From saturation to high $p_t$ jets:
toward a unified picture of parton production at all transverse momenta

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High $p_t$ particle production: $pp$ collisions

collinear factorization: separation of soft (long distance) and hard (short distance)

$$
\frac{d\sigma^{pp\to hX}}{d^2p_t \, dy} \sim f_a(x_1) \otimes f_b(x_2) \otimes \hat{\sigma} \otimes D_f^h(z) + \ldots \ldots$$

$x \equiv \frac{p^+}{P^+}$

power corrections
pQCD: the standard paradigm

$$E \frac{d\sigma}{d^3p} \sim f_1(x) \otimes f_2(x) \otimes \frac{d\sigma}{dt} \otimes D(z)$$

bulk of QCD phenomena happens at low $p_t$ (small $x$)
collinear factorization breaks down at small $x$

“attractive” bremsstrahlung vs. “repulsive” recombination

$$\text{energy} \sim \frac{1}{x}$$

$$\frac{\alpha_s}{Q^2} \frac{x G(x, Q^2)}{\pi r^2} \sim 1$$

saturation scale $Q_s^2(x, b_t, A)$

included in pQCD \hspace{4cm} not included in pQCD
A hadron/nucleus at high energy: gluon saturation

- high gluon density: multiple scattering via Wilson lines
  - $p_t$ broadening
- energy dependence: x-evolution via JIMWLK
  - suppression of spectra/away side peaks

$$Q_s^2(x, b_t, A) \sim A^{1/3} \left( \frac{1}{x} \right)^{0.3}$$

$$Q_s^2(x = 3 \times 10^{-4}) \sim 1 \text{ GeV}^2$$

for a proton target (quarks)

A framework for multi-particle production in QCD at small x/low $p_t$

- Initial conditions for hydro
- Thermalization?
- Long range rapidity correlations
- Azimuthal angular correlations
- Nuclear modification factor
eliminate/minimize medium effects (proton-nucleus)

Eikonal approximation

\[ J_\mu^a \simeq \delta^{\mu-} \rho_a \]

\[ D_\mu J^{\mu} = D_- J^- = 0 \]

\[ \partial_- J^- = 0 \quad \text{(in } A^+ = 0 \text{ gauge)} \]

does not depend on \( x^- \)

solution to classical EOM:

\[ A^-(a)(x^+, x_t) \equiv n^- S_a(x^+, x_t) \]

with

\[ n^\mu = (n^+ = 0, n^- = 1, n_\perp = 0) \]

\[ n^2 = 2 n^+ n^- - n_\perp^2 = 0 \]

recall (eikonal limit):

\[ \bar{u}(q) \gamma^\mu u(p) \to \bar{u}(p) \gamma^\mu u(p) \sim p^\mu \]

\[ \bar{u}(q) A^- u(p) \to p \cdot A \sim p^+ A^- \]

multiple scattering of a quark from background color field

\[ A^-_a(x^+, x_t) \]
\[ iM_1 = (ig) \int d^4x_1 e^{i(q-p)\cdot x_1} \bar{u}(q) \left[ \frac{\hbar S(x_1)}{p_1} \right] u(p) \]

\[ = (ig)(2\pi)\delta(p^+-q^+) \int d^2x_{1t} \, dx_1^+ \, \bar{u}(q) \left[ \frac{\hbar S(x_2^+, x_{1t})}{\hbar S(x_1^+, x_{1t})} \right] u(p) \]

\[ iM_2 = (ig)^2 \int d^4x_1 \, d^4x_2 \, \int \frac{d^4p_1}{(2\pi)^4} e^{i(p_1-p)\cdot x_1} e^{i(q-p_1)\cdot x_2} \bar{u}(q) \left[ \frac{i\hbar p_1}{p_1^2 + i\epsilon} \right] \frac{\hbar S(x_2)}{\hbar S(x_1)} u(p) \]

\[ \int \frac{dp_1^-}{(2\pi)^2 p_1^+} \frac{e^{ip_1^- (x_1^--x_2^+)}}{2p_1^- \left( p_1^- - \frac{p_{1t}^- - i\epsilon}{2p_1^+} \right)} = \frac{-i}{2p_1^+} \theta(x_2^+ - x_1^+) e^{i \frac{p_1^2}{2p_1^+} (x_1^+ - x_2^+)} \]

contour integration over the pole leads to path ordering of scattering

ignore all terms: \( O(\frac{p_t}{p^+}, \frac{q_t}{q^+}) \) and use \( \frac{\hbar p_1}{2n \cdot p} \hbar = \hbar \)

