Threshold effects beyond leading power in Drell-Yan production

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PARTICLEFACE Meeting on Next-to-Leading Power Corrections

- Introduction to threshold logarithms.
- Drell-Yan production: why is it useful?
- Next-to-leading power (NLP) threshold logs up to N^3LO.
- Relation to NLP factorisation.

- If ξ is a kinematic variable that is zero at threshold for some process:

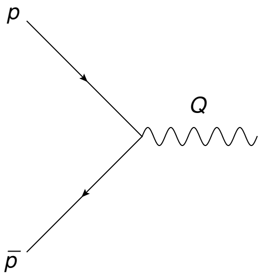
$$\frac{d\sigma}{d\xi} = \sum_{n,m} \alpha^n \left[c_{nm}^{(0)} \left(\frac{\ln^m \xi}{\xi} \right)_+ + c_{nm}^{(1)} \ln^m \xi + \ldots \right].$$

- First set of terms correspond to (leading) threshold logs: pure soft and / or collinear.
- Second set of terms is next-to-leading power (NLP) threshold logs: next-to-soft and / or collinear.
- As $\xi \to 0$, must sum these terms to all orders in perturbation theory (*resummation*) to get sensible results.

Why study threshold effects?

- There are many (hundreds?) of observables for which LP resummation is important.
- NLP effects can be numerically sizeable e.g. Higgs production (Kramer, Laenen, Spira; Herzog; Mistlberger).
- There are three main uses of NLP threshold effects:
  1. Resummation may be necessary close to threshold.
  2. Can be used to obtain (good) approximate higher order cross-sections.
  3. Can improve numerical convergence / stability of fixed-order cross-section codes (e.g. NLO, NNLO).
- Precision LHC data makes this increasingly relevant!

- Drell-Yan production has been a traditional testing ground for resummation ideas.



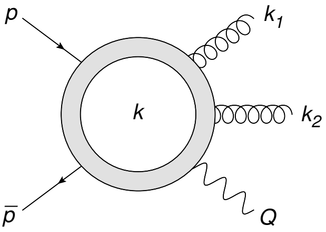
- Let $z = Q^2/s$ be the fraction of (squared) energy s carried by the vector boson.
- At threshold, $\xi \equiv (1-z) \to 1$.

- For threshold logs, real radiation must be (next-to-) soft.
- No final state *collinear* radiation, thus a simpler playground for threshold effects.

- Differential cross-section in z known at NLO in 1979 (Altarelli, Ellis, Martinelli), and NNLO in 1991 (Hamberg, van Neerven, Matsuura).
- More differential quantities also calculated in 2000s (e.g. Anastasiou, Dixon, Melnikov, Petriello).
- LP logs resummed up to next-to-next-to-leading logarithmic (NNLL) order using a variety of methods (1980s-2010s).
- Small z resummation also known at NLL order, 2008 (Marzani, Ball).
- NLP logs (conjecturally) resummed up to NLL using *physical evolution kernels*, 2009 (Moch, Vogt).
- NLP logs at LL resummed using Soft Collinear Effective Theory (SCET), 2018 (Beneke, Broggio, Garny, Jaskiewicz, Szafron, Vernazza, Wang).

- Higher precision in DY is needed for precision SM tests and background modelling.
- Threshold corrections are particularly important (e.g. W and Z mass measurements).
- This motivates the calculation of (N)LP threshold contributions at N^{3}LO:
  1. Threshold corrections are a precursor to a full N^{3}LO calculation.
  2. They can be used to test existing conjectures about NLP logs.
  3. They can be used to formulate *new* conjectures / theorems about NLP logs.
- In a given process at fixed order in α_s, we can use the *method of regions* to classify all threshold effects (Beneke, Smirnov, Pak, Jantzen).

- Let us focus on the 1-loop, 2-real contribution to DY production at N^{3}LO.



