Artin Billiard Exponential Decay of Correlation Functions

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Plan

- Artin Dynamical System with Quasi-Ergodic Trajectories
- Construction of Periodic Geodesic Trajectories
- Summary

Let us consider the model of the Lobachevsky plane realized in the upper half-plane $y > 0$ of the complex plane $z = x + iy \in \mathcal{C}$ with the Poincaré metric which is given by the line element

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{dz\, d\bar{z}}{(\Im z)^2}\,, \tag{2.1}$$

where $\bar{z}$ is the complex conjugate of z and $\Im z$ is the imaginary part of z. The Lobachevsky plane is a surface of a constant negative curvature, its Gaussian curvature is $K = -1$. This metric has two well known properties: 1) it is invariant with respect to all linear substitutions, which form the group $g \in G$ of isometries of the Lobachevsky plane:

$$w = g \cdot z \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot z \equiv \frac{\alpha z + \beta}{\gamma z + \delta}, \tag{2.2}$$

where $\alpha, \beta, \gamma, \delta$ are *real coefficients of the matrix* g and the determinant of g is positive, $\alpha\delta - \beta\gamma > 0$.

2) The geodesic lines are either semi-circles orthogonal to the real axis or rays perpendicular to the real axis. The geodesic equation has the form

$$\frac{d^2x}{dt^2} - \frac{2}{y}\frac{dx}{dt}\frac{dy}{dt} = 0\,, \qquad \frac{d^2y}{dt^2} + \frac{1}{y}\left(\frac{dx}{dt}\right)^2 - \frac{1}{y}\left(\frac{dy}{dt}\right)^2 = 0$$

and has two solutions

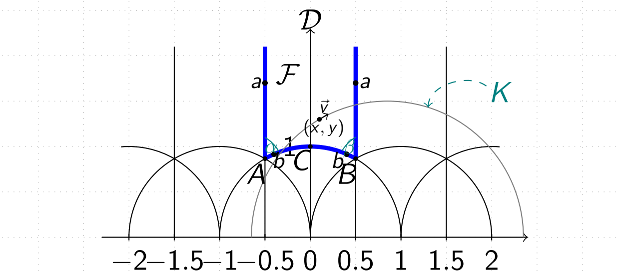
$$\begin{array}{lll} x(t) - x_0 = r\tanh(t)\,, & y(t) = \frac{r}{\cosh(t)} & \leftarrow \text{orthogonal semi-circles} \\ x(t) = x_0, & y(t) = e^t & \leftarrow \text{perpendicular rays} \quad . \end{array}$$

Here $x_0 \in (-\infty, +\infty)$, $t \in (-\infty, +\infty)$ and $r \in (0, \infty)$. The points on the geodesics curves move with a unit velocity $\frac{ds}{dt} = 1$.

In order to construct a compact surface $\mathcal{F}$ on the Lobachevsky plane, one can identify all points in the upper half of the plane which are related to each other by the substitution (2.2) with the integer coefficients and a unit determinant. These transformations form a modular group $d \in D$. Thus we consider two points z and w to be "identical" if:

$$w = \frac{mz+n}{pz+q}, \quad d = \begin{pmatrix} m & n \\ p & q \end{pmatrix}, \quad d \in D \tag{2.3}$$

with integers m, n, p, q constrained by the condition $mq - pn = 1$. The D is the discrete subgroup of the isometry transformations G of (2.2) The identification creates a regular tessellation of the Lobachevsky plane by congruent hyperbolic triangles in. The Lobachevsky plane is covered by the infinite-order triangular tiling. One of these triangles can be chosen as a fundamental region (denoted by $\mathcal{F}$).



Inside the modular triangle $\mathcal{F}$ there is exactly one representative among all equivalent points of the Lobachevsky plane with the exception of the points on the triangle edges which are opposite to each other. These points should be identified in order to form a *closed compact surface* $\bar{\mathcal{F}}$ by "gluing" the opposite edges of the modular triangle together. On the figure above one can see the pairs of points on the edges of the triangle which are identified.

In order to describe the behavior of the geodesic trajectories on the surface $\bar{\mathcal{F}}$ one can use the knowledge of the geodesic trajectories on the whole Lobachevsky plane. Let us consider an arbitrary point $(x, y) \in \mathcal{F}$ and the velocity vector $\vec{v} = (\cos\theta, \sin\theta)$. These are the coordinates of the phase space $(x, y, \theta) \in \mathcal{M}$, and they uniquely determine the geodesic trajectory as the orthogonal circle K in the whole Lobachevsky plane. As this trajectory "hits" the edges of the fundamental region $\mathcal{F}$ and goes outside of it, one should apply the modular transformation to that parts of the circle K which lie outside of $\mathcal{F}$ in order to return them back to the $\mathcal{F}$. That algorithm will define the whole trajectory on $\bar{\mathcal{F}}$ for $t \in (-\infty, +\infty)$.

One should observe that this description of the trajectory on $\bar{\mathcal{F}}$ is equivalent to the set of geodesic circles $\{K'\}$ which appear under the action of the modular group on the initial circle K. One should join together the parts of the geodesic circles $\{K'\}$ which lie inside $\mathcal{F}$ into a unique continuous trajectory on $\bar{\mathcal{F}}$.

In this context the quasi-ergodicity of the trajectory K on $\bar{\mathcal{F}}$ will mean that among all geodesic circles $\{K'\}$ there are those which are approaching arbitrarily close to any given circle C. Now notice that the geodesic circle K is determined by its *base points* ξ and η, which lie on the real axis. Under the action of the modular group the coordinates ξ and η will be mapped into the base points ξ' and η' of the transformed circle $K^{'}$:

$$\xi' = \frac{m\xi + n}{p\xi + q}, \qquad \eta' = \frac{m\eta + n}{p\eta + q} \ . \tag{2.4}$$

In this context the geodesic circles can be considered "close" to each other if their base points lie in the infinitesimal neighborhood.

