Threshold resummation of the Drell-Yan process at NLP: soft functions and RGE

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Next-to-leading power corrections in particle physics
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Leading-logarithmic threshold resummation of the Drell-Yan process at next-to-leading power

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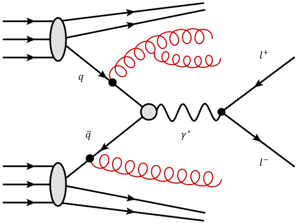
- Operator basis and renormalization
  ↪ see talk by Jian
- Factorization theorem and evaluation of collinear functions
  ↪ see talk by Sebastian

Outline

- hard function
- kinematic corrections
- soft function
- scale dependence and fixed order expansion

DY cross-section



$$A(p_A)B(p_B) \to \mathrm{DY}(Q) + X$$

$z = \frac{Q^2}{\hat{s}}$ threshold $z \to 1$

$$\Omega \sim Q(1-z)$$

$$\hat{\sigma}(z) = H(\hat{s}) \times Q^2 \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}} \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x}$$
$$\times \left\{ \widetilde{S}_0(x) + 2 \cdot \frac{1}{2} \int d\omega\, J^{(O)}_{2\xi}(x_a n_+ p_A;\omega)\, \widetilde{S}_{2\xi}(x,\omega) + \bar{c}\text{-term} \right\}$$

Scales:

- hard $\mu_h \sim Q$
- collinear $\mu_c \sim \sqrt{Q\Omega}$
- soft $\mu_s \sim \Omega$

$Q \gg \Omega$

Hard Function

$$\begin{aligned}\hat{\sigma}(z) &= \boxed{H(\hat{s})} \\ &\times \quad Q^2 \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}}\, \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x} \\ &\times \quad \left\{ \widetilde{S}_0(x) + 2 \cdot \frac{1}{2} \int d\omega\, J^{(O)}_{2\xi}(x_a n_+ p_A; \omega)\, \widetilde{S}_{2\xi}(x,\omega) + \bar{c}\text{-term} \right\}\end{aligned}$$

Hard function

When considering power corrections we have to be careful about kinematic factors. Consider the hard function

$$\bar{\psi}\gamma_\mu\psi(0) = \int dt\, d\bar{t}\, \widetilde{C}^{A0}(t,\bar{t}\,)\, J_\mu^{A0}(t,\bar{t}\,), \qquad H(\hat{s},\mu_h) = |C^{A0}(-\hat{s})|^2$$

we can obtain power corrections from expansion

$$H(\hat{s}) = H(Q^2) + Q^2(1-z)H'(Q^2) + \ldots$$

The LP factorization is

$$\hat{\sigma}(z) = H(Q^2)\, Q S_{\mathrm{DY}}(Q(1-z))$$

with

$$H(\hat{s}) = 1 + \mathcal{O}(\alpha_s) \quad \text{and} \quad S_{\mathrm{DY}}(\Omega) = \delta(\Omega) + \mathcal{O}(\alpha_s)$$

so at the LL accuracy it is enough to consider $H(Q^2)$

Hard function running

Well known RGE for two-jet operator

$$\frac{d}{d\ln\mu}H(Q^2,\mu) = \left(2\Gamma_{\text{cusp}}\ln\frac{Q^2}{\mu^2} + 2\gamma\right)H(Q^2,\mu)$$

$$\Gamma_{\text{cusp}} = \frac{\alpha_s}{\pi}C_F + \mathcal{O}(\alpha_s^2), \qquad \gamma = -\frac{3}{2}\frac{\alpha_s}{\pi}C_F + \mathcal{O}(\alpha_s^2),$$

The general solution RGE reads

$$H(Q^2,\mu) = \exp\left[4S(\mu_h,\mu) - 2a_\gamma(\mu_h,\mu)\right]\left(\frac{Q^2}{\mu_h^2}\right)^{-2a_\Gamma(\mu_h,\mu)}H(Q^2,\mu_h)$$

where

$$S(\nu,\mu) = -\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\,\frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)}\int\limits_{\alpha_s(\nu)}^{\alpha}\frac{d\alpha'}{\beta(\alpha')},$$

$$a_\Gamma(\nu,\mu) = -\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\,\frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)}, \qquad a_\gamma(\nu,\mu) = -\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\,\frac{\gamma(\alpha)}{\beta(\alpha)}$$

Kinematic corrections

$$
\begin{aligned}
\hat{\sigma}(z) &= H(\hat{s}) \\
&\times \boxed{Q^2 \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}}\, \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x}} \\
&\times \left\{ \widetilde{S}_0(x) + 2 \cdot \frac{1}{2} \int d\omega\, J_{2\xi}^{(O)}(x_a n_+ p_A; \omega)\, \widetilde{S}_{2\xi}(x,\omega) + \bar{c}\text{-term} \right\}
\end{aligned}
$$

