Lattice studies of maximally supersymmetric Yang–Mills theories

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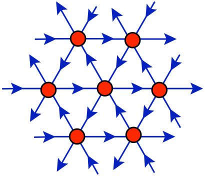
CERN Lattice Seminar, 7 June 2018

arXiv:1505.03135 arXiv:1611.06561 arXiv:1709.07025

& more to come with Simon Catterall, Raghav Jha and Toby Wiseman

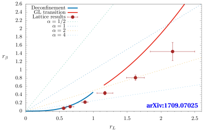
Overview and plan

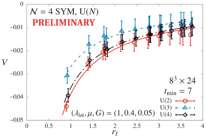


Central idea

Preserve (some) susy in discrete space-time

to make lattice investigations practical



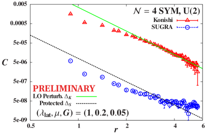


Goals

1) Reproduce reliable results in

perturbative, holographic, etc. regimes

2) Use lattice to access new domains

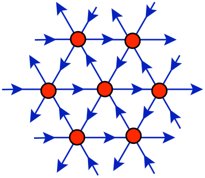


Overview and plan

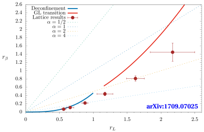
Preserve (some) susy in discrete space-time

Reproduce reliable analytic results

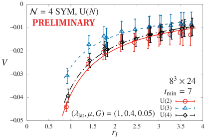
Use lattice to access new domains



Lattice supersymmetry

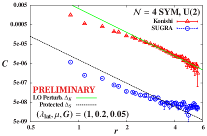


Lattice $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM)



Selected results as time permits

- Dimensionally reduced (2d) thermodynamics
- Static potential Coulomb coefficient
- Anomalous dimension of Konishi operator



Prospects and future directions

Motivation: Why lattice supersymmetry

Dualities, holography, confinement, conformality, BSM, ...

Lattice promises non-perturbative insights from first principles

Many potential lattice susy applications...

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Many potential lattice susy applications. . .

- Compute Wilson loops, spectrum, scaling dimensions, etc.,
  going beyond perturbation theory, holography, bootstrap
- New non-perturbative tests of conjectured dualities
- Predict low-energy constants from dynamical susy breaking
- Validate or refine holographic models
  for QCD phase diagram, non-Fermi liquids, etc.

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. . . relatively little exploration

Obstruction: Why not lattice supersymmetry

Supersymmetry extends 4d Poincaré symmetry

by $4\mathcal{N}$ spinor generators Q_α^{I} and $\overline{Q}_{\dot\alpha}^{\mathrm{I}}$ $\quad (\mathrm{I} = 1, \cdots, \mathcal{N})$

Super-Poincaré algebra includes $\left\{Q_\alpha^{\mathrm{I}}, \overline{Q}_{\dot\alpha}^{\mathrm{J}}\right\} = 2\delta^{\mathrm{IJ}}\sigma^{\mu}_{\alpha\dot\alpha} P_\mu$

$\longrightarrow$ infinitesimal translations that don't exist in discrete space-time

Consequences for lattice calculations

Explicitly broken supersymmetry $\Longrightarrow$ relevant susy-violating operators

Typically many such operators, especially with scalar fields

Fine-tuning to recover supersymmetric continuum limit

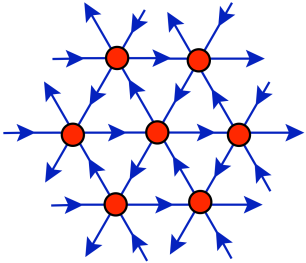
generally not practical in numerical lattice calculations

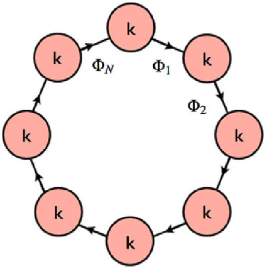
Solution: Exact supersymmetry on the lattice

If 2^d supersymmetries in d dimensions,
can preserve susy sub-algebra at non-zero lattice spacing

$\Longrightarrow$ Correct continuum limit with little or no fine tuning

Equivalent constructions arXiv:0903.4881
from 'topological' twisting and dimensional deconstruction





$d = 4$ picks out maximally supersymmetric Yang–Mills ($\mathcal{N} = 4$ SYM)

$\mathcal{N} = 4$ SYM — the fruit fly of QFT

Widely used to develop continuum QFT tools & techniques,
from scattering amplitudes to holography

Arguably simplest non-trivial 4d field theory

SU(N) gauge theory with four fermions Ψ^{I} and six scalars Φ^{IJ},
all massless and in adjoint rep.

Action consists of kinetic, Yukawa and four-scalar terms
with coefficients related by symmetries

Maximal 16 supersymmetries Q_α^{I} and $\overline{Q}_{\dot\alpha}^{\mathrm{I}}$ $(\mathrm{I} = 1, \cdots, 4)$
transform under global SU(4) $\sim$ SO(6) R symmetry

Conformal: β function is zero for any 't Hooft coupling $\lambda = g^2 N$

Topological twisting for $\mathcal{N} = 4$ SYM

Intuitive picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q_\alpha^1 & Q_\alpha^2 & Q_\alpha^3 & Q_\alpha^4 \\ \overline{Q}_{\dot\alpha}^1 & \overline{Q}_{\dot\alpha}^2 & \overline{Q}_{\dot\alpha}^3 & \overline{Q}_{\dot\alpha}^4 \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu\gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_\nu + \overline{\mathcal{Q}}_\mu\gamma_\mu\gamma_5 + \overline{\mathcal{Q}}\gamma_5$$

$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a\gamma_a + \mathcal{Q}_{ab}\gamma_a\gamma_b$$

with $a, b = 1, \cdots, 5$

Kähler–Dirac muliplet of 'twisted' supersymmetries $\mathcal{Q}$

transform with integer spin under 'twisted rotation group'

$$\text{SO}(4)_{tw} \equiv \text{diag}\left[\text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R\right] \qquad \text{SO}(4)_R \subset \text{SO}(6)_R$$

Change of variables $\longrightarrow$ closed subalgebra $\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$

that can be **exactly preserved on the lattice**

Susy subalgebra from twisted $\mathcal{N} = 4$ SYM

Fields & $\mathcal{Q}$s transform with integer spin under $\text{SO}(4)_{tw}$ — no spinors

$$Q_\alpha \text{ and } \overline{Q}_{\dot\alpha} \longrightarrow \mathcal{Q},\ \mathcal{Q}_a \text{ and } \mathcal{Q}_{ab}$$

$$\Psi \text{ and } \overline{\Psi} \longrightarrow \eta,\ \psi_a \text{ and } \chi_{ab}$$

$$A_\mu \text{ and } \Phi^{\mathrm{I}} \longrightarrow \text{complexified gauge field } \mathcal{A}_a \text{ and } \overline{\mathcal{A}}_a$$

$$\longrightarrow \text{U}(N) = \text{SU}(N) \otimes \text{U}(1) \text{ gauge theory}$$

Schematically, under $\text{SO}(d)_{tw} = \text{diag}\left[\text{SO}(d)_{\text{euc}} \otimes \text{SO}(d)_R\right]$

$$A_\mu \sim \text{vector} \otimes \text{scalar} \longrightarrow \text{vector}$$

$$\Phi^{\mathrm{I}} \sim \text{scalar} \otimes \text{vector} \longrightarrow \text{vector}$$