- We can use a *Sudakov decomposition* for the loop momentum k.
- Introduce n_+, n_- via

$$p^\nu = \frac{\sqrt{\hat{s}}}{2} n_+^\mu, \quad \bar{p}^\mu = \frac{\sqrt{\hat{s}}}{2} n_-^\mu .$$

- Then write

$$k^\mu = \underbrace{\frac{(n_- \cdot k)}{2} n_+^\mu}_{k_+} + \underbrace{\frac{(n_+ \cdot k)}{2} n_-^\mu}_{k_-} + k_\perp^\mu$$

- Here $k_\perp = (0, \boldsymbol{k}_\perp, 0)$ is transverse to p and $\bar{p}$ i.e. $k_\perp \cdot n_\pm = 0$.

Method of Regions

- One may introduce a book-keeping parameter $\lambda \sim (1 - z)$ that keeps track of which components of k^{μ} are small.
- Then the singular regions of the loop momentum (k) integration can be phrased in terms of $(k_+, \boldsymbol{k}_\perp, k_-)$:

$$\text{Hard}: \quad k \sim \sqrt{\hat{s}}\,(1,1,1)\ ; \qquad \text{Soft}: \quad k \sim \sqrt{\hat{s}}\left(\lambda^2, \lambda^2, \lambda^2\right)\ ;$$

$$\text{Collinear}: \quad k \sim \sqrt{\hat{s}}\left(1, \lambda, \lambda^2\right)\ ; \quad \text{Anticollinear}: \quad k \sim \sqrt{\hat{s}}\left(\lambda^2, \lambda, 1\right)\ .$$

- These are the only relevant regions for (inclusive) threshold production.
- Expansion of the loop integrand in λ amounts to LP, NLP... logs in the final result for the cross-section.

- Loop integrals are invariant under shifts of the loop momentum:

$$k^\mu \to k^\mu + \sum_i \alpha_i p_i^\mu .$$

- This invariance is broken by the expansion in λ.
- Singular regions correspond to poles in propagators.
- For some choices of k, these poles do not straightforwardly correspond to the scaling behaviours of k outlined before.
- One may naïvely "miss" regions, so that care is needed.
- Discussed already by Beneke & Smirnov; for an explicit example in DY, see arXiv:1807.09246 (Bahjat-Abbas, Sinninghe Damsté, Vernazza, White).

Method of Regions: another subtlety

- We use dimensional regularisation in $d = 4 - 2\epsilon$ dimensions.
- One then finds that the hard region has poles in ϵ, whereas similar contributions to the soft region vanish.
- However, the λ expansion in the soft region introduces *spurious UV poles*.
- Introducing counterterms would shift the singular contribution from the hard to the soft region.
- Then the hard region is IR finite, and the soft region IR singular, as we would expect.
- Instead we can choose not to, so that the hard region remains singular...

Singular stuff in the hard region is soft stuff in disguise!

Application to DY production

- Now let us return to the 1-virtual, 2-real contributions to DY.
- We will further restrict to abelian-like contributions, with a colour factor $\sim C_F^3$.
- Feynman diagrams for the amplitude were generated with `QGRAF` (Nogueira), and reduced to scalar integrals using `Reduze` (von Manteuffel, Studerus).
- Method of regions applied to each scalar integral, using `Asy` to cross-check (Jantzen, Pak, Smirnov^2).
- Contribution to the cross-section is through the interference term

$$\mathcal{M} = \int \frac{d^d k}{(2\pi)^d} \mathcal{A}_{\mathrm{2r,1v}} \mathcal{A}_{2r}^{\dagger}.$$

- Two independent calculations, with full agreement.

- For the cross-section, we must integrate the squared matrix element over the phase space:

$$\frac{d\sigma}{dz} \sim \int d\Phi^{(3)} \delta\left(z - \frac{Q^2}{s}\right) \mathcal{M}.$$

- To discuss the results, it is useful to define some invariants:

$$t_{2,3} = (p - k_{1,2})^2, \quad u_{2,3} = (\bar{p} - k_{1,2})^2, \quad s_{12} = 2k_1 \cdot k_2.$$

- The squared matrix elements in each region have a relatively compact form...
- ...although the integrals over the phase space are extremely challenging!