Kinematic corrections

Soft function for generic x

$$\widetilde{S}_0(x) = \frac{1}{N_c} \operatorname{Tr} \langle 0 | \bar{\mathbf{T}}(Y_+^\dagger(x) Y_-(x)) \, \mathbf{T}(Y_-^\dagger(0) Y_+(0)) | 0 \rangle$$

Use partonic center-of-mass frame $x_a \vec{p}_A + x_b \vec{p}_B = 0$
Momentum $\vec{p}_{X_s}$ of the soft hadronic final state is balanced by the lepton-pair $\vec{q} + \vec{p}_{X_s} = 0$

$$\vec{q} \sim \lambda^2, \quad q^0 = \sqrt{\hat{s}} + \mathcal{O}(\lambda^2)$$

Energy of the soft radiation

$$[x_1 p_1 + x_2 p_2 - q]^0 = p^0_{X_s} = \sqrt{\hat{s}} - \sqrt{Q^2 + \vec{q}^{\,2}} = \frac{\Omega_*}{2} - \frac{\vec{q}^{\,2}}{2Q} + \mathcal{O}\left(\lambda^6\right)$$

with

$$\Omega_* = 2Q \frac{1 - \sqrt{z}}{\sqrt{z}} = Q(1-z) + \frac{3}{4} Q (1-z)^2 + \mathcal{O}\left(\lambda^6\right)$$

Kinematic corrections II

Expansion of the kinematic factors leads to

$$Q \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}}\, \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x}\, \widetilde{S}_0(x) \rightarrow$$

$$\int \frac{dx^0}{4\pi}\, e^{ix^0\Omega_*/2} \left(1 + \frac{ix^0 \partial^2_{\vec{x}}}{2Q} + \mathcal{O}\left(\lambda^4\right)\right) \widetilde{S}_0(x^0,\vec{x})_{|\vec{x}=0} \rightarrow$$

$$S_{\mathrm{DY}}(Q(1-z)) + \frac{1}{Q} S_{K1}(Q(1-z)) + \frac{1}{Q} S_{K2}(Q(1-z)) + \mathcal{O}(\lambda^4)$$

NLP kinematic soft functions

$$S_{K1}(\Omega) = \frac{\partial}{\partial\Omega} \partial^2_{\vec{x}} S_0\left(\Omega,\vec{x}\right)_{|\vec{x}=0}$$

$$S_{K2}(\Omega) = \frac{3}{4}\,\Omega^2 \frac{\partial}{\partial\Omega} S_0(\Omega,\vec{x})_{|\vec{x}=0}$$

Kinematic corrections III

It is more convenient to introduce

$$\Delta_{ab}(z) = \frac{\hat{\sigma}_{ab}(z)}{z}$$

$\Delta_{ab}^{\rm LP}(z) = \hat{\sigma}_{ab}^{\rm LP}(z)$ but $\Delta_{ab}^{\rm NLP}(z)$ receives additional NLP correction

$$(1-z) \times \hat{\sigma}_{\rm LP}(z)$$

which leads to

$$S_{K3}(\Omega) = \Omega\, S_0(\Omega, \vec{x})_{|\vec{x}=0}$$

Factorization theorem for $\Delta(z) = \Delta_{q\bar{q}}(z)$:

$$\begin{aligned} \Delta(z) &= H(Q^2) \\ &\times Q\left\{ S_{\rm DY}(Q(1-z)) + \sum_{i=1}^{3} \frac{1}{Q} S_{Ki}(Q(1-z)) \right. \\ &\left. +2 \cdot \frac{1}{2} \int d\omega\, J_{2\xi}^{(O)}(x_a n_+ p_A; \omega)\, \widetilde{S}_{2\xi}(x,\omega) + \bar{c}\text{-term} \right\} \end{aligned}$$

No further expansion in λ is needed!

Kinematic soft functions at $\mathcal{O}(\alpha_s)$

Expanding the kinematic factors in the factorization formula we obtain further corrections related to the LP soft function

$$
\begin{aligned}
S_{K1}(\Omega) &= \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\epsilon} + 2\ln\frac{\mu}{\Omega} - 2\right)\theta(\Omega) \\
S_{K2}(\Omega) &= \frac{\alpha_s C_F}{2\pi}\left(\frac{3}{\epsilon} + 6\ln\frac{\mu}{\Omega} + 6\right)\theta(\Omega) \\
S_{K3}(\Omega) &= \frac{\alpha_s C_F}{2\pi}\left(-\frac{4}{\epsilon} - 8\ln\frac{\mu}{\Omega}\right)\theta(\Omega)
\end{aligned}
$$

$$
\sum_{i=1}^{3} S_{Ki}(\Omega) = 2\,\frac{\alpha_s C_F}{\pi}\,\theta(\Omega)
$$

At $\mathcal{O}(\alpha_s)$ no LL kinematic corrections!