Easiest to see by dimensionally reducing from 5d

$$\mathcal{A}_a = A_a + i\Phi_a \longrightarrow (A_\mu, \phi) + i(\Phi_\mu, \overline{\phi})$$

Susy subalgebra from twisted $\mathcal{N} = 4$ SYM

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Twisted-scalar supersymmetry $\mathcal{Q}$

correctly interchanges bosonic $\longleftrightarrow$ fermionic d.o.f. with $\mathcal{Q}^2 = 0$

$$\mathcal{Q}\,\mathcal{A}_a = \psi_a \qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad \mathcal{Q}\,\overline{\mathcal{A}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad \mathcal{Q}\,d = 0$$

$\nwarrow$ bosonic auxiliary field with e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{A}_a$

Lattice $\mathcal{N} = 4$ SYM

Lattice theory looks nearly the same despite breaking $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Covariant derivatives $\longrightarrow$ finite difference operators

Complexified gauge fields $\mathcal{A}_a \longrightarrow$ gauge links $\mathcal{U}_a \in \mathfrak{gl}(N, \mathbb{C})$

$$\mathcal{Q}\,\mathcal{A}_a \longrightarrow \mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a \longrightarrow \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

(geometrically η on sites, ψ_a on links, etc.)

Susy lattice action $(\mathcal{Q}S = 0)$ from $\mathcal{Q}^2 \cdot = 0$ and Bianchi identity

$$S = \frac{N}{4\lambda_{\text{lat}}} \text{Tr}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de}\right]$$

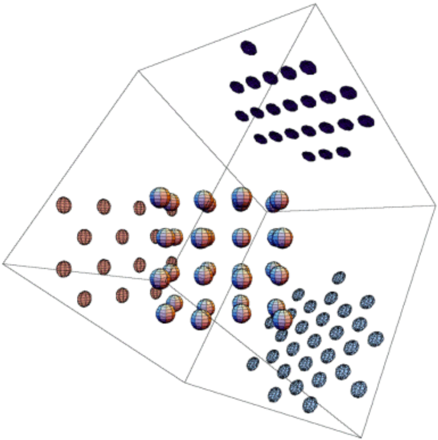
Five links in four dimensions $\longrightarrow$ A_4^* lattice

Again easiest to dimensionally reduce from 5d,
treating all five gauge links $\mathcal{U}_a$ symmetrically

Start with hypercubic lattice
in 5d momentum space

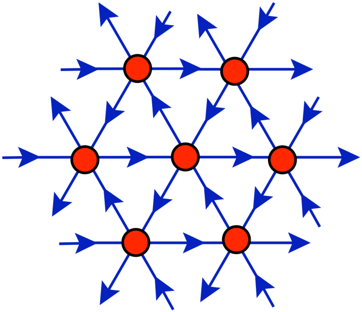
Symmetric constraint $\sum_a \partial_a = 0$
projects to 4d momentum space

Result is A_4 lattice
$\longrightarrow$ dual A_4^* lattice in real space



Twisted SO(4) symmetry on the A_4^* lattice

Can view A_4^* lattice
as 4d analog of 2d triangular lattice



Basis vectors linearly dependent
and non-orthogonal $\longrightarrow \lambda = \lambda_{\text{lat}}/\sqrt{5}$

Preserves S_5 point group symmetry

S_5 irreps precisely match onto irreps of twisted SO(4)$_{tw}$

$$\mathbf{5} = \mathbf{4} \oplus \mathbf{1} : \quad \psi_a \longrightarrow \psi_\mu, \;\; \overline{\eta}$$

$$\mathbf{10} = \mathbf{6} \oplus \mathbf{4} : \quad \chi_{ab} \longrightarrow \chi_{\mu\nu}, \;\; \overline{\psi}_\mu$$

$S_5 \longrightarrow$ SO(4)$_{tw}$ in continuum limit restores $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Summary of twisted $\mathcal{N} = 4$ SYM on the A_4^* lattice

U(N) gauge invariance + $\mathcal{Q}$ + S_5 lattice symmetries
$\longrightarrow$ several significant analytic results

Moduli space preserved to all orders of lattice perturbation theory
$\longrightarrow$ no scalar potential induced by radiative corrections

β function vanishes at one loop in lattice perturbation theory

Real-space RG blocking transformations preserving $\mathcal{Q}$ and S_5
$\longrightarrow$ no new terms in long-distance effective action

Only one log. tuning to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Not quite suitable for numerical calculations

Exact zero modes and flat directions must be regulated,
especially important in U(1) sector

Regulating SU(N) flat directions

$$S = \frac{N}{4\lambda_{\text{lat}}}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de} + \mu^2 V\right]$$

Scalar potential $V = \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2$ lifts SU(N) flat directions

and ensures $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$ in continuum limit

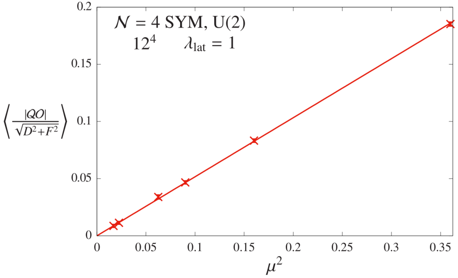
Softly breaks $\mathcal{Q}$ — all susy violations $\propto \mu^2 \to 0$ in continuum limit

Ward identity violations, $\langle\mathcal{Q}\mathcal{O}\rangle \neq 0$,

show $\mathcal{Q}$ breaking and restoration

Here considering

$$\mathcal{Q}\left[\eta\mathcal{U}_a\overline{\mathcal{U}}_a\right] = d\mathcal{U}_a\overline{\mathcal{U}}_a - \eta\psi_a\overline{\mathcal{U}}_a$$



Full $\mathcal{N} = 4$ SYM lattice action

arXiv:1505.03135

$$S = \frac{N}{4\lambda_{\text{lat}}} \left[\mathcal{Q} \left(\chi_{ab}\mathcal{F}_{ab} + \quad \downarrow \quad - \frac{1}{2}\eta d \right) - \frac{1}{4}\epsilon_{abcde}\ \chi_{ab}\overline{\mathcal{D}}_c\ \chi_{de} + \mu^2 V \right]$$

$$\eta \left\{ \overline{\mathcal{D}}_a \mathcal{U}_a + G \sum_{a<b} \left[\det \mathcal{P}_{ab} - 1\right] \mathbb{I}_N \right\}$$