\[ iM_2 = (ig)^2 (-i)(i) 2\pi \delta(p^+-q^+) \int dx_1^+ \, dx_2^+ \, \theta(x_2^+ - x_1^+) \int d^2x_{1t} \, e^{-i(q_t-p_t)\cdot x_{1t}} \bar{u}(q) \left[ S(x_2^+, x_{1t}) \hbar S(x_1^+, x_{1t}) \right] u(p) \]
\[ iM_n = 2\pi\delta(p^+ - q^+) \bar{u}(q) \hbar \int d^2x_t e^{-i(q_t - p_t) \cdot x_t} \left\{ (ig)^n (-i)^n(i)^n \int dx_1^+ dx_2^+ \cdots dx_n^+ \theta(x_n^+ - x_{n-1}^+) \cdots \theta(x_2^+ - x_1^+) \left[ S(x_n^+, x_t) S(x_{n-1}^+, x_t) \cdots S(x_2^+, x_t) S(x_1^+, x_t) \right] \right\} u(p) \]

sum over all scatterings \[ iM = \sum_{n} iM_n \]

\[ iM(p, q) = 2\pi\delta(p^+ - q^+) \bar{u}(q) \hbar \int d^2x_t e^{-i(q_t - p_t) \cdot x_t} [V(x_t) - 1] u(p) \]

with \[ V(x_t) \equiv \hat{P} \exp \left\{ ig \int_{-\infty}^{+\infty} dx^+ n^- S_a(x^+, x_t) t_a \right\} \]

\[ \frac{d\sigma_{qT\rightarrow qX}}{d^2p_t dy} \sim |iM|^2 \sim F.T. \quad < Tr V(x_t) V^\dagger(y_t) > \text{ a dipole} \]
1-loop correction: energy dependence

basic ingredient: soft radiation vertex (LC gauge)

\[ g \bar{u}(q) t^a \gamma_\mu u(p) \epsilon_\mu^\lambda(k) \rightarrow 2 g t^a \frac{\epsilon_\lambda(k) \cdot k_t}{k_t^2} \]

coordinate space:

\[
\int \frac{d^2 k_t}{(2\pi)^2} e^{ik_t \cdot (x_t-z_t)} 2 g t^a \frac{\epsilon_\lambda(k) \cdot k_t}{k_t^2} = \frac{2i g}{2\pi} t^a \frac{\epsilon_\lambda(x_t-z_t)}{(x_t-z_t)^2}
\]

virtual corrections:

\[ \rightarrow Tr V(x_t) V^\dagger(y_t) \]

real corrections:

\[ \frac{1}{(x_t-z_t)^2} \frac{(x_t-z_t) \cdot (y_t-z_t)}{(x_t-z_t)^2(y_t-z_t)^2} \]

\[ \rightarrow Tr V(x_t) V^\dagger(z_t) Tr V(z_t) V^\dagger(y_t) \]

the S matrix

\[ S(x_t, y_t) \equiv \frac{1}{N_c} Tr V(x_t) V^\dagger(y_t) \]
1-loop correction: BK eq.

\[
\frac{d}{dy} T(x_t, y_t) = \frac{N_c \alpha_s}{2\pi^2} \int d^2 z_t \frac{(x_t - y_t)^2}{(x_t - z_t)^2(y_t - z_t)^2} \left[ T(x_t, z_t) + T(z_t, y_t) - T(x_t, y_t) - T(x_t, z_t)T(z_t, y_t) \right]
\]

\[
T \equiv 1 - S
\]

\[
\tilde{T}(p_t) \sim \frac{1}{p_t^2} \left[ \frac{Q_s^2}{p_t^2} \right] \quad Q_s^2 \ll p_t^2
\]

\[
\tilde{T}(p_t) \sim \log \left[ \frac{Q_s^2}{p_t^2} \right] \quad Q_s^2 \gg p_t^2
\]

\[
\tilde{T}(p_t) \sim \frac{1}{p_t^2} \left[ \frac{Q_s^2}{p_t^2} \right] \gamma \quad Q_s^2 < p_t^2
\]

nuclear modification factor

\[
R_{pA} \equiv \frac{d\sigma_{pA}^A}{d^2 p_t dy} / A^{1/3} \frac{d\sigma_{pp}^{pp}}{d^2 p_t dy}
\]

suppression of $p_t$ spectra
nuclear shadowing
centrality dependence
Particle production in high energy collisions

$pQCD$ and collinear factorization at high $p_t$

breaks down at low $p_t$ (small $x$)

$CGC$ at low $p_t$

breaks down at large $x$ (high $p_t$)

**need a unified formalism**
Particle production at RHIC: kinematics

Collinear factorization

CGC

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\[ p_T = 1.5 \text{ GeV} \]
\[ \eta = 3.2 \]

\[ \int_{x_{\text{min}}}^{1} dx \ xG(x, Q^2) \rightarrow x_{\text{min}} G(x_{\text{min}}, Q^2) \]

This is an extreme approximation with potentially severe consequences!
how to tackle this problem?

what should be the starting point/expression/operator?

pQCD: quark and gluon operators

\[ \overline{\Psi}(y^-, 0_t)\gamma^+\Psi(0^-, 0_t) \]

renormalization lead to DGLAP evolution eq.

