qquad\qquad [\sigma|\tau\rangle = \langle\sigma|\tau] = \delta_{\sigma,\tau}$$

not orthonormal

"scalar product matrix"

$$\mathrm{Tr}\, A = \sum_{\tau,\sigma} [\tau|A|\sigma]\langle\sigma|\tau\rangle$$

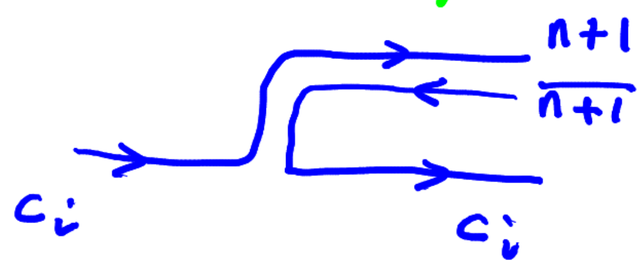
$$|\sigma_3\rangle[\sigma_3|D|\sigma_2\rangle[\sigma_2|V|\sigma_1\rangle[\sigma_1|H|\tau_1]\langle\tau_1|V^{\dagger}|\tau_2]\langle\tau_2|D^{\dagger}|\tau_3]\langle\tau_3|$$



$\sigma_3 = (3\,1\,2)$ $\quad\sigma_2 = (2\,1)$ $\quad\sigma_1 = (1\,2)$ $\quad\tau_1 = (2\,1)$ $\quad\tau_2 = (2\,1)$ $\quad\tau_3 = (2\,3\,1)$

Real emissions:

$$T_i = \lambda_i t_{c_i} - \bar{\lambda}_i \bar{t}_{\bar{c}_i} - \frac{1}{N_c}(\lambda_i - \bar{\lambda}_i)\, S$$



$$t_\alpha \left| \begin{matrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{matrix} \right\rangle = \left| \begin{matrix} 1 & \cdots & \alpha & \cdots & n & n+1 \\ \sigma(1) & \cdots & n+1 & \cdots & \sigma(n) & \sigma(\alpha) \end{matrix} \right\rangle$$

$$S \left| \begin{matrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{matrix} \right\rangle = \left| \begin{matrix} 1 & 2 & \cdots & n & n+1 \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) & n+1 \end{matrix} \right\rangle$$

$\lambda_i = 1/\sqrt{2}$ out q / in $\bar{q}$

$\bar{\lambda}_i = 0$

$\lambda_i = 0$ in q

$\bar{\lambda}_i = 1/\sqrt{2}$ out $\bar{q}$

$\lambda_i = \bar{\lambda}_i = 1/\sqrt{2}$ g

$$T\,|\sigma_n\rangle \cdots\cdots\cdots \langle\tau_n|\,T = |\sigma_{n+1}\rangle \cdots\cdots \langle\tau_{n+1}|$$

if σ_n & τ_n differ by n transpositions

then σ_{n+1} & τ_{n+1} differ by n or $n+2$ transpositions

or $n+1$ transpositions

$t \ldots s$ $\;\;1/N_c$

$t \ldots t$

$s \ldots s$ $\;\;1/N_c^2$

$t \ldots t$

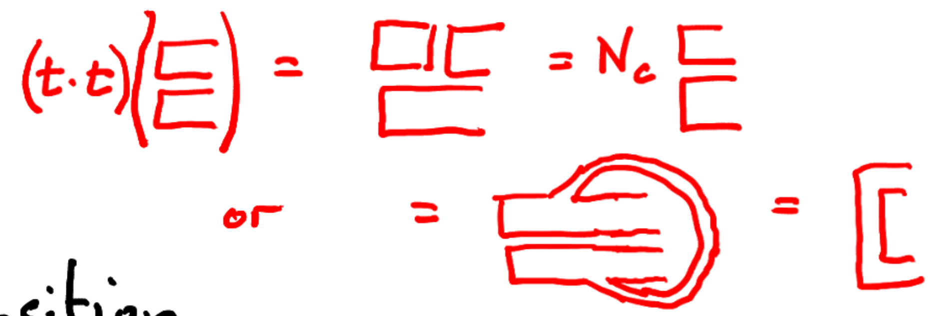
Note: cannot reduce # transpositions via real emissions

Virtual corrections

$$s\cdot t = t\cdot s = \mathbb{1}$$

$$s\cdot s = N_c\,\mathbb{1}$$

$$t\cdot t = N_c\,\mathbb{1} \quad \text{or} \quad 1 \text{ transposition}$$



$$\Gamma_\tau \left(\Gamma \middle| \sigma \right\rangle = N_c\,\delta_{\tau\sigma}\,\Gamma_\sigma + \Sigma_{\sigma\tau} + \frac{1}{N_c}\,\delta_{\tau\sigma}\,\rho$$