Modify e.o.m. for d to constrain plaquette determinant

$\longrightarrow$ lifts U(1) zero mode & flat directions without susy breaking

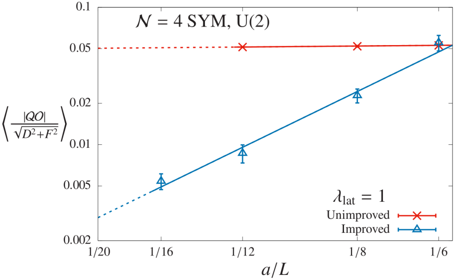
Much better than adding

another soft $\mathcal{Q}$-breaking term

$\langle \mathcal{Q}\mathcal{O} \rangle \propto (a/L)^2$,

effective '$O(a)$ improvement',

since $\mathcal{Q}$ forbids all dim-5 operators



Advertisement: Public code for lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

$$S_{\text{imp}} = S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \tag{3.10}$$

$$S'_{\text{exact}} = \frac{N}{2\lambda_{\text{lat}}} \sum_n \text{Tr}\Bigg[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{b]}(n) - \eta(n)\overline{\mathcal{D}}^{(-)}_a\psi_a(n) + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_a\mathcal{U}_a(n) + G\sum_{a\neq b}\left(\det\mathcal{P}_{ab}(n) - 1\right)\mathbb{I}_N\right)^2\Bigg] - S_{\text{det}}$$

$$S_{\text{det}} = \frac{N}{2\lambda_{\text{lat}}} G \sum_n \text{Tr}\left[\eta(n)\right] \sum_{a\neq b}\left[\det\mathcal{P}_{ab}(n)\right] \text{Tr}\left[\mathcal{U}_b^{-1}(n)\psi_b(n) + \mathcal{U}_a^{-1}(n+\widehat{\mu}_b)\psi_a(n+\widehat{\mu}_b)\right]$$

$$S_{\text{closed}} = -\frac{N}{8\lambda_{\text{lat}}} \sum_n \text{Tr}\left[\epsilon_{abcde}\ \chi_{de}(n+\widehat{\mu}_a+\widehat{\mu}_b+\widehat{\mu}_c)\overline{\mathcal{D}}^{(-)}_c\chi_{ab}(n)\right],$$

$$S'_{\text{soft}} = \frac{N}{2\lambda_{\text{lat}}}\mu^2 \sum_n \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a(n)\overline{\mathcal{U}}_a(n)\right] - 1\right)^2$$

The full $\mathcal{N} = 4$ SYM lattice action is somewhat complicated

($\gtrsim$100 inter-node data transfers in the fermion operator)

To reduce barriers to entry our parallel code is publicly developed at

github.com/daschaich/susy

Evolved from MILC lattice QCD code, presented in arXiv:1410.6971

Application I: Thermodynamics on a 2-torus

Improve arXiv:1008.4964 with new parallel code

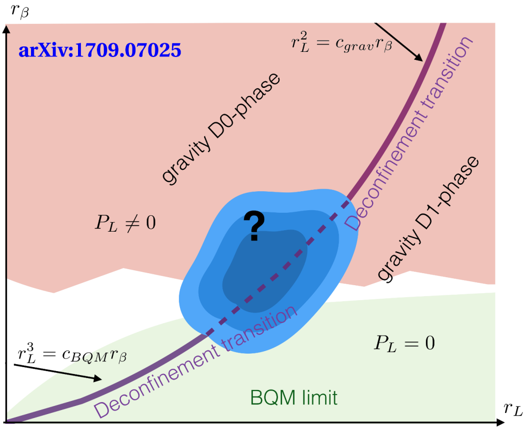
Dimensionally reduce to 2d $\mathcal{N} = (8,8)$ SYM with four scalar $\mathcal{Q}$,
study low temperatures $t = 1/r_\beta \longleftrightarrow$ black holes in dual supergravity

For decreasing r_L **at large *N***

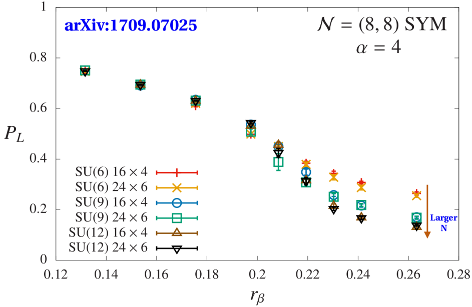
homogeneous black string (D1)
$\longrightarrow$ localized black hole (D0)

$\updownarrow$

"spatial deconfinement"
signalled by Wilson line P_L



$\mathcal{N} = (8, 8)$ SYM lattice phase diagram results



Fix aspect ratio $\alpha = r_L/r_\beta$,

scan in $r_\beta = r_L/\alpha = \beta\sqrt{\lambda}$

Clear transition in Wilson line

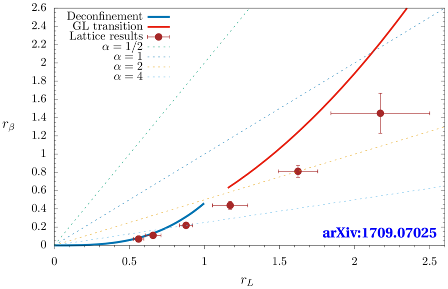
and its susceptibility

Lower-temperature transitions

at smaller $\alpha < 1 \longrightarrow$ larger errors

Results consistent with holography

and high-temp. bosonic QM

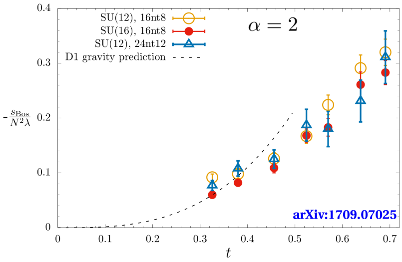


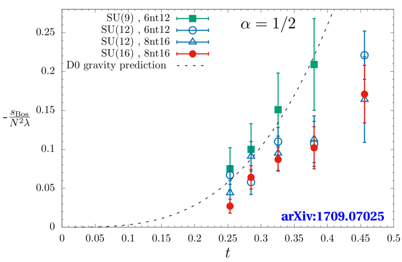
Dual black hole thermodynamics

Holography predicts bosonic action for corresponding dual black holes

$\propto t^3$ for large-r_L D1 phase $\qquad \propto t^{3.2}$ for small-r_L D0 phase

Lattice results consistent with holography for sufficiently low $t \lesssim 0.4$



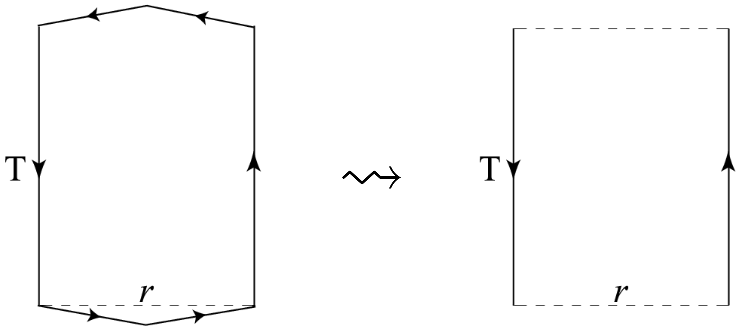


Need larger $N > 16$ to avoid instabilities at lower temperatures

Application II: Static potential $V(r)$

Static probes $\longrightarrow$ $r \times T$ Wilson loops $W(r, T) \propto e^{-V(r)\,T}$

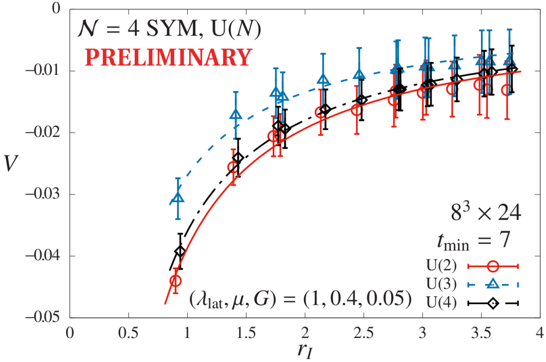
Coulomb gauge trick reduces A_4^* lattice complications



Static potential is Coulombic at all λ

Fits to confining $V(r) = A - C/r + \sigma r \longrightarrow$ vanishing string tension σ

$\Longrightarrow$ Fit to just $V(r) = A - C/r$ to extract Coulomb coefficient $C(\lambda)$