Evolution of the LP soft function

In position space, renormalization of the LP soft function is multiplicative

$$\frac{d}{d\ln\mu}\widetilde{S}_0(x) = \left[2\Gamma_{\text{cusp}}L - 2\gamma_W\right]\widetilde{S}_0(x)$$

$$L \equiv \ln\left(-\frac{1}{4}n_-x n_+x\mu^2 e^{2\gamma_E}\right)$$

$$\gamma_W = \mathcal{O}(\alpha_s^2)$$

Consider expansion of the soft function

$$\widetilde{S}_0(x) = \widetilde{S}_0(x_0) + \ldots + \frac{1}{2}\vec{\partial}_3^{\,2}\widetilde{S}_0(x)_{|\vec{x}=0}(x^3) + \ldots$$

Expansion of the log generates inhomogeneous term

$$L = L_0 - \frac{(x^3)^2}{(x^0)^2} + \mathcal{O}\left(\frac{(x^3)^4}{(x^0)^4}\right)$$

$$L_0 \equiv \ln\left(-\frac{1}{4}(x^0)^2\mu^2 e^{2\gamma_E}\right).$$

Example: expansion of the soft function RGE

Coefficient of $(x^3)^2$ gives

$$\frac{d}{d\ln\mu}\,\frac{1}{2}\vec{\partial}_3^{\,2}\widetilde{S}_0(x)_{|\vec{x}=0} = \Big[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\Big]\,\frac{1}{2}\vec{\partial}_3^{\,2}\widetilde{S}_0(x)_{|\vec{x}=0} - \frac{2}{(x^0)^2}\widetilde{S}_0(x_0)$$

Define soft functions

$$\begin{aligned}\widetilde{S}_3(x_0) &= \frac{ix_0}{2}\vec{\partial}_3^{\,2}\widetilde{S}_0(x)_{|\vec{x}=0}\,,\\ \widetilde{S}_{x_0}(x_0) &= \frac{-2i}{x^0 - i\varepsilon}\,\widetilde{S}(x_0)\end{aligned}$$

Soft functions mix

$$\begin{aligned}\frac{d}{d\ln\mu}\,\widetilde{S}_3(x_0) &= \Big[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\Big]\,\widetilde{S}_3(x_0) + \widetilde{S}_{x_0}(x_0)\\ \frac{d}{d\ln\mu}\,\widetilde{S}_{x_0}(x_0) &= \Big[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\Big]\,\widetilde{S}_{x_0}(x_0)\end{aligned}$$

Note: $\widetilde{S}_3(x_0) = \mathcal{O}\left(\alpha_s L_0\right)$ and $\widetilde{S}_{x_0}(x_0) = 1 + \mathcal{O}\left(\alpha_s L_0^2\right)$

$\widetilde{S}_{x_0}(x_0)$ corresponds to θ-soft function defined in JHEP 1808 (2018) 013 by Ian Moult, Iain W. Stewart, Gherardo Vita, Hua Xing Zhu

Kinematic correction: soft functions

To compute all kinematic corrections we define

$$\widetilde{S}_{\perp}(x_0) = \frac{ix_0}{2}\vec{\partial}_{\perp}^{2}\widetilde{S}_0(x)_{|\vec{x}=0}$$

$$\widetilde{S}_{3}(x_0) = \frac{ix_0}{2}\vec{\partial}_{3}^{2}\widetilde{S}_0(x)_{|\vec{x}=0}$$

$$\widetilde{S}_{K2}(x_0) = \frac{3}{4}\,(2i)^2\,\partial_{x^0}^2\left[\frac{ix^0}{2}\widetilde{S}(x_0)\right]$$

$$\widetilde{S}_{K3}(x_0) = 2i\partial_{x^0}\widetilde{S}(x_0)$$

$$\widetilde{S}_{x_0}(x_0) = \frac{-2i}{x^0 - i\varepsilon}\,\widetilde{S}(x_0)$$

Only $\widetilde{S}_{x_0}(x_0)$ starts at tree level!

$$\vec{S}(x^0) = \left(\widetilde{S}_{\perp}, \widetilde{S}_3, \widetilde{S}_{K2}, \widetilde{S}_{K3}, \widetilde{S}_{x_0}\right)^T$$

RGE for kinematic soft functions

Proceeding like in the example we obtain

$$\frac{d}{d\ln\mu}\,\vec{S}(x^0) \quad = \quad \Big[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\Big]\,\mathbf{1}\,\vec{S}(x) \\ + \quad \Gamma_{\rm cusp}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & -6 & +3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\vec{S}(x^0)$$

$$\frac{d}{d\ln\mu}\,\widetilde{S}_{K1+K2+K3}(x^0) = \Big[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\Big]\,\widetilde{S}_{K1+K2+K3}(x^0) - 6\,\Gamma_{\rm cusp}\,\widetilde{S}_{K3}(x^0)\,,$$

Note: $\widetilde{S}_{K1+K2+K3}(x^0) = \mathcal{O}\left(\alpha_s\right)$

No LL kinematic corrections to all orders!