- Squared matrix elements in hard / collinear regions:

$$\mathcal{M}^{\rm LP}_{\rm hard} = \mathcal{N}\left(\frac{\mu^2_{\overline{\rm MS}}}{-s}\right)^{\epsilon} f_1^{\rm H}\,\frac{s^3}{t_2\,t_3\,u_2\,u_3}, \quad \mathcal{N} = 128\pi\alpha_s^3(1-\epsilon)C_F^3 e_q^2 N_c;$$

$$\mathcal{M}^{\rm NLP}_{\rm hard} = \mathcal{N}\left(\frac{\mu^2_{\overline{\rm MS}}}{-s}\right)^{\epsilon} \frac{s^2(t_2+t_3+u_2+u_3)}{t_2\,t_3\,u_2\,u_3}\left[f_2^{\rm H} + \frac{1}{2}\,\frac{t_2\,u_3+t_3\,u_2-s_{12}\,s}{(t_2+t_3)(u_2+u_3)}\,f_1^{\rm H}\right];$$

$$\mathcal{M}^{\rm LP}_{\rm col.} = 0;$$

$$\begin{aligned}\mathcal{M}^{\rm NLP}_{\rm col.} = {}& \mathcal{N}(\mu^2_{\overline{\rm MS}})^{\epsilon}\,\frac{s^2}{t_2 t_3 u_2 u_3}\Bigg\{\Big[u_2(-t_2)^{-\epsilon} + u_3(-t_3)^{-\epsilon}\Big] f_1^{\rm C} \\ &+ \frac{t_3 u_2 + t_2 u_3 - s_{12} s}{t_2+t_3}\Bigg[\Big((-t_2)^{-\epsilon} - 2(-t_2-t_3)^{-\epsilon} + (-t_3)^{-\epsilon}\Big) f_2^{\rm C} \\ &- \left(\frac{t_2}{t_3}(-t_2)^{-\epsilon} - \frac{(t_2^2+t_3^2)}{t_2 t_3}(-t_2-t_3)^{-\epsilon} + \frac{t_3}{t_2}(-t_3)^{\epsilon}\right) f_3^{\rm C}\Bigg]\Bigg\}.\end{aligned}$$

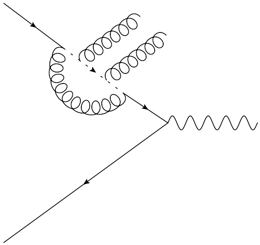
- Each f_i^X is a Laurent series in ϵ, with constant coefficients.
- Full results in arXiv:1807.09246 (Bahjat-Abbas, Sinninghe Damsté, Vernazza, White).

Hard and collinear regions

- The matrix elements do not look too bad at first glance.
- However, they contain fractional powers of the invariants t_i.
- Nevertheless, it is possible to carry out all phase space integrals *analytically*...
- ...after finding a particularly nice parametrisation.
- This turns out to be a Sudakov decomposition for the loop momentum k.
- Similar techniques have been considered previously for NLO subtraction applications (Campbell, Ellis, Mondini, Williams).

Soft Region

- Soft contributions $\propto (p \cdot \bar{p})^{-\epsilon}$ sit in the hard region (see earlier).
- However, there is an interesting non-vanishing soft region that occurs for the first time at N^3LO.



- An incoming hard fermion can become soft by emitting a hard gluon.
- The soft fermion can then emit two soft gluons.

- The somewhat peculiar nature of this contribution suggests it will be heavily suppressed in logarithmic order.
- We will see that this is indeed the case!

- The squared matrix element in the soft region is NLP only:

$$
\begin{aligned}
\mathcal{M}_{\rm soft}^{\rm NLP} = \mathcal{N}\left(\frac{\mu_{\overline{\rm MS}}^2}{-s_{12}}\right)^{\epsilon} \frac{s^2}{t_2 t_3 u_2 u_3} \\
\times \Bigg\{ \frac{t_3\, f_1^{\rm S}}{t_2(t_2+t_3)^2}\bigg[(s_{12}s - t_2 u_3 - t_3 u_2)\Big(t_2 + t_3 - t_3\, {}_2F_1\Big(1,1,1-\epsilon,\tfrac{t_2}{t_2+t_3}\Big)\Big)\bigg] \\
+ \frac{f_2^{\rm S}}{s\, s_{12}(t_2+t_3)}\left[(t_2 u_3 - t_3 u_2)^2 - s_{12}s(t_2 u_3 + t_3 u_2)\right] \\
+ \frac{f_3^{\rm S}}{s\, s_{12}\, t_2(t_2+t_3)^2}\bigg[s_{12}^2 s^2 t_3(t_2 - t_3) + t_3(t_2+t_3)(t_2 u_3 - t_3 u_2)^2 \\
+ s_{12} s t_2(t_2+t_3)(t_2 u_3 - 3 t_3 u_2) - t_3\Big(s_{12}^2 s^2 (t_2 - t_3) + (t_2+t_3)(t_2 u_3 - t_3 u_2)^2 \\
- 2 s_{12} s t_2 (t_2 u_3 + t_3 u_2)\Big)\, {}_2F_1\Big(1,1,1-\epsilon,\tfrac{t_2}{t_2+t_3}\Big)\bigg] \\
+ \{t_2, t_3 \leftrightarrow u_2, u_3\} + \{t_2, t_3 \leftrightarrow u_3, u_2\} + \{t_2, u_2 \leftrightarrow t_3, u_3\}\Bigg\}.
\end{aligned}
$$

- It is proportional to $(2k_1 \cdot k_2)^{-\epsilon}$.
- This requires at least two real gluons, and one virtual, hence why this is new at N^{3}LO.

- Terms without hypergeometric functions can be integrated using similar techniques to NNLO (van Neerven, Hamberg, Matsuura).
- Some terms with a hypergeometric function can be integrated similarly, for arbitrary d.
- Others require Mellin-Barnes techniques: we encounter up to six-fold Mellin-Barnes integrals.
- Expanding in ϵ, these reduce via Barnes lemmas.
- Various software packages were useful e.g. `MB` (Czakon), `MBresolve` (Smirnov^2), `barnesroutines` (Kosower), Eule (`Gürdoğan`), xSummer (Moch, Uwer), FORM (Vermaseren).
- Similar techniques were used for Higgs production at N^3LO (Anastasiou, Duhr, Dulat, Herzog, Mistlberger).

Results for the K factor

- It is convenient to factor out the LO cross-section and define the *K factor* at $\mathcal{O}(\alpha_s^n)$

$$\left(\frac{\alpha_s}{4\pi}\right)^n K^{(n)}(z) = \frac{1}{\sigma_0}\frac{d\sigma^{(n)}(z)}{dz}.$$

- Then the NLP terms at $\mathcal{O}(\epsilon^0)$ in the hard / collinear regions:

$$K^{(3),H}_{q\bar{q}}\big|_{C_F^3} = 128\left[\frac{128}{15}L^5 - \frac{128}{3}L^4 + \left(\frac{248}{3} - 112\zeta_2\right)L^3 + (-144 + 336\zeta_2 + 184\zeta_3)L^2 + \left(144 - \frac{651}{2}\zeta_2 - 368\zeta_3 + \frac{1017}{4}\zeta_4\right)L\right];$$

$$K^{(3),C}_{q\bar{q}}\big|_{C_F^3} = 32\left[-\frac{625}{24}L^4 + \frac{625}{24}L^3 + \left(-\frac{75}{4} + \frac{525\zeta_2}{4}\right)L^2 + \left(10 - \frac{525}{8}\zeta_2 - 205\zeta_3\right)L\right].$$

- Note that the collinear region is NLL, matching results at lower orders (Bonocore, Laenen, Magnea, Vernazza, White).
- See arXiv:1807.09246 for full results.

- The contribution from the soft region is remarkably simple...

$$K_{q\bar{q}}^{(3),\mathrm{S}}\Big|_{C_F^3} = 32\left\{\frac{1}{\epsilon}\left(\frac{2}{3}\zeta_2 + \frac{1}{3}\zeta_3\right) - (4\zeta_2 + 2\zeta_3)L\right\}.$$

- ...and agrees with our earlier expectation that it would be heavily suppressed.
- In fact it starts at NNNNLL order!
- Interestingly, it has mixed transcendentality weight - connection to Wilson line calculations in QCD?
- Also suggests it would vanish in $\mathcal{N} = 4$ SYM.