$T_i\cdot T_j$

$\#(\sigma,\tau) = 1$

Plätzer: Eur. Phys. J. C (2014) 74, arXiv: 1312.2448

$$[\tau|e^{\Gamma}|\sigma\rangle = \sum_{l=0}^{\overset{d}{\infty}} \frac{(-1)^l}{N_c^l} \sum_{\sigma_0,\sigma_1,\ldots\sigma_l}^{\qquad l} \delta_{\tau\sigma_0}\,\delta_{\sigma_l\sigma} \left(\prod_{\alpha=0}^{l-1} \Sigma_{\sigma_\alpha\sigma_{\alpha+1}}\right) R(\{\sigma_0,\sigma_1,\ldots,\sigma_l\})$$

e.g. $d=1$

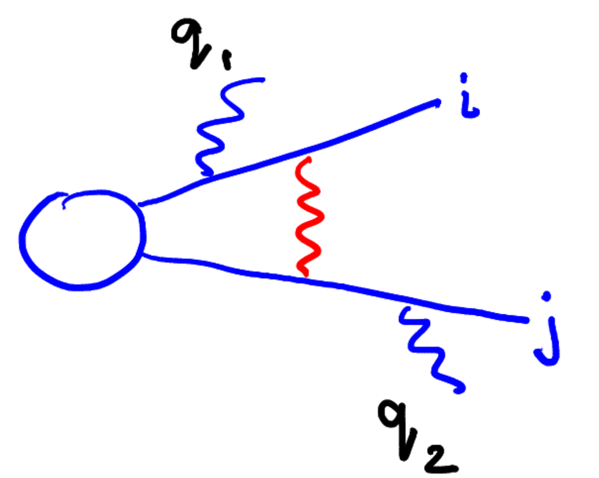
$$[\tau|e^{\Gamma}|\sigma\rangle = \delta_{\tau\sigma}\, e^{-N_c\Gamma'_\sigma} - \frac{1}{N_c}\Sigma_{\tau\sigma}\underbrace{\frac{e^{-N_c\Gamma'_\tau} - e^{-N_c\Gamma'_\sigma}}{\Gamma'_\tau - \Gamma'_\sigma}}_{= -N_c e^{-N_c\Gamma'_\sigma} \text{ if } \sigma=\tau} \qquad \left(\Gamma' = \Gamma - \frac{\rho}{N_c^2}\right)$$

"N^dLC"

= much more than N^dLC for observables

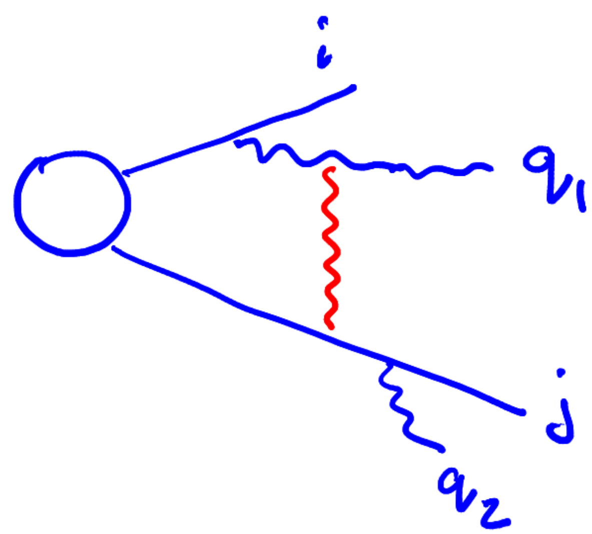
A remarkable result?

The ordering variable



$$q_{1\perp}^{ij} = \left[\frac{2\, p_i\cdot q_1 \; p_j\cdot q_1}{p_i\cdot p_j}\right]^{1/2}$$

$$\mathbf{T}_j \frac{p_j}{p_j\cdot k} \int_{\tilde{q}_2}^{\tilde{q}_1} \frac{dk_\perp}{k_\perp} \{\ldots\} \; \mathbf{T}_i \frac{p_i}{p_i\cdot k}$$



$$\mathbf{T}_j \frac{p_j}{p_j\cdot k} \int_{\tilde{q}_2}^{q_{1\perp}^{ij}} \frac{dk_\perp}{k_\perp} \{\ldots\} \; \mathbf{T}_i \left(\frac{p_i}{p_i\cdot k} - \frac{p_j}{p_j\cdot k}\right)$$