Soft function

$$
\begin{aligned}
\hat{\sigma}(z) &= H(\hat{s}) \\
&\times Q^2 \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}}\, \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x} \\
&\times \left\{ \widetilde{S}_0(x) + 2\cdot\frac{1}{2}\int d\omega\, J^{(O)}_{2\xi}(x_a n_+ p_A;\omega)\, \boxed{\widetilde{S}_{2\xi}(x,\omega)} + \bar{c}\text{-term} \right\}
\end{aligned}
$$

Lagrangian insertion

Soft operator in position space

$$\widetilde{\mathcal{S}}_{2\xi}\,(x, z_-) = \bar{\mathbf{T}}\left[Y_+^\dagger(x) Y_-(x)\right] \mathbf{T}\left[Y_-^\dagger(0) Y_+(0) \frac{i\partial_\perp^\nu}{in_-\partial} \mathcal{B}^+_{\perp\nu}(z_-)\right]$$

with decoupled soft fields

$$\mathcal{B}^\mu_\pm = Y_\pm^\dagger \left[iD_s^\mu Y_\pm \right]$$

Lagrangian is already multipole expanded → soft fields depend only on z_-

$$\mathcal{L}^{(2)}_{2\xi} = \frac{1}{2} \bar{\chi}_c z_\perp^\mu z_\perp^\nu \left[i\partial_\nu in_-\partial \mathcal{B}^+_\mu \right] \frac{\not{n}_+}{2} \chi_c$$

Lagrangian insertion

Soft operator in position space

$$\widetilde{\mathcal{S}}_{2\xi}\,(x, z_-) = \bar{\mathbf{T}}\left[Y_+^\dagger(x) Y_-(x)\right] \mathbf{T}\left[Y_-^\dagger(0) Y_+(0) \frac{i\partial_\perp^\nu}{in_-\partial} \mathcal{B}_{\perp\nu}^+(z_-)\right]$$

In the factorization theorem we need only vacuum matrix element

$$S_{2\xi}(\Omega,\omega) = \int \frac{dx^0}{4\pi} \int \frac{d(n_+z)}{4\pi}\, e^{ix^0\Omega/2 - i\omega(n_+z)/2}\, \frac{1}{N_c}\, \mathrm{Tr}\,\langle 0|\widetilde{\mathcal{S}}_{2\xi}(x^0, z_-)|0\rangle$$

Lagrangian insertion

Soft operator in position space

$$\widetilde{\mathcal{S}}_{2\xi}\,(x,z_-) = \bar{\mathbf{T}}\left[Y_+^\dagger(x)Y_-(x)\right]\mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\,\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}_{\perp\nu}^+(z_-)\right]$$

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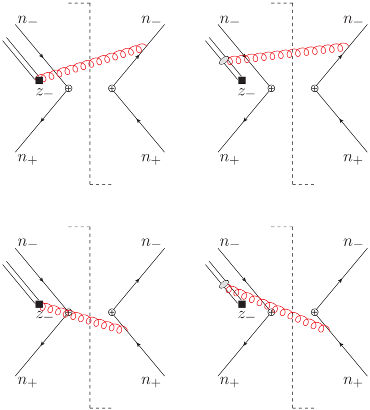

Lagrangian insertion

Soft operator in position space

$$\widetilde{\mathcal{S}}_{2\xi}\,(x, z_-) = \bar{\mathbf{T}}\left[Y_+^\dagger(x)Y_-(x)\right]\mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\perp\nu}(z_-)\right]$$

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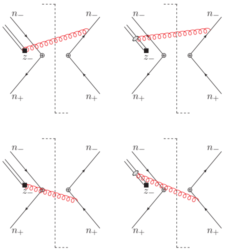
$$S_{2\xi}(\Omega,\omega) = \int\frac{dx^0}{4\pi}\int\frac{d(n_+z)}{4\pi}\,e^{ix^0\Omega/2-i\omega(n_+z)/2}\,\frac{1}{N_c}\,\mathrm{Tr}\,\langle 0|\widetilde{\mathcal{S}}_{2\xi}(x^0,z_-)|0\rangle$$



$$S_{2\xi}(\Omega,\omega) = \frac{\alpha_s C_F}{2\pi}\left\{\theta(\Omega)\delta(\omega)\left(-\frac{1}{\epsilon}+\ln\frac{\Omega^2}{\mu^2}\right)+\left[\frac{1}{\omega}\right]_+\theta(\omega)\theta(\Omega-\omega)\right\}$$

Soft function renormalization

We assume that renormalization in the momentum space is a convolution in Ω and ω

$$S_{2\xi}(\Omega,\omega)_{|\mathrm{ren}} = \int d\Omega' \int d\omega' \, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega') \, S_{2\xi}(\Omega',\omega')_{|\mathrm{bare}} + \int d\Omega' \, Z_{2\xi,x_0}(\Omega,\omega;\Omega') \, S_{x_0}(\Omega')_{|\mathrm{bare}}$$

Renormalization through mixing

$$Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega') = \delta(\Omega-\Omega')\delta(\omega-\omega') + \mathcal{O}(\alpha_s)\,,$$

$$Z_{2\xi,x_0}(\Omega,\omega;\Omega') = \frac{\alpha_s C_F}{2\pi}\frac{1}{\epsilon}\delta(\Omega-\Omega')\delta(\omega) + \mathcal{O}(\alpha_s^2)\,.$$

Soft function renormalization

We assume that renormalization in the momentum space is a convolution in Ω and ω

$$S_{2\xi}(\Omega,\omega)_{|\text{ren}} = \int d\Omega' \int d\omega'\, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega')\, S_{2\xi}(\Omega',\omega')_{|\text{bare}} + \int d\Omega'\, Z_{2\xi,x_0}(\Omega,\omega;\Omega')\, S_{x_0}(\Omega')_{|\text{bare}}$$

Aside:

Is the convolution assumption too strong?