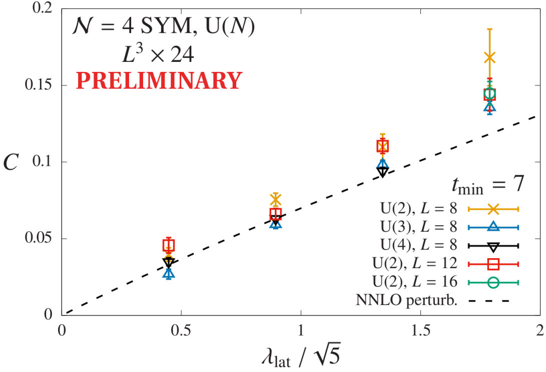


Recent progress: Incorporating tree-level improvement into analysis

Coupling dependence of Coulomb coefficient

Continuum perturbation theory predicts $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

Holography predicts $C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty$ and $\lambda \to \infty$ with $\lambda \ll N$



Surprisingly good agreement with perturbation theory for $\lambda_{\text{lat}} \leq 4$

Application III: Konishi operator scaling dimension

Conformality $\longrightarrow$ spectrum of scaling dimensions $\Delta(\lambda)$
govern power-law decays of correlation functions

Konishi is simplest conformal primary operator

$$\mathcal{O}_K(x) = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}(x)\Phi^{\mathrm{I}}(x)\right] \qquad C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

Predictions for Konishi scaling dimension $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$

- From weak-coupling perturbation theory,
  related to strong coupling by $\frac{4\pi N}{\lambda} \longleftrightarrow \frac{\lambda}{4\pi N}$ S duality
- From holography for $N \to \infty$ and $\lambda \to \infty$ with $\lambda \ll N$
- Upper bounds from conformal bootstrap

Only lattice gauge theory can access nonperturbative λ at moderate N

Konishi operator on the lattice

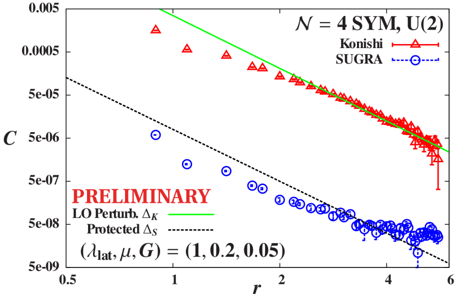
Scalar fields $\varphi(n)$ from polar decomposition of complexified links

$$\mathcal{U}_a(n) \longrightarrow e^{\varphi_a(n)} U_a(n) \qquad\qquad \mathcal{O}_K^{\text{lat}}(n) = \sum_a \text{Tr}\left[\varphi_a(n)\varphi_a(n)\right] - \text{vev}$$

Also looking at 'SUGRA' (20′)
$\mathcal{O}_S \sim \varphi_a \varphi_b$ with protected $\Delta_S = 2$

Challenging systematics from
directly fitting power-law decay

Better lattice tools to find Δ:
Finite-size scaling
Monte Carlo RG



Need lattice RG blocking transformation to carry out MCRG. . .

Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve $\mathcal{Q}$ and S_5 symmetries $\longleftrightarrow$ geometric structure

Simple transformation constructed in arXiv:1408.7067

$$\mathcal{U}'_a(n') = \xi\,\mathcal{U}_a(n)\mathcal{U}_a(n+\widehat{\mu}_a) \qquad\qquad \eta'(n') = \eta(n)$$
$$\psi'_a(n') = \xi\left[\psi_a(n)\mathcal{U}_a(n+\widehat{\mu}_a) + \mathcal{U}_a(n)\psi_a(n+\widehat{\mu}_a)\right] \qquad\qquad \text{etc.}$$

Doubles lattice spacing $a \longrightarrow a' = 2a$, with ξ a tunable rescaling factor

Scalar fields from polar decomposition $\mathcal{U}(n) = e^{\varphi(n)}U(n)$
are shifted, $\varphi \longrightarrow \varphi + \log\xi$, since blocked U must remain unitary

$\mathcal{Q}$-preserving RG blocking needed
to show only one log. tuning to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Scaling dimensions from MCRG stability matrix

System as (infinite) sum of operators $\quad H = \sum_i c_i\, \mathcal{O}_i$

Couplings c_i flow under RG blocking R_b

n-times-blocked system $\quad H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Fixed point defined by $H^\star = R_b H^\star$ with couplings $c_i^\star$

Linear expansion around fixed point defines stability matrix $T_{ij}^\star$

$$c_i^{(n)} - c_i^\star = \sum_k \left.\frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}}\right|_{H^\star} \left(c_k^{(n-1)} - c_k^\star\right) \equiv \sum_j T_{ik}^\star \left(c_k^{(n-1)} - c_k^\star\right)$$

Correlators of $\mathcal{O}_i$, $\mathcal{O}_k$ $\longrightarrow$ elements of stability matrix [Swendsen, 1979]

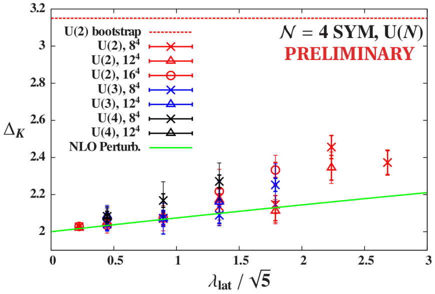
Eigenvalues of $T_{ik}^\star$ $\longrightarrow$ scaling dimensions of corresponding operators

Preliminary Δ_K results from Monte Carlo RG

MCRG stability matrix
includes both $\mathcal{O}_K^{\text{lat}}$ and $\mathcal{O}_S^{\text{lat}}$

Impose protected $\Delta_S = 2$

Systematic uncertainties from
different amounts of smearing



Complication: Twisted $\text{SO}(4)_{tw}$ involves only $\text{SO}(4)_R \subset \text{SO}(6)_R$

$\Longrightarrow$ Lattice Konishi operator mixes with $\text{SO}(4)_R$-singlet part
of the $\text{SO}(6)_R$-nonsinglet SUGRA operator

Working on variational analyses to disentangle operators

Recapitulation and outlook

- Lattice promises non-perturbative insights from first principles
- Lattice $\mathcal{N} = 4$ SYM is practical thanks to exact $\mathcal{Q}$ susy
- Public code to reduce barriers to entry

Significant progress toward goals of lattice investigations

- 2d $\mathcal{N} = (8, 8)$ SYM thermodynamics consistent with holography
- Static potential Coulomb coefficient $C(\lambda)$ at weak coupling
- Preliminary conformal scaling dimension of Konishi operator

Many more directions are being — or can be — pursued

- Understanding the (absence of a) sign problem
- Systems with less supersymmetry, in lower dimensions, including matter fields, exhibiting spontaneous susy breaking, . . .

Thank you!