$\sum_i$ 

$$= \sum_{i\neq j} \frac{i f^{abc}\, T_i^b\, T_j^c}{C_A} \left(\frac{p_i}{p_i\cdot k} - \frac{p_j}{p_j\cdot k}\right)$$

Bierenbaum, Czakon & Mitov arXiv: 1107.4384

Thanks to Mike Seymour & René Angeles Martinez

Highly non-trivial

- full 3 & 4 gluon vertices
- exact Θ functions

R. Angeles Martinez
JF & Seymour
arXiv: 1510.07998
1602.00623

"
$$\int_0^{2\pi} \frac{d\phi}{2\pi} \frac{1 + \alpha \cos\phi}{1 + 2\alpha\cos\phi + \alpha^2} = \Theta(1 - |\alpha|)$$
"

$$\alpha = k_\perp / (q_\perp^{ij})$$

A bit more detail….

Double emission & one-loop case

- Limit 1: Both emissions are at wide angle but one gluon is much softer than the other, i.e. $(q_1^\pm \sim q_{1T}) \gg (q_2^\pm \sim q_{2T})$. Specifically, we take $q_2 \to \lambda q_2$ and keep the leading term for small λ.

- Limit 2: One emission (q_2) collinear with p_i by virtue of its small transverse momentum and the other (q_1) at a wide angle, i.e. $q_2^+ \gg q_{2T}$ and $q_1^+ \sim q_{1T} \gg q_{2T}$. Specifically, we take $q_2 \to (q_2^+, \lambda^2 q_{2T}^2/(2q_2^+), \lambda \mathbf{q}_{2T})$ and keep the leading term for small λ.

- Limit 3: One emission (q_1) collinear with p_i by virtue of its high energy and the other (q_2) at a wide angle, i.e. $q_1^+ \gg q_{1T}$ and $q_{1T} \gg q_{2T} \sim q_2^+$. Specifically, we take[3] $q_1 \to (q_1^+/\lambda, \lambda q_{1T}^2/(2q_1^+), \mathbf{q}_{1T})$ and $q_2 \to \lambda q_2$, and keep the leading term for small λ.

Limit-1

Limit-2

Limit-3

René Ángeles-Martínez: PhD thesis

Ángeles-Martínez , JRF, Seymour: JHEP 1512 (2015) 091; Physical Review Letters 116 (2016) 21 212003

Eikonal cuts

e.g. 1st row of graphs

$$G_{11} = \frac{q_1^-}{(q_2^- + q_1^-)} \int_0^{Q^2} \frac{\mathrm{d}k_T^2}{k_T^2} \qquad G_{13} = \frac{q_2^-}{(q_1^- + q_2^-)} \int_0^{Q^2} \frac{\mathrm{d}k_T^2}{k_T^2}$$

$$G_{11} + G_{13} = \int_0^{Q^2} \frac{\mathrm{d}k_T^2}{k_T^2}$$

as expected

$$G_{12} = -\left[\int_0^{2q_1^- q_2^+} \frac{\mathrm{d}k_T^2}{k_T^2} + \frac{q_2^- - q_1^-}{q_2^- + q_1^-} \int_0^{2(q_1^+ + q_2^-)^2 q_2^+ / q_1^-} \frac{\mathrm{d}k_T^2}{k_T^2} \right]$$