CGC: correlators of Wilson lines (DIS, Hybrid,....)

\[ F_2 \sim Tr V(x_t) V^\dagger(y_t) \]

renormalization leads to JIMWLK/BK evolution eq.
toward unifying small and large $x$

(multiple scattering)

scattering from small $x$ modes of the target field $A^− ≡ n^− S$ involves only small transverse momenta exchange (small angle deflection)

$$p^\mu = (p^+, \sim \sqrt{s}, p^− = 0, p_t = 0)$$

$$S = S(p^+ \sim 0, p^- / P^- \ll 1, p_t)$$

allow hard scattering by including one all $x$ field $A_\mu^a(x^+, x^−, x_t)$ during which there is large momenta exchanged and quark can get deflected by a large angle.

include eikonal multiple scattering before and after (along a different direction) the hard scattering
hard scattering: large deflection
scattered quark travels in the new “z” direction: \( \bar{z} \)

\[
\begin{pmatrix}
\bar{x} \\
\bar{y} \\
\bar{z}
\end{pmatrix} = \mathcal{O}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]

\[i \mathcal{M}_1 = (ig) \int d^4x \, e^{i(p-p)x} \, \bar{u}(\bar{q}) \, [\bar{A}(x)] \, u(p)\]

\[i \mathcal{M}_2 = (ig)^2 \int d^4x \, d^4x_1 \, \int \frac{d^4p_1}{(2\pi)^4} \, e^{i(p_1-p)x_1} \, e^{i(q-p_1)x} \bar{u}(\bar{q}) \left[ \bar{A}(x) \frac{i\phi_1}{\bar{p}_1^2 + i\epsilon} \, \hat{\not{p}} \, S(x_1) \right] u(p)\]

\[i \mathcal{M}_2 = (ig)^2 \int d^4x \, d^4\bar{x}_1 \, \int \frac{d^4\bar{p}_1}{(2\pi)^4} \, e^{i(\bar{p}_1-p)x} \, e^{i(q-\bar{p}_1)\bar{x}_1} \bar{u}(\bar{q}) \left[ \hat{\not{\bar{p}}} \, \bar{S}(\bar{x}_1) \frac{i\phi_1}{\bar{p}_1^2 + i\epsilon} \, A(x) \right] u(p)\]

with \( \bar{u}^\mu = \Lambda^\mu_\nu \, v^\nu \)
summing all the terms gives:

\[ i\mathcal{M}_1 = \int d^4x \, d^2z_t \, d^2\bar{z}_t \int \frac{d^2k_t}{(2\pi)^2} \frac{d^2\bar{k}_t}{(2\pi)^2} e^{i(\bar{k}-k) \cdot x} e^{-i(\bar{q}_t-\bar{k}_t) \cdot \bar{z}_t} e^{-i(k_t-p_t) \cdot z_t} \]

\[
\bar{u}(\bar{q}) \left[ \overline{V}_{AP}(x^+, \bar{z}_t) \frac{k}{2k^+} \left[ igA(x) \frac{k}{2k^+} \gamma V_{AP}(z_t, x^+) \right] u(p) \right.
\]

with

\[
\overline{V}_{AP}(x^+, \bar{z}_t) \equiv \hat{P} \exp \left\{ ig \int_{x^+}^{+\infty} d\bar{z}^+ \, \bar{S}^-_{\alpha}(\bar{z}_t, \bar{z}^+) \, t_\alpha \right\}
\]

\[
V_{AP}(z_t, x^+) \equiv \hat{P} \exp \left\{ ig \int_{-\infty}^{x^+} d\bar{z}^+ \, S^-_{\alpha}(z_t, \bar{z}^+) \, t_\alpha \right\}
\]

can extract the effective quark propagator

\[
i\mathcal{M}(p, \bar{q}) = \bar{u}(\bar{q}) \tau_F u(p)
\]

this quark propagator is the building block for DIS structure functions, single inclusive particle production in pA,....
but there is more to do: interactions of large and small $x$ modes