Collaborators

Simon Catterall, Raghav Jha, Toby Wiseman

also Georg Bergner, Poul Damgaard, Joel Giedt, Anosh Joseph

Funding and computing resources

Supplement: Lattice superQCD in 2d & 3d

Add fundamental matter multiplets without breaking $\mathcal{Q}^2 = 0$

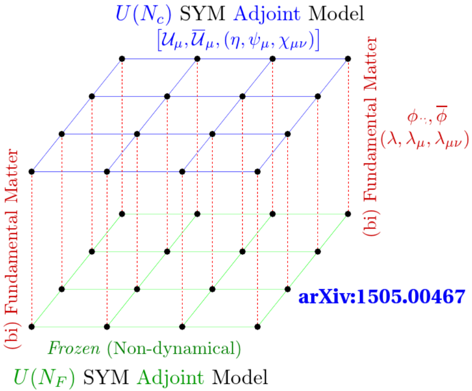
Proposed by Matsuura [arXiv:0805.4491] and Sugino [arXiv:0807.2683], first numerical study by Catterall & Veernala [arXiv:1505.00467]

2-slice lattice SYM
with U(N) $\times$ U(F) gauge group

Adj. fields on each slice

Bi-fundamental in between

Set U(F) coupling to zero
$\longrightarrow$ U(N) SQCD in $d-1$ dims.
with F fund. hypermultiplets



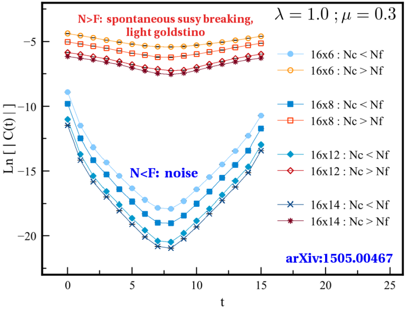
Dynamical susy breaking in 2d quiver superQCD

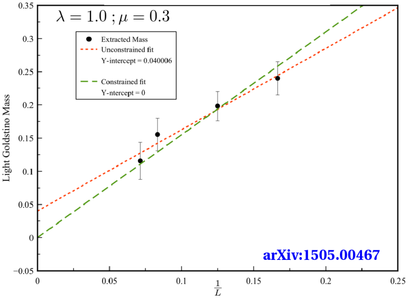
Auxiliary field e.o.m. $\longrightarrow$ Fayet–Iliopoulos D-term potential

$$d = \overline{\mathcal{D}}_a \mathcal{U}_a + \sum_{i=1}^{F} \phi_i \overline{\phi}_i + r\mathbb{I}_N \qquad \longrightarrow \qquad S_D \propto \sum_{i=1}^{F} \left(\text{Tr}\left[\phi_i \overline{\phi}_i + r\mathbb{I}_N\right] \right)^2$$

Zero out N diagonal elements via F scalar vevs

or else susy breaking, $\langle \mathcal{Q}\eta \rangle = \langle d \rangle \neq 0 \longleftrightarrow \langle 0 \left| H \right| 0 \rangle > 0$





Supplement: Potential sign problem

Observables: $\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}][d\overline{\mathcal{U}}] \; \mathcal{O} \; e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]} \; \text{pf}\, \mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$

Pfaffian can be complex for lattice $\mathcal{N} = 4$ SYM, $\text{pf}\, \mathcal{D} = |\text{pf}\, \mathcal{D}| e^{i\alpha}$

Complicates interpretation of $\left\{ e^{-S_B} \, \text{pf}\, \mathcal{D} \right\}$ as Boltzmann weight

RHMC uses phase quenching, $\text{pf}\, \mathcal{D} \longrightarrow |\text{pf}\, \mathcal{D}|$, needs reweighting

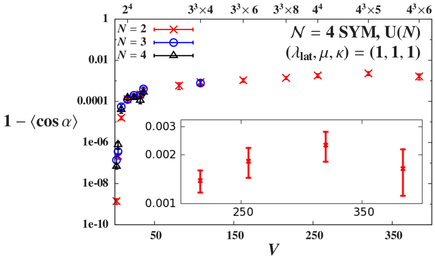
$$\langle \mathcal{O} \rangle = \frac{\left\langle \mathcal{O} e^{i\alpha} \right\rangle_{pq}}{\left\langle e^{i\alpha} \right\rangle_{pq}} \qquad \text{with} \quad \left\langle \mathcal{O} e^{i\alpha} \right\rangle_{pq} = \frac{1}{\mathcal{Z}_{pq}} \int [d\mathcal{U}][d\overline{\mathcal{U}}] \, \mathcal{O} e^{i\alpha} \, e^{-S_B} \, |\text{pf}\, \mathcal{D}|$$

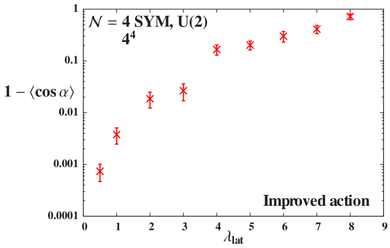
$\Longrightarrow$ Monitor $\left\langle e^{i\alpha} \right\rangle_{pq}$ as function of volume, coupling, N

Pfaffian phase dependence on volume and coupling

Left: $1 - \langle\cos(\alpha)\rangle_{pq} \ll 1$ independent of volume and N at $\lambda_{\text{lat}} = 1$

Right: Larger $\lambda_{\text{lat}} \geq 4 \longrightarrow$ much larger phase fluctuations





To do: Analyze more volumes and N with improved action

Extremely expensive $\mathcal{O}(n^3)$ computation

$\sim$50 hours $\times$ 16 cores for single U(2) 4^4 measurement

Two puzzles posed by the sign problem

Periodic temporal boundary conditions for the fermions

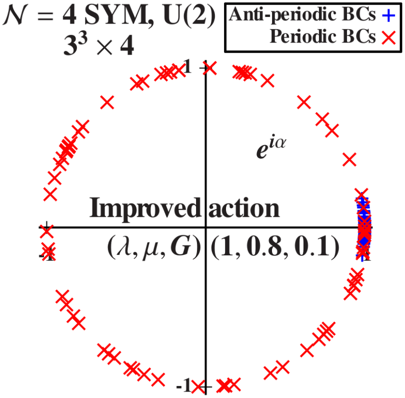
$\longrightarrow$ obvious sign problem, $\left\langle e^{i\alpha} \right\rangle_{pq} \approx 0$

Anti-periodic BCs $\longrightarrow e^{i\alpha} \approx 1$, phase reweighting negligible

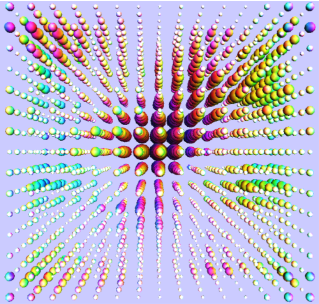
Why such sensitivity to the BCs?

Other pq observables are nearly identical for these two ensembles

Why doesn't the sign problem affect other observables?



Backup: Essence of numerical lattice calculations


(Image credit: Claudio Rebbi)

Evaluate observables from functional integral via importance sampling Monte Carlo

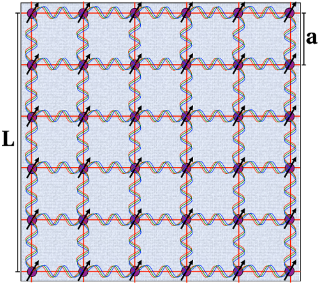
$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}U \;\; \mathcal{O}(U) \;\; e^{-S[U]}$$

$$\longrightarrow \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(U_i) \;\; \text{with uncert.} \;\; \propto \sqrt{\frac{1}{N}}$$

U are field configurations in discretized euclidean space-time, sampled with probability $\propto e^{-S}$

$S[U]$ is lattice action, should be real and positive $\longrightarrow \frac{1}{\mathcal{Z}} e^{-S}$ as probability distribution

Backup: More features of lattice calculations



P. Vranas LLNL

Spacing "a" between lattice sites
$\longrightarrow$ UV cutoff scale $1/a$

Removing cutoff: $a \to 0$ (with $L/a \to \infty$)