- Dependence of Z on Ω' cannot be uniquely determined - at LP we determine it from the known properties of Wilson loop renormalization in position space – multiplicative renormalization in position space
- Dependence on ω' can by determined under additional assumptions

Soft function renormalization

We assume that renormalization in the momentum space is a convolution in Ω and ω

$$S_{2\xi}(\Omega,\omega)_{|\mathrm{ren}} = \int d\Omega' \int d\omega'\, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega')\, S_{2\xi}(\Omega',\omega')_{|\mathrm{bare}} + \int d\Omega'\, Z_{2\xi,x_0}(\Omega,\omega;\Omega')\, S_{x_0}(\Omega')_{|\mathrm{bare}}$$

Renormalization through mixing

$$Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega') = \delta(\Omega-\Omega')\delta(\omega-\omega') + \mathcal{O}(\alpha_s)\,,$$

$$Z_{2\xi,x_0}(\Omega,\omega;\Omega') = \frac{\alpha_s C_F}{2\pi}\frac{1}{\epsilon}\delta(\Omega-\Omega')\delta(\omega) + \mathcal{O}(\alpha_s^2)\,.$$

How to determine $\mathcal{O}(\alpha_s)$ of the diagonal Z-factor?

Soft operator

Let us consider an operator rather than its matrix element

$$\mathcal{S}_{2\xi}(\Omega,\omega) = \int \frac{dx^0}{4\pi}\int \frac{d(n_+z)}{4\pi} e^{i\left(x^0\Omega - n_+z\omega\right)/2}\, \overline{\mathbf{T}}\left[Y_+^\dagger(x_0)\, Y_-(x_0)\right] \times \mathbf{T}\left[Y_-^\dagger(0)\, Y_+(0)\, \frac{i\partial_{\perp\mu}}{in_-\partial}\mathcal{B}_+^\mu(z_-)\right]$$

Generalize renormalization equation to

$$\left[\mathcal{S}_A(\Omega,\omega_i)\right]_{\text{ren}} = \sum_B \int d\Omega' d\omega_j' \mathcal{Z}_{AB}\left(\Omega,\omega_i;\Omega',\omega_j'\right)\left[\mathcal{S}_B\left(\Omega',\omega_j'\right)\right]_{\text{bare}}$$

$$Z_{2\xi\,2\xi} = \frac{1}{N_c}\sum_{a,c}(\mathcal{Z}_{2\xi\,2\xi})_{aa,cc}$$

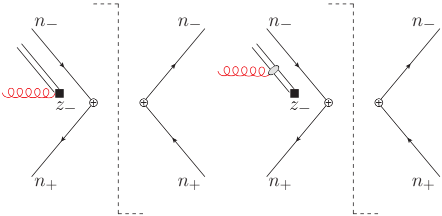
For the leading $1/\epsilon^2$ pole we will find that

$$(\mathcal{Z}_{2\xi\,2\xi})_{ab,cd} \equiv \delta_{ac}\delta_{bd}\mathcal{Z}_{2\xi\,2\xi} + \mathcal{O}(\epsilon^{-1})$$

$$Z_{2\xi\,2\xi} = \mathcal{Z}_{2\xi\,2\xi} + \mathcal{O}(\epsilon^{-1})$$

Soft matrix elements

Problem of finding Z-factor reduced to operator renormalization

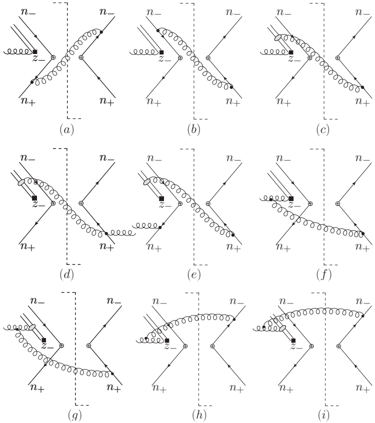


tree level matrix element is not zero

$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}} = g_s T^A \left(\frac{p_\perp \cdot \epsilon_\perp^*}{n_- p} - \frac{p_\perp^2 n_- \epsilon^*}{(n_- p)^2}\right)\delta(\Omega)\delta(\omega - n_- p).$$

Dependence on the external momentum allows to determine full dependence on ω'