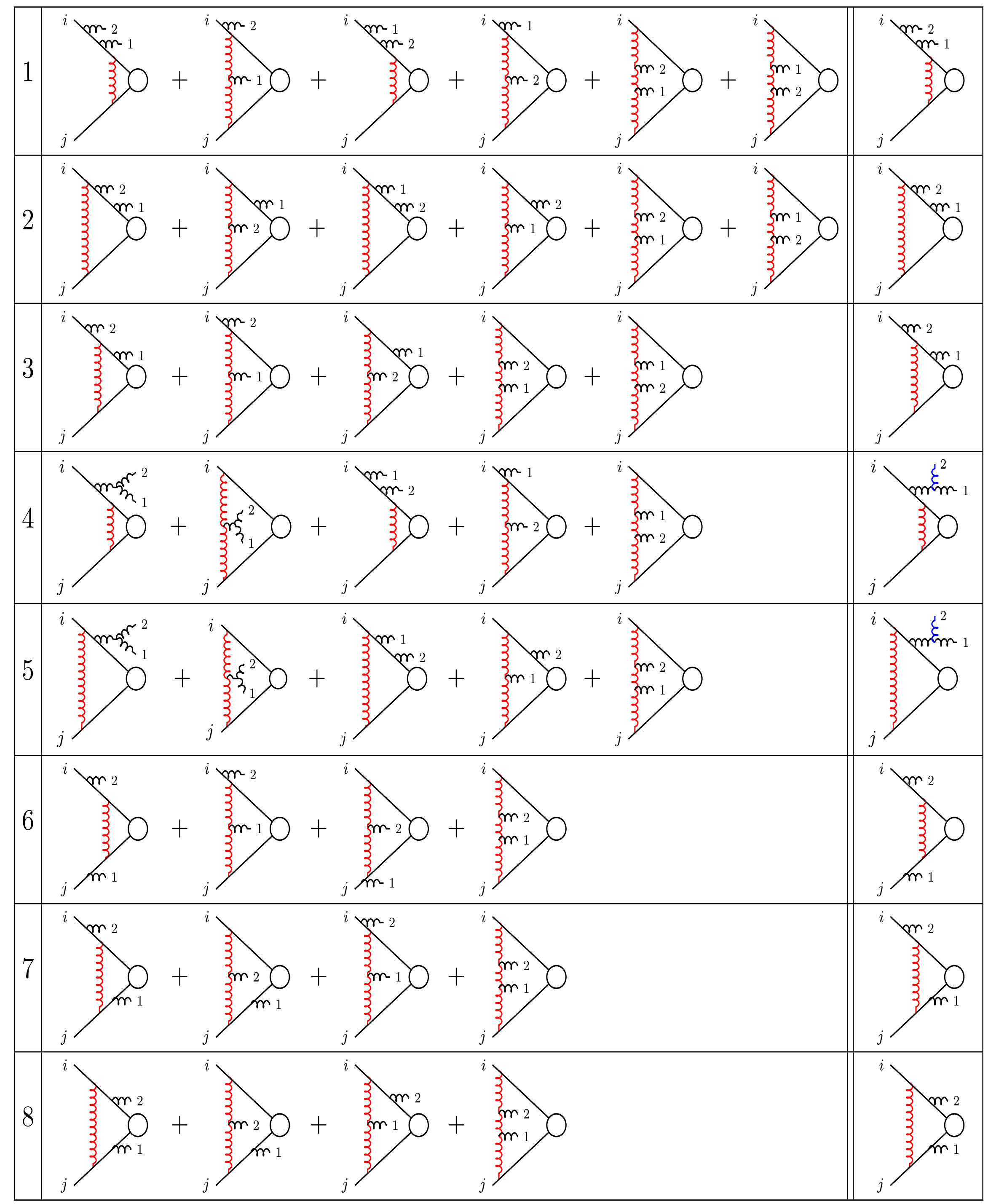
$$G_{12} + G_{14} = -\frac{1}{(q_1^- + q_2^-)} \left[q_2^- \int_0^{q_{2T}^2} \frac{\mathrm{d}k_T^2}{k_T^2} + q_1^- \int_0^{q_{1T}^2} \frac{\mathrm{d}k_T^2}{k_T^2} \right]$$