Lattice cutoff preserves hypercubic subgroup
$\longrightarrow$ restore Poincaré in continuum limit

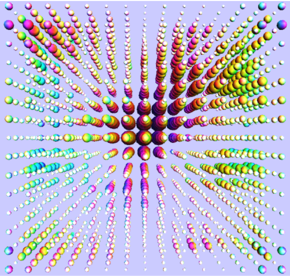
Lattice action S defined by bare lagrangian at the UV cutoff $1/a$

After generating and saving ensembles $\{U_n\}$ distributed $\propto e^{-S}$
often quick and easy to measure many observables $\langle \mathcal{O} \rangle$

Changing the action (generally) requires generating new ensembles

Backup: Hybrid Monte Carlo (HMC) algorithm

Goal: Sample field configurations U with probability $\frac{1}{\mathcal{Z}} e^{-S[U]}$


(Image credit: Claudio Rebbi)

HMC is Markov process based on Metropolis–Rosenbluth–Teller

Fermions $\longrightarrow$ extensive action computation

$\Longrightarrow$ Global updates using fictitious molecular dynamics

1. Introduce fictitious "MD time" τ and stochastic canonical momenta for fields
2. Inexact MD evolution along trajectory in τ $\longrightarrow$ new configuration
3. Accept/reject test on MD discretization error

Backup: Failure of Leibnitz rule in discrete space-time

$\left\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\right\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$ is problematic

$\longrightarrow$ try $\left\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\right\} = 2i\sigma^\mu_{\alpha\dot{\alpha}}\nabla_\mu$ for a discrete translation

$$\nabla_\mu \phi(x) = \frac{1}{a}\left[\phi(x + a\widehat{\mu}) - \phi(x)\right] = \partial_\mu \phi(x) + \frac{a}{2}\partial_\mu^2 \phi(x) + \mathcal{O}(a^2)$$

Essential difference between ∂_μ and ∇_μ on the lattice, $a > 0$

$$\begin{aligned}\nabla_\mu\left[\phi(x)\eta(x)\right] &= a^{-1}\left[\phi(x + a\widehat{\mu})\eta(x + a\widehat{\mu}) - \phi(x)\eta(x)\right] \\ &= \left[\nabla_\mu\phi(x)\right]\eta(x) + \phi(x)\nabla_\mu\eta(x) + a\left[\nabla_\mu\phi(x)\right]\nabla_\mu\eta(x)\end{aligned}$$

Only recover Leibnitz rule $\partial_\mu(fg) = (\partial_\mu f)g + f\partial_\mu g$ when $a \to 0$

$\Longrightarrow$ "Discrete supersymmetry" breaks down on the lattice

(Dondi & Nicolai, "Lattice Supersymmetry", 1977)

Backup: Twisting $\longleftrightarrow$ Kähler–Dirac fermions

Kähler–Dirac representation related to spinor $Q^{\rm I}_{\alpha}$, $\overline{Q}^{\rm I}_{\dot\alpha}$ by

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu\gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_\nu + \overline{\mathcal{Q}}_\mu\gamma_\mu\gamma_5 + \overline{\mathcal{Q}}\gamma_5$$
$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a\gamma_a + \mathcal{Q}_{ab}\gamma_a\gamma_b$$
$$\text{with } a, b = 1, \cdots, 5$$

The 4×4 matrix involves R symmetry transformations along each row, (euclidean) Lorentz transformations along each column

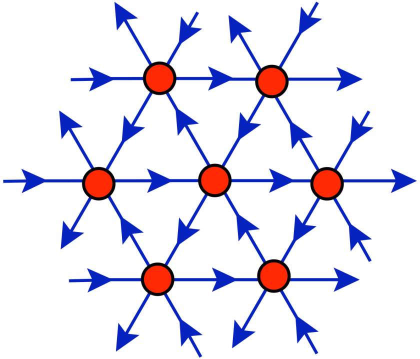
$\Longrightarrow$ Kähler–Dirac components transform under "twisted rotation group"

$$\text{SO}(4)_{tw} \equiv \text{diag}\Big[\text{SO}(4)_{\rm euc} \otimes \underbrace{\text{SO}(4)_R}_{\uparrow}\Big]$$

only $\text{SO}(4)_R \subset \text{SO}(6)_R$

Backup: Hypercubic representation of A_4^* lattice

In the code it is very convenient to represent the A_4^* lattice as a hypercube plus one backwards diagonal link



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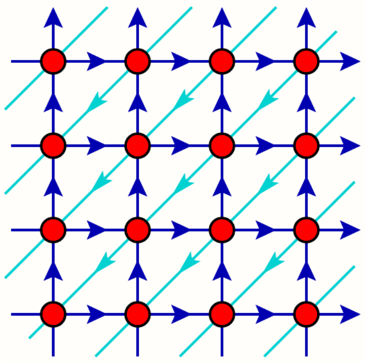


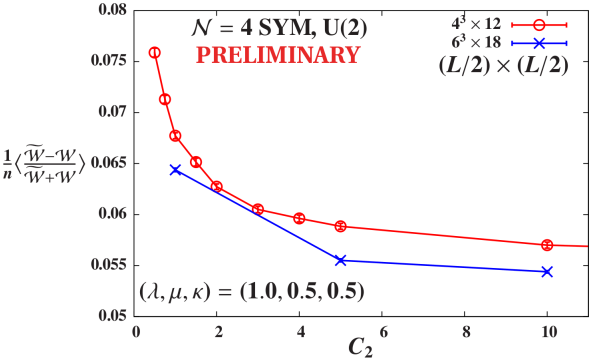
Backup: Restoration of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries

$\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ from restoration of R symmetry (motivation for A_4^* lattice)

Modified Wilson loops test R symmetries at non-zero lattice spacing

Parameter c_2 may need logarithmic tuning in continuum limit

Results from arXiv:1411.0166 to be revisited with improved action



Backup: More on flat directions

Complexified links $\longrightarrow$ U(N) = SU(N) $\otimes$ U(1) gauge invariance

Supersymmetry transformation $\mathcal{Q}\,\mathcal{U}_a = \psi_a$

$\Longrightarrow$ links must be in algebra with continuum limit $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$

Flat directions in SU(N) sector are physical,

those in U(1) sector decouple only in continuum limit

Both must be regulated in calculations $\longrightarrow$ two deformations

Scalar potential $\propto \mu^2 \sum_a \left(\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - N\right)^2$ for SU(N) sector

Plaquette determinant $\sim G\sum_{a<b}\left(\det \mathcal{P}_{ab} - 1\right)$ for U(1) sector

Scalar potential **softly** breaks $\mathcal{Q}$ supersymmetry

susy-violating operators vanish as $\mu^2 \to 0$

Plaquette determinant can be made $\mathcal{Q}$-invariant $\longrightarrow$ improved action

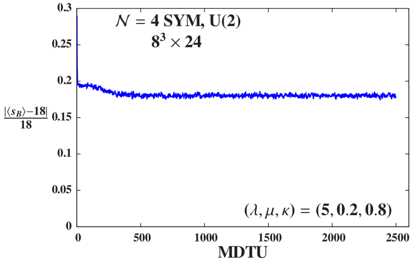
Backup: Problem with SU(N) flat directions