One loop “real” diagrams

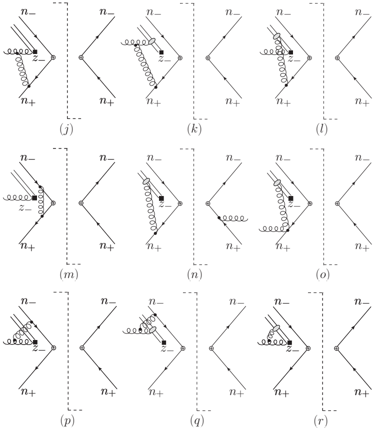


$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{a)}_{\text{1-loop}} =$$
$$\left[\frac{\alpha_s}{2\pi}\frac{C_F}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{b)}_{\text{1-loop}} =$$
$$\left[\frac{\alpha_s}{2\pi}\frac{C_F}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{c)}_{\text{1-loop}} =$$
$$\left[-\frac{\alpha_s}{4\pi}\frac{C_A}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

One loop "virtual" diagrams



$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{j)+k)}_{\text{1-loop}} =$$

$$\left[\frac{\alpha_s}{4\pi}\frac{C_A}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

Diagonal part of the anomalous dimension

We find the sum of virtual and real contribution to give a result exactly equal to the corresponding cusp anomalous dimension of the leading power soft function

$$Z^{(1)}_{2\xi\,2\xi}\left(\Omega,\omega;\Omega',\omega'\right) = -\frac{\alpha_s C_F}{\pi}\frac{1}{\epsilon^2}\delta\left(\Omega-\Omega'\right)\delta\left(\omega-\omega'\right)$$

$$\Gamma_{2\xi\,2\xi}\left(\Omega,\omega;\Omega',\omega'\right) = 4\frac{\alpha_s C_F}{\pi}\ln\frac{\mu}{\mu_s}\delta\left(\Omega-\Omega'\right)\delta\left(\omega-\omega'\right)$$

- C_A part cancels!
- leading pole is diagonal in color indices
- result is proportional to the tree level but the dependence on Ω' must be extrapolated from the LP result

Alternative approach without operator renormalization

Renormalization condition for the two-loop soft function $S_{2\xi}^{(2)}$

$$\begin{aligned}
S_{2\xi}^{(2)} + Z_{2\xi\, x_0}^{(1)} S_{x_0}^{(1)} + Z_{2\xi\, x_0}^{(2)} S_{x_0}^{(0)} + Z_{2\xi\, 2\xi}^{(1)} S_{2\xi}^{(1)} &= \text{finite} \\
S_{x_0}^{(1)} + Z_{x_0\, x_0}^{(1)} S_{x_0}^{(0)} &= \text{finite} \\
S_{2\xi}^{(1)} + Z_{2\xi\, x_0}^{(1)} S_{x_0}^{(0)} &= \text{finite}
\end{aligned}$$

Following structure

$$\Gamma = \alpha_s(\mu) \begin{pmatrix} \Gamma_{AA} \ln \frac{\mu}{\mu_s} + \gamma_{AA} & \gamma_{AB} \\ \gamma_{BA} & \Gamma_{BB} \ln \frac{\mu}{\mu_s} + \gamma_{BB} \end{pmatrix}$$

implies

$$Z_{AB}^{(2)} = \frac{1}{4} Z_{AB}^{(1)} \left(Z_{AA}^{(1)} + 3 Z_{BB}^{(1)} \right) + \mathcal{O}\left(\frac{1}{\epsilon^2} \right) \quad A \neq B.$$

$$S_{2\xi}^{(2)} - \frac{1}{4} Z_{2\xi\, x_0}^{(1)} \left(3 Z_{2\xi\, 2\xi}^{(1)} + Z_{x_0\, x_0}^{(1)} \right) S_{x_0}^{(0)} = \mathcal{O}\left(\frac{1}{\epsilon^2} \right)$$

LL soft function RGE

Both methods lead to the same AD matrix → non-trivial check of

- the choice of S_{x_0}
- the correctness of our procedure to extract leading poles
- the relation between soft operator and soft function renormalization

At the LL we have

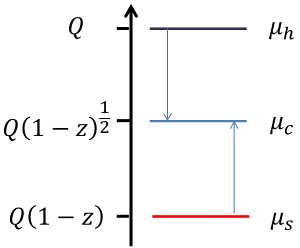
$$\frac{d}{d\ln\mu}\begin{pmatrix} S_{2\xi}(\Omega,\omega) \\ S_{x_0}(\Omega) \end{pmatrix} = \frac{\alpha_s}{\pi}\begin{pmatrix} 4C_F\ln\frac{\mu}{\mu_s} & -C_F\delta(\omega) \\ 0 & 4C_F\ln\frac{\mu}{\mu_s} \end{pmatrix}\begin{pmatrix} S_{2\xi}(\Omega,\omega) \\ S_{x_0}(\Omega) \end{pmatrix}$$

with a solution

$$\begin{aligned} S_{2\xi}^{\rm LL}(\Omega,\omega,\mu) &= \frac{2C_F}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\exp\left[-4S^{\rm LL}(\mu_s,\mu)\right]\theta(\Omega)\delta(\omega) \\ &= C_F\frac{\alpha_s}{\pi}\ln\frac{\mu_s}{\mu}\exp\left[-2C_F\frac{\alpha_s}{\pi}\ln^2\frac{\mu_s}{\mu}\right]\theta(\Omega)\delta(\omega) \end{aligned}$$