- We have calculated threshold effects in the 2-real, 1-virtual contribution to DY at N^{3}LO.
- The calculation proceeds up to NLP level, and considers abelian-like contributions $\sim C_F^3$ only.
- Work in progress includes:
  1. Full non-abelian contributions (other colour structures).
  2. Other initial states.
  3. Triple and single real emission contributions.
- However, the results obtained thus far are already useful for classifying the general structure of NLP effects.

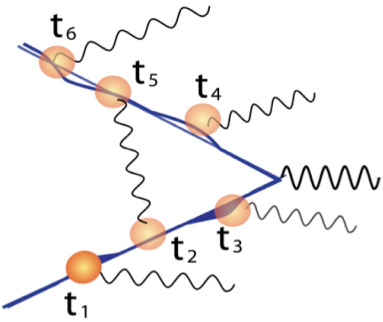
Systematic structure of NLP effects

- Ideally we would like to classify NLP effects in *arbitrary processes*.
- At LP for example, we can write *factorisation formulae* containing *universal functions*.
- It is still an open question whether or not this is fully possible at NLP level.
- An interesting field theory question by itself, but there are also many practical applications!
- This has a long history...

Low-Burnett-Kroll theorem

- Next-to-soft effects were first studied in gauge theory (QED) by Low (1958).
- He considered external scalars; generalised to fermions by Burnett and Kroll (1968).
- Both groups only considered massive particles: all threshold effects soft.
- Del Duca (1990) potentially generalised the Low-Burnett-Kroll result to include collinear effects.

- Next-to-soft effects has been considered using path-integral methods (Laenen, Stavenga, White).

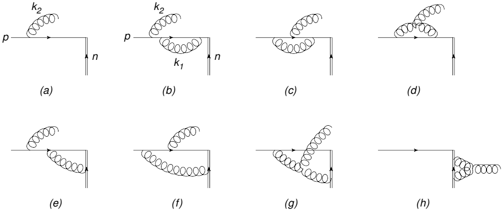


- Can replace propagators for external legs by quantum mechanics path integrals.
- Leading term in perturbative expansion is classical trajectory (soft limit).
- First-order wobbles give next-to-soft behaviour.

- At least some NLP effects exponentiate ("webs").
- Works for gravity too (White)!

- In certain processes, can resum NLP logs to all orders using physical evolution kernels (Almasi, Lo Presti, Moch, Soar, Vermaseren, Vogt).
- One can also use SCET (Chang, Feige, Kolodrubetz, Larkoski, Moult, Neill, Rothen, Stewart, Tackmann, Vita, Zhu; Beneke, Broggio, Garny, Jaskiewicz, Szafron, Vernazza, Wang)...
- ...or diagrammatic approaches (Gervais; Bonocore, Del Duca, Laenen, Magnea, Melville, Sinninghe Damsté, Vernazza, White).
- Will focus briefly on the latter here.

- Recently, a general factorisation formula was presented for a single extra gluon emission up to NLP level (Bonocore, Laenen, Magnea, Melville, Vernazza, White).
- Contains a *jet emission function* (Del Duca), new at NLP.



- Calculated at one-loop order for quarks; gluon jet in progress (Sinninghe Damsté, Vernazza).
- At NNLO, the jet emission function contributes NLP logs only at NLL order.

- It is an open question whether this jet is sufficient to describe NLP effects at higher orders.
- Also, whether emissions from inside the jet remain NLL or beyond.
- The method of regions calculation sheds light on this.
- We saw that the collinear region was NLL rather than LL, suggesting that indeed jet emission functions are not needed at LL.
- Further work in examining the implications of the N^3LO results is in progress.

- Threshold effects at next-to-leading power are important for precision physics...
- ...whether one resums them or not!
- We have calculated a large class of NLP effects in Drell-Yan production, at N^3LO.
- The results are of interest in themselves, but also useful for exploring NLP factorisation formulae.
- Much further work still to be done...

Open Questions

- Can remaining contributions to DY at N^3LO be calculated using similar techniques?
- Can we understand the general structure of NLP effects:
  (I) at fixed order?
  (II) at all orders?
- Can the diagrammatic and SCET approaches be compared? What are their relative strengths and weaknesses?
- What are the consequences of NLP threshold effects for LHC (or other) physics?