these re-sum to
\[ iM_2 = \frac{2i}{(p - \bar{q})^2} \int d^4 x \ e^{i(\bar{q} - p)x} \bar{u}(\bar{q}) \left[ (ig t^a) \left[ \partial_x + U_{AP}^\dagger(x_t, x^+) \right]^{ab} 
\right. \\
\left. \left[ n \cdot (p - \bar{q}) A_b(x) - (p - \bar{q}) \cdot A_b(x) \gamma^i \right] \right] u(p) \]

with

\[ U_{AP}(x_t, x^+) \equiv \hat{P} \exp \left\{ ig \int_{-\infty}^{x^+} dz^+ S_a^{-}(z^+, x_t) T_a \right\} \]

but there is more!
integration over $p_1^-$

$$\int \frac{dp_1^-}{2\pi} \frac{e^{ip_1^- (x_1^+-x^+)}}{[p_1^2 + i\epsilon] \left[(p_1 - \bar{q})^2 + i\epsilon\right]}$$

both poles are below the real axis, we get

$$e^{i\frac{p_{1t}^2}{2p^+} (x_1^+-x^+)} \left[\frac{p_{1t}^2}{2p^+} - \bar{q}^- - \frac{(p_{1t} - \bar{q}_t)^2}{2(p^+-\bar{q}^+)}\right] + e^{i\left[\bar{q}^- + \frac{(p_{1t} - \bar{q}_t)^2}{2(p^+-\bar{q}^+)}\right]} (x_1^+-x^+)$$

ignoring phases we get a cancellation!

this can be shown to hold to all orders whenever both initial state quark and hard gluon scatter from the soft fields!
how about the final state quark interactions?

integration over $\bar{p}_{1}$

$$
\int \frac{d\bar{p}_{1}}{2\pi} \frac{e^{ip_{1-}(x_{1}^{+} - x^{+})}}{[\bar{p}_{1}^{2} + i\epsilon] [(p_{1} - \bar{p}_{1})^{2} + i\epsilon]}
$$

now the poles are on the opposite side of the real axis, we get both ordering

$$
\theta(x^{+} - \bar{x}_{1}^{+}) \text{ and } \theta(\bar{x}_{1}^{+} - x^{+})
$$

ignoring the phases the contribution of the two poles add!

path ordering is lost!

however further rescatterings are still path-ordered before/after $x_{1}^{+}, \bar{x}_{1}^{+}$
these contributions resum to

\[ i\mathcal{M}_3 = -2i \int d^4x \, d^2\bar{x}_t \, d\bar{x}^+ \frac{d^2\bar{p}_{1t}}{(2\pi)^2} \, e^{i(\bar{q}^+ - p^+)x^-} \, e^{-i(\bar{p}_{1t} - p_t) \cdot x_t} \, e^{-i(q_t - \bar{p}_{1t}) \cdot \bar{x}_t} \]

\[ \bar{u}(\bar{q}) \left[ \left[ \partial_{\bar{x}+} \bar{V}_{AP}(\bar{x}^+, \bar{x}_t) \right] \bar{\eta} \bar{p}_1 (igt^a) \left[ \partial_{x+} U^\dagger_{AP}(x_t, x^+) \right]^{ab} \right. \]

\[ \frac{\left[ n \cdot (p - \bar{q}) A^b(x) - (p - \bar{p}_1) \cdot A^b(x) \bar{\eta} \right]}{[2n \cdot \bar{q} \, 2n \cdot (p - \bar{q}) \, p^- - 2n \cdot (p - \bar{q}) \, \bar{p}_{1t}^2 - 2n \cdot \bar{q} \, (\bar{p}_{1t} - p_t)^2]} \left. \right] u(p) \]
full amplitude:  \[ iM = iM_{\text{eik}} + iM_1 + iM_2 + iM_3 \]

soft (eikonal) limit:  \[ A^\mu(x) \rightarrow n^- S(x^+, x_t) \quad n \cdot \bar{q} \rightarrow n \cdot p \]

\[ iM \rightarrow iM_{\text{eik}} \]
Need to clarify, further steps to be taken

The limit of intermediate/large $x$ ?
  ignore $S^{-}$?

Gauge invariance ?

Matching between small and large $x$ ?
  brute force?

Gluon scattering and radiation

One-loop correction: a new evolution equation
QCD kinematic phase space

unifying saturation with high $p_t$ (large $x$) physics?

kinematics of saturation: where is saturation applicable?

*jet physics*, high $p_t$ (polar and azimuthal) angular correlations

cold matter energy loss, spin physics?, .......
single inclusive pion production

DHJ, NPA765 (2006) 57-70

$p + p \rightarrow \pi^0 + X$