LL resummation

We run hard and soft functions to the collinear scale → no large logs in the collinear function



Inserting tree-level jet function we find

$$\Delta^{\rm LL}(z) = \Delta^{\rm LL}_{\rm LP}(z) - \exp\left[4S^{\rm LL}(\mu_h,\mu_c) - 4S^{\rm LL}(\mu_s,\mu_c)\right] \times \frac{8C_F}{\beta_0} \ln\frac{\alpha_s(\mu_c)}{\alpha_s(\mu_s)}\,\theta(1-z)$$

Can we restore full scale dependence?

Scale dependence

$$\frac{d}{d\ln\mu}\hat{\sigma}_{ab}(z,\mu) = -\sum_c \int_z^1 dx\,\left(P_{ca}(x)\hat{\sigma}_{cb}\left(\frac{z}{x},\mu\right) + P_{cb}(x)\hat{\sigma}_{ac}\left(\frac{z}{x},\mu\right)\right)$$

Restoring scale dependence

We know that the scale dependence of the $\hat{\sigma}_{ab}(z,\mu)$ is canceled by the PDF

$$\frac{d}{d\ln\mu}\hat{\sigma}_{ab}(z,\mu) = -\sum_c \int_z^1 dx \left(P_{ca}(x)\hat{\sigma}_{cb}\left(\frac{z}{x},\mu\right) + P_{cb}(x)\hat{\sigma}_{ac}\left(\frac{z}{x},\mu\right)\right)$$

AP splitting kernel should be expanded around $x = 1$

$$P_{ab}(x) = P_{ab}^{\mathrm{LP}}(x) + P_{ab}^{\mathrm{NLP}} + \mathcal{O}(1-x)$$

where

$$P_{ab}^{\mathrm{LP}}(x) = \left(2\Gamma_{\mathrm{cusp}}(\alpha_s)\,\frac{1}{[1-x]_+} + 2\gamma^{\phi}(\alpha_s)\delta(1-x)\right)\delta_{ab}\,,$$
$$P_{ab}^{\mathrm{NLP}} = \gamma_{ab}^{\mathrm{NLP}}(\alpha_s)$$

Our $\Delta(z,\mu)$ obeys RGE

$$\frac{d}{d\ln\mu}\Delta(z,\mu) = -2\int_z^1 \frac{dx}{x} P_{qq}(x)\Delta\left(\frac{z}{x},\mu\right) + \mathcal{O}(\lambda^4)$$

We can formally expand

$$\Delta(z,\mu) = \Delta_{\mathrm{LP}}(z,\mu) + \Delta_{\mathrm{NLP}}(z,\mu) + \ldots$$

Restoring scale dependence II

$\Delta_{\mathrm{NLP}}(z,\mu)$ obeys RGE

$$\frac{d}{d\ln\mu}\Delta_{\mathrm{NLP}}(z,\mu) = -4\left[\Gamma_{\mathrm{cusp}}(\alpha_s)\ln(1-z)+\ldots\right]\Delta_{\mathrm{NLP}}(z,\mu) + K(z,\mu)$$

with the inhomogeneous term given by

$$K(z,\mu) = -2\gamma_{qq}^{\mathrm{NLP}}(\alpha_s)\int_z^1 dy\,\Delta_{\mathrm{LP}}(y,\mu) - 4\,\Gamma_{\mathrm{cusp}}(\alpha_s)(1-z)\Delta_{\mathrm{LP}}(z,\mu)$$

The solution is

$$\Delta_{\mathrm{NLP}}(z,\mu) = e^{\hat{S}(z,\mu_c,\mu)}\,\Delta_{\mathrm{NLP}}(z,\mu_c) + \int_{\ln\mu_c}^{\ln\mu} d\ln\mu'\, e^{\hat{S}(z,\mu',\mu)}\,K(z,\mu')$$

with

$$\hat{S}(z,\nu,\mu) = 4a_\Gamma(\nu,\mu)\ln(1-z)\,.$$

Fixed order check

For arbitrary μ we then find

$$\Delta^{\rm LL}_{\rm NLP}(z,\mu) \quad = \quad \exp\left[4S^{\rm LL}(\mu_h,\mu) - 4S^{\rm LL}(\mu_s,\mu)\right] \times \frac{-8C_F}{\beta_0} \ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\,\theta(1-z)$$

Note $\Delta^{\rm LL}_{\rm NLP}(z,\mu_c)$ has the same form $\rightarrow$ *no LL in collinear function!*

$$S^{\rm LL}(\mu_1,\mu_2) = -\frac{\alpha_s C_F}{2\pi}\ln^2\frac{\mu_2}{\mu_1} \quad \text{and} \quad \frac{1}{\beta_0}\ln\frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)} = \frac{\alpha_s}{2\pi}\ln\frac{\mu_2}{\mu_1}$$