subleading in limits 1 & 2

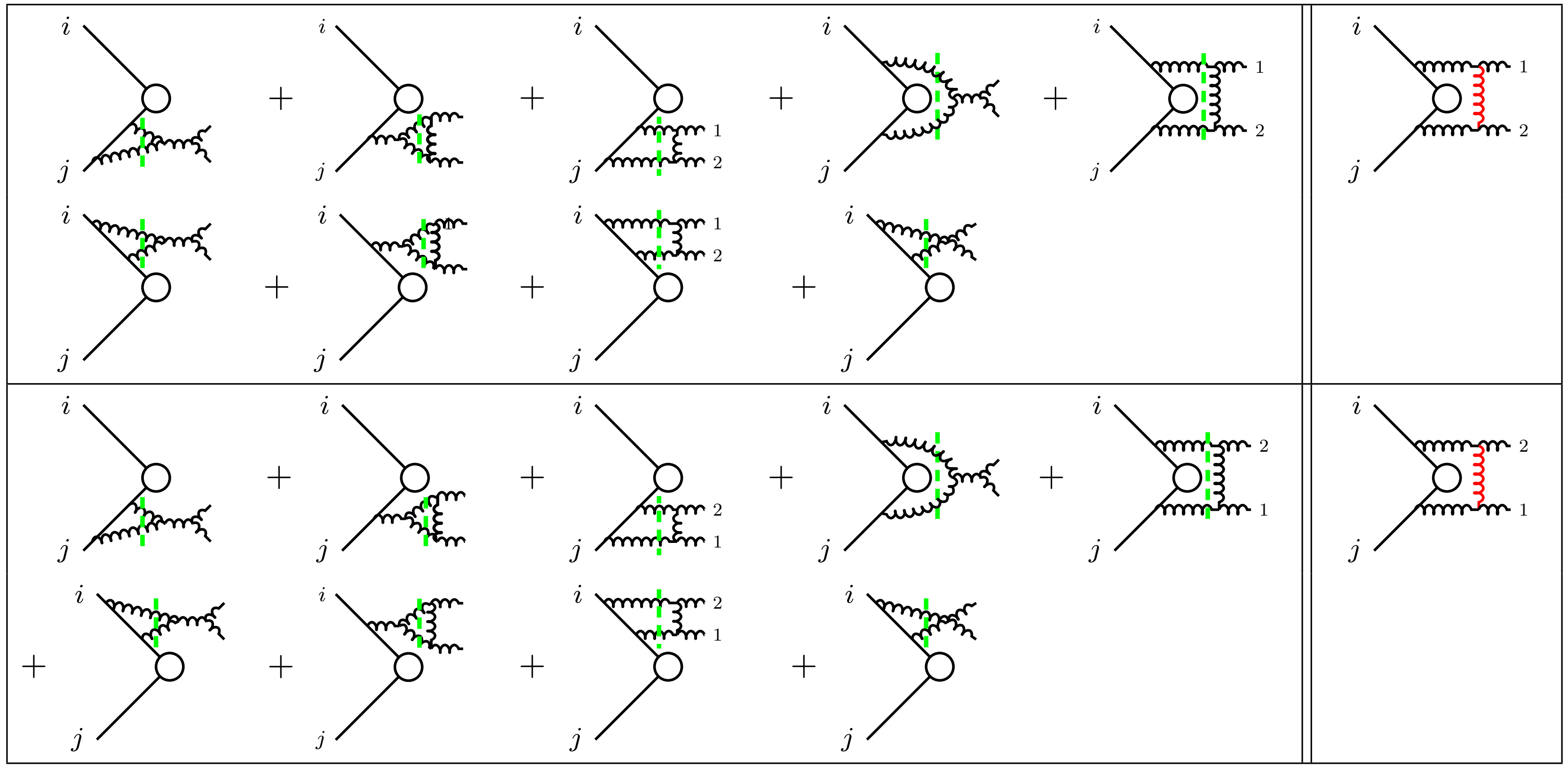
$$G_{15} + G_{16} \approx -\frac{q_2^-}{(q_1^- + q_2^-)} \int_{q_{2T}^2}^{q_{1T}^2} \frac{\mathrm{d}k_T^2}{k_T^2}$$

only leading in limit 3

In all 3 limits the sum over all graphs = $\displaystyle\int_{q_{1T}^2}^{Q^2} \frac{\mathrm{d}k_T^2}{k_T^2}$



Soft-gluon cuts



e.g. $\displaystyle -\frac{i\pi}{8\pi^2}\frac{p_j\cdot\varepsilon_1}{p_j\cdot q_1}\frac{q_1\cdot\varepsilon_2}{q_1\cdot q_2}$ in limit 1

$$G_{1c} = -\frac{3}{2}\int_{p_j\cdot q_1}^{p_j\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2} - \frac{3}{2}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2},$$

$$G_{1d} = \frac{3}{2}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2},$$

$$G_{1e} = -\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2} + \frac{1}{2}\int_{p_j\cdot q_1}^{p_j\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2}.$$

$$G_{2c} = \frac{3}{4}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2},$$

$$G_{2d} = -\frac{3}{2}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2},$$

$$G_{2e} = \frac{7}{4}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2} + \int_{p_i\cdot q_1}^{p_i\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2}.$$

$$\text{sum} = -\int_0^{(q_2^{(1j)})^2}\frac{\mathrm{d}l_T^2}{l_T^2}$$

$$\text{sum} = \int_0^{(q_2^{(1i)})^2}\frac{\mathrm{d}l_T^2}{l_T^2}$$

- Note this is ***NOT*** the dipole ordering that has previously appeared in the literature.*

- This is occurring at amplitude level.

- No statement on the ordering of the real emissions.

- Originally proved for imaginary part of loops and Drell-Yan but now proved for real part too and for general hard processes at one-loop with any number of real emissions.

Ángeles Martínez, JF, Seymour, in preparation
"A new aspect of QCD coherence"

* e.g. Caron-Huot, Neill and Vaidya, Höche and Prestel

Conclusions

- Amplitude level evolution in colour-flow basis with systematic 1/N improvements is implemented for electron-positron collisions.
  -> see Simon's talk

- Loop integrals simplify remarkably assuming eikonal couplings only to the initial hard partons.