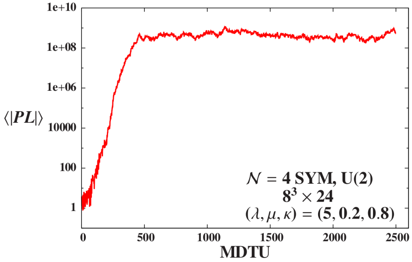
$\mu^2/\lambda_{\text{lat}}$ too small $\longrightarrow$ $\mathcal{U}_a$ can move far from continuum form $\mathbb{I}_N + \mathcal{A}_a$

Example: $\mu = 0.2$ and $\lambda_{\text{lat}} = 5$ on $8^3 \times 24$ volume

Left: Bosonic action stable $\sim$18% off its supersymmetric value

Right: Complexified Polyakov ('Maldacena') loop wanders off to $\sim 10^9$

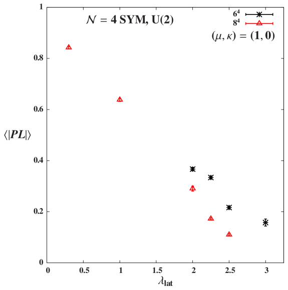


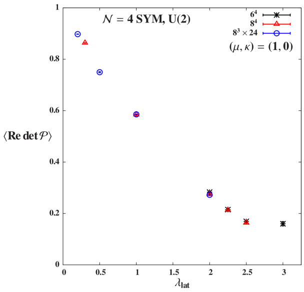


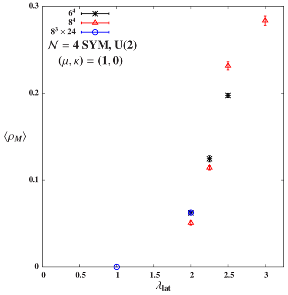
Backup: Problem with U(1) flat directions

Monopole condensation $\longrightarrow$ confined lattice phase

not present in continuum $\mathcal{N} = 4$ SYM







Around the same $\lambda_{\text{lat}} \approx 2\ldots$

Left: Polyakov loop falls towards zero

Center: Plaquette determinant falls towards zero

Right: Density of U(1) monopole world lines becomes non-zero

Backup: More on soft supersymmetry breaking

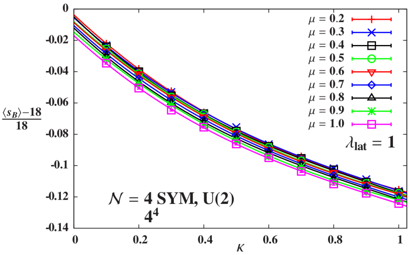
Until 2015 $(\det \mathcal{P} - 1)$ was another soft susy-breaking term

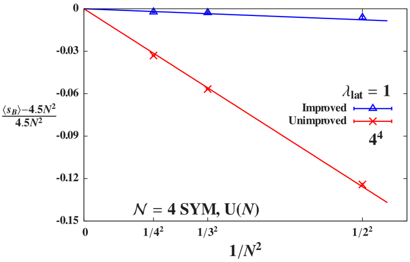
$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}}\mu^2 \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2 + \kappa \sum_{a<b} \left|\det \mathcal{P}_{ab} - 1\right|^2$$

Much larger $\mathcal{Q}$-breaking effects than scalar potential

Left: $\mathcal{Q}$ Ward identity from bosonic action $\langle s_B \rangle = 9N^2/2$

Right: Soft susy breaking suppressed $\propto 1/N^2$





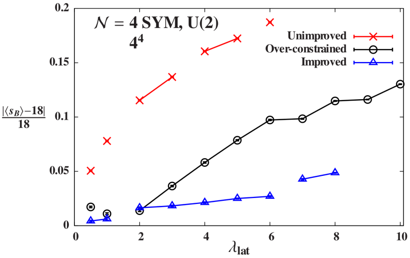
Backup: Supersymmetric moduli space modification

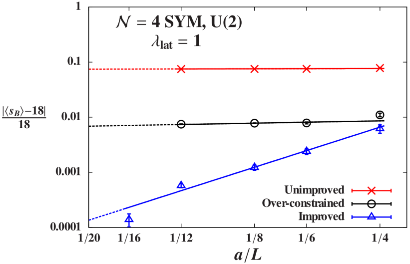
arXiv:1505.03135 introduces method to impose $\mathcal{Q}$-invariant constraints

Modify auxiliary field equations of motion $\longrightarrow$ moduli space

$$d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) \qquad \longrightarrow \qquad d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) + G\mathcal{O}(n)\mathbb{I}_N$$

Including both plaquette determinant and scalar potential in $\mathcal{O}(n)$
over-constrains system $\longrightarrow$ sub-optimal Ward identity violations



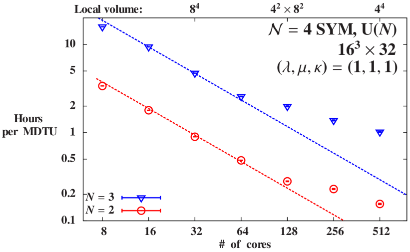


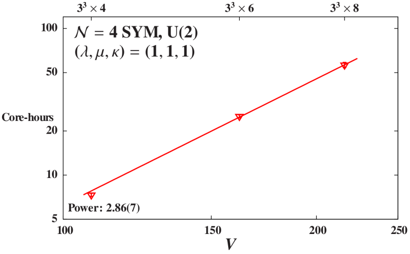
Backup: Code performance—weak and strong scaling

Results from arXiv:1410.6971 to be revisited with improved action

Left: Strong scaling for U(2) and U(3) $16^3 \times 32$ RHMC

Right: Weak scaling for $\mathcal{O}(n^3)$ pfaffian calculation (fixed local volume)
$n \equiv 16N^2V$ is number of fermion degrees of freedom





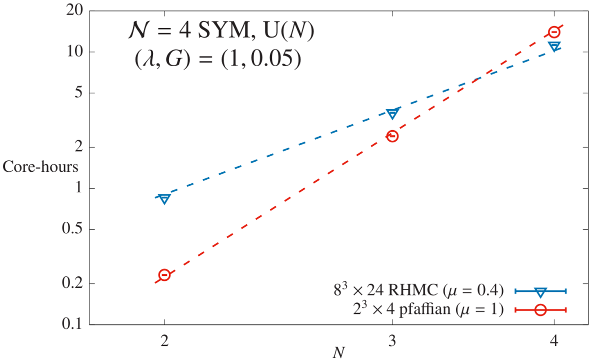
Dashed lines are optimal scaling

Solid line is power-law fit

Backup: Numerical costs for $N = 2$, 3 and 4 colors

Red: Original RHMC cost scaling $\sim N^5$ now improved to $\sim N^{3.5}$
Plot from arXiv:1410.6971 to be updated

Blue: Pfaffian cost scaling consistent with expected N^6



Backup: Dimensional reduction to $\mathcal{N} = (8, 8)$ SYM

Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

A_4^* lattice $\longrightarrow$ A_2^* (triangular) lattice

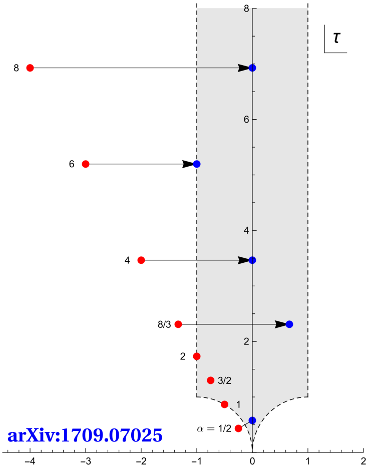
$\Longrightarrow$ Torus **skewed** depending on α