Our result

$$\Delta^{\rm LL}_{\rm NLP}(z,\mu) = \frac{\hat{\sigma}^{\rm LL}_{\rm NLP}(z,\mu)}{z} \quad = \quad \exp\left[2\frac{\alpha_s C_F}{\pi}\ln^2\frac{\mu}{\mu_s} - 2\frac{\alpha_s C_F}{\pi}\ln^2\frac{\mu}{\mu_h}\right] \times(-4)\frac{\alpha_s C_F}{\pi}\ln\frac{\mu_s}{\mu}\,\theta(1-z)$$

agrees with

- R. Hamberg, W. L. van Neerven and T. Matsuura, 1991, full fixed order NNLO computation
- D. de Florian, J. Mazzitelli, S. Moch and A. Vogt, 2014 approximate results for $\mu=\mu_h$ up to N^4LO

Fixed order expanded result

- R. Hamberg, W. L. van Neerven and T. Matsuura, 1991
- D. de Florian, J. Mazzitelli, S. Moch and A. Vogt, 2014

$$\begin{aligned}
\Delta^{\mathrm{LL}}_{\mathrm{NLP}}(z,\mu) \quad = \quad & -\theta(1-z)\Bigg\{4C_F\frac{\alpha_s}{\pi}\Big[\ln(1-z)-L_\mu\Big] \\
& +8C_F^2\left(\frac{\alpha_s}{\pi}\right)^2\Big[\ln^3(1-z)-3L_\mu\ln^2(1-z)+2L_\mu^2\ln(1-z)\Big] \\
& +8C_F^3\left(\frac{\alpha_s}{\pi}\right)^3\Big[\ln^5(1-z)-5L_\mu\ln^4(1-z)+8L_\mu^2\ln^3(1-z)-4L_\mu^3\ln^2(1-z)\Big] \\
& +\frac{16}{3}C_F^4\left(\frac{\alpha_s}{\pi}\right)^4\Big[\ln^7(1-z)-7L_\mu\ln^6(1-z)+18L_\mu^2\ln^5(1-z)-20L_\mu^3\ln^4(1-z) \\
& \qquad +8L_\mu^4\ln^3(1-z)\Big] \\
& +\frac{8}{3}C_F^5\left(\frac{\alpha_s}{\pi}\right)^5\Big[\ln^9(1-z)-9L_\mu\ln^8(1-z)+32L_\mu^2\ln^7(1-z)-56L_\mu^3\ln^6(1-z) \\
& \qquad +48L_\mu^4\ln^5(1-z)-16L_\mu^5\ln^4(1-z)\Big]\Bigg\}+\mathcal{O}(\alpha_s^6\times(\log)^{11}),
\end{aligned}$$

$L_\mu = \ln(\mu/Q)$.

Summary and Conclusions

- NLP LL threshold resummation achieved thanks to systematic expansion in position space SCET
- Next step: extension to NLL
  - Anomalous dimension of the hard function is known but rather complicated beyond LL
  - Renormalization of the soft functions must be better understood
    - what is the complete basis of functions/operators that is closed under RGE
    - is there a choice of variables for which the renormalization becomes multiplicative?
  - Are the ω_i convolution integrals convergent beyond LL
  - RGE for the collinear functions

Auxiliary slide: soft function in position space

At the one-loop order in dimensional regularization with $d = 4 - 2\epsilon$, the bare soft function must have a simple dependence

$$\tilde{S}_{0,\text{bare}}(x) = 1 + \frac{\alpha_s}{\pi}\left(-n_- x n_+ x \mu^2\right)^{\epsilon} f\left(\epsilon, \frac{x^2}{n_+ x n_- x}\right)$$

Explicit evaluation gives

$$\begin{aligned}
\widetilde{S}_{0,\text{bare}}(x) &= 1 + \frac{\alpha_s C_F}{\pi}\frac{\Gamma(1-\epsilon)}{\epsilon^2} e^{-\epsilon\gamma_E} \\
&\times \left(-\frac{1}{4} n_- x n_+ x \mu^2 e^{2\gamma_E}\right)^{\epsilon}\left(\frac{x^2}{n_- x n_+ x}\right)^{1+\epsilon} {}_2F_1\left(1, 1, 1-\epsilon; 1 - \frac{x^2}{n_- x n_+ x}\right) \\
&= 1 + \frac{\alpha_s C_F}{\pi}\left(\frac{1}{\epsilon^2} + \frac{L}{\epsilon} + \frac{L^2}{2} + \frac{\pi^2}{12} + \text{Li}_2\left(1 - \frac{x^2}{n_- x n_+ x}\right) + \mathcal{O}(\epsilon)\right)
\end{aligned}$$

where we defined

$$L \equiv \ln\left(-\frac{1}{4} n_- x n_+ x \mu^2 e^{2\gamma_E}\right).$$