Modular trans. into fund. domain

can make skewed torus rectangular

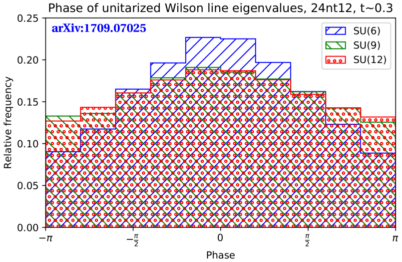
Also need to stabilize compactified links

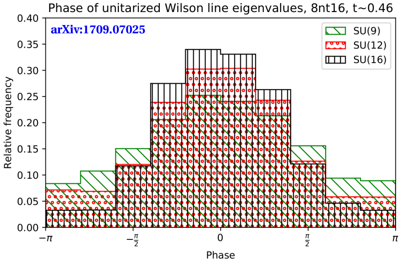
to ensure broken center symmetries



Backup: $\mathcal{N} = (8, 8)$ SYM Wilson line eigenvalues

Check ‘spatial deconfinement’ through histograms of Wilson line eigenvalue phases



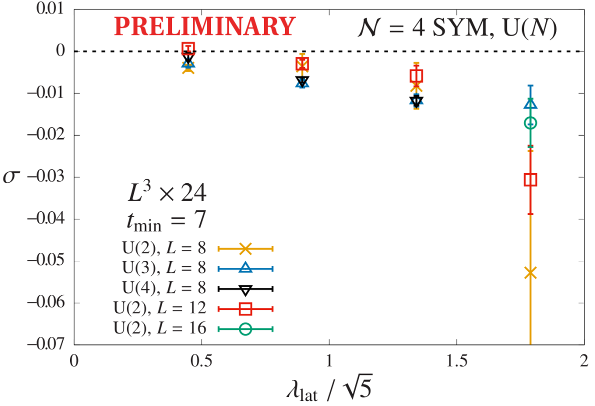


Left: $\alpha = 2$ distributions more extended as N increases
$\longrightarrow$ dual gravity describes homogeneous black string (D1 phase)

Right: $\alpha = 1/2$ distributions more compact as N increases
$\longrightarrow$ dual gravity describes localized black hole (D0 phase)

Backup: Static potential is Coulombic at all λ

String tension σ from fits to confining form $V(r) = A - C/r + \sigma r$



Slightly negative values flatten $V(r_I)$ for $r_I \lesssim L/2$

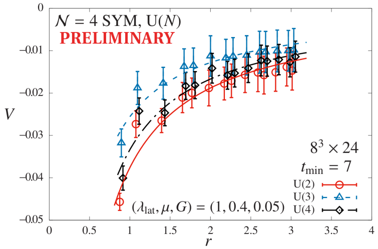
$\sigma \to 0$ as accessible range of r_I increases on larger volumes

Backup: Discretization artifacts in static potential

Discretization artifacts visible at short distances

where Coulomb term in $V(r) = A - C/r$ is most significant

Right: Highlight artifacts by extracting fluctuations around Coulomb fit



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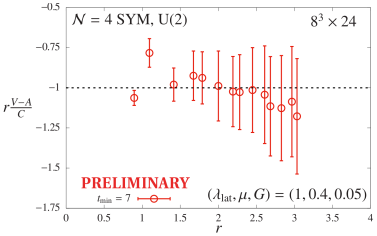


Danger of potential contamination in results for Coulomb coefficient C

Backup: Tree-level improvement

Classic trick to reduce discretization artifacts in static potential
(Lang & Rebbi '82; Sommer '93; Necco '03)

Associate $V(r)$ data with r from Fourier transform of gluon propagator

Recall $\displaystyle \frac{1}{4\pi^2 r^2} = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{e^{ir\cdot k}}{k^2}$ where $\displaystyle \frac{1}{k^2} = G(k)$ in continuum

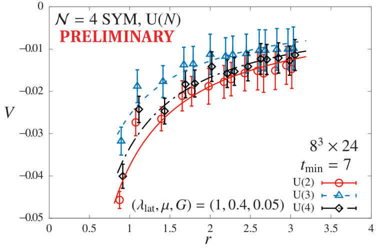
$$\text{On } A_4^* \text{ lattice} \longrightarrow \frac{1}{r_I^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4\widehat{k}}{(2\pi)^4} \frac{\cos\left(ir_I \cdot \widehat{k}\right)}{4\sum_{\mu=1}^4 \sin^2\left(\widehat{k}\cdot\widehat{e}_\mu \,/\, 2\right)}$$

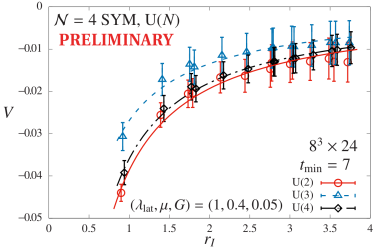
Tree-level perturbative lattice propagator from arXiv:1102.1725

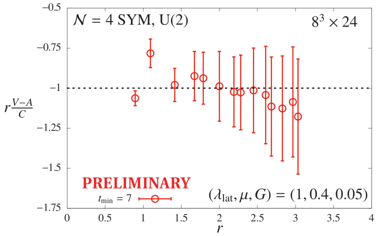
$\widehat{e}_\mu$ are A_4^* lattice basis vectors
while momenta $\widehat{k} = \frac{2\pi}{L}\sum_{\mu=1}^4 n_\mu \widehat{g}_\mu$ depend on dual basis vectors

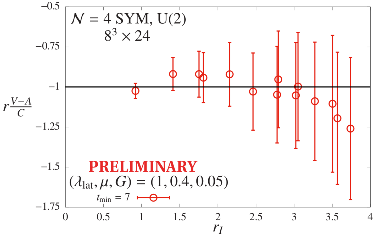
Backup: Tree-level-improved static potential

Tree-level improvement significantly reduces discretization artifacts





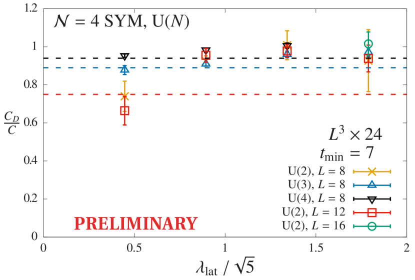


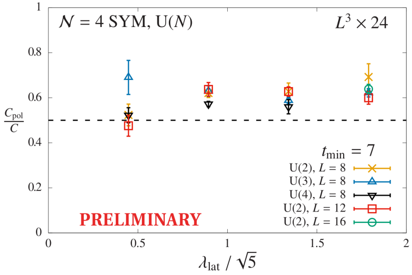


Backup: More $\mathcal{N} = 4$ SYM static potential tests

Left: Projecting Wilson loops from U(N) $\longrightarrow$ SU(N) $\Longrightarrow$ factor of $\frac{N^2-1}{N^2}$

Right: Unitarizing links removes scalars $\Longrightarrow$ factor of $1/2$





Several ratios end up above expected values

Cause not clear — seems insensitive to lattice volume and μ

Backup: Smearing for Konishi analyses

As for glueballs, smear to enlarge operator basis

APE-like smearing: $\;-\;\longrightarrow\;(1-\alpha)-\;+\;\frac{\alpha}{8}\sum\sqcap,$

staples built from unitary parts of links but no final unitarization

(unitarized smearing — e.g. stout — doesn't affect Konishi)

Average plaquette stable upon smearing (**right**)

while minimum plaquette steadily increases (**left**)