It is convenient to introduce the plane $\mathcal{E}$ with the coordinates $(\xi, \eta) \in \mathcal{E}$. To each point $(\xi,\ \eta)$ of the $\mathcal{E}$ plane corresponds a geodesic circle on the z-plane, with $\xi,\ \eta$ being its base coordinates. And conversely to every circle of the z-plane one can assign two points $(\xi,\ \eta)$ or $(\eta,\ \xi)$ on the $\mathcal{E}$ plane. The reason for this ambiguity lies in the fact that the ordering of the base point coordinates after the action of the modular transformation can be inverted. The geodesic circles were considered to be in the infinitesimal neighborhood if their base coordinates were close. This implies now that the points $(\xi' = d\xi,\ \eta' = d\eta)$ on the $\mathcal{E}$ plane resulting from the action of the full group of modular transformation on $(\xi,\ \eta)$ must be everywhere dense on the $\mathcal{E}$ plane if the trajectory K is quasi-ergodic.

In order to have quasi-ergodic behavior of a trajectory it is necessary and sufficient to have an everywhere dense distribution of points (ξ', η') in the subregion of the $\mathcal{E}$ plane defined by the condition

$$-1 < \eta < 0 \,, \qquad 1 < \xi \,. \tag{2.5}$$

The necessity of this condition is obvious. Provided this condition is fulfilled, one can apply to the points of this region the modular transformations $\xi' = \xi + n \,, \qquad \eta' = \eta + n$ with some integer n and to see that the areas $n < \xi$, $n - 1 < \eta < n$ are covered everywhere dense. One can fix the arbitrariness of the base points coordinates, mentioned above, by ordering them as $\eta < \xi$. It follows then that it is sufficient to require an everywhere dense distribution of points on the $\mathcal{E}$ plane for the geodesic circles which are crossing the fundamental region $\mathcal{F}$. Thus one can assume that the base point coordinates of the initial circle K lie in the region $-1 < \eta < 0$ and $\xi > 1$.

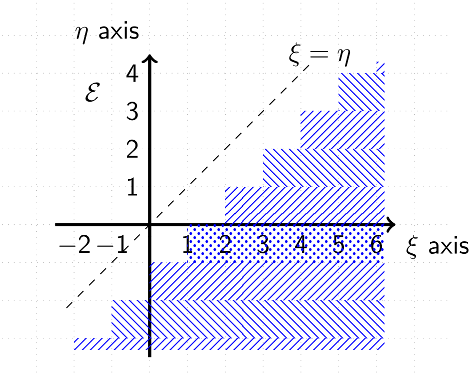


Figure: The $\mathcal{E}$ plane. For a trajectory to be a quasi-ergodic it is necessary and sufficient to have an everywhere dense distribution of the points (ξ', η') (2.4) in the subregion of the $\mathcal{E}$ plane defined by the conditions $-1 < \eta < 0\,, 1 < \xi$. The region between the "stairs" and the diagonal $\xi = \eta$ corresponds to the geodesic circles which are not crossing the fundamental region $\mathcal{F}$.

The coordinates ξ and $-\eta$ in the region $-1 < \eta < 0$ and $\xi > 1$ can be represent as the continued fractions with positive integers a_i

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}, \quad -\eta = \cfrac{1}{a_{-1} + \cfrac{1}{a_{-2} + \cfrac{1}{a_{-3} + \ddots}}},$$

which means that the geodesic trajectory K on $\bar{\mathcal{F}}$ can be represented as an infinite A-chain of the form

$$......., a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, ... \tag{2.6}$$

Let us define for any integer $n \geq 0$ the coordinates η_n and ξ_n by the relations

$$\xi_n = a_n + \cfrac{1}{a_{n+1} + \cfrac{1}{a_{n+2} + \cfrac{1}{a_{n+3} + \ddots}}}, \quad -\eta_n = \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \cfrac{1}{a_{n-3} + \ddots}}}.$$

This corresponds to the shift on the A-chain expansion to the right. These coordinates satisfy the relations

$$\xi = \frac{P_n \xi_n + P_{n-1}}{Q_n \xi_n + Q_{n-1}}, \qquad \eta = \frac{P_n \eta_n + P_{n-1}}{Q_n \eta_n + Q_{n-1}}, \tag{2.7}$$

where P_n and Q_n are positive integers.

The coefficients satisfy the recursion relations given below

$$P_{n+1} = P_n a_n + P_{n-1}\,, \tag{2.8}$$
$$Q_{n+1} = Q_n a_n + Q_{n-1}$$

P_n and Q_n satisfy also the relation

$$P_n Q_{n-1} - Q_n P_{n-1} = (-1)^n\,. \tag{2.9}$$

Let us consider the integer matrices constructed in terms of P_n and Q_n as

$$d_n = \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix}. \tag{2.10}$$

Because of the (2.9) the determinant of d_n is equal to $(-1)^n$ and for even n they represent the matrices of the modular transformations (2.3). It follows therefore from (2.7) and (2.10) that the points (ξ_n, η_n) appear under the action of the modular transformations d_n^{-1} on the point (ξ, η)

$$\xi_n = d_n^{-1} \cdot \xi, \quad \eta_n = d_n^{-1} \cdot \eta \ . \tag{2.11}$$

The geodesic circle K was given by its base coordinated (ξ, η) and, as it was just demonstrated, the points (ξ_n, η_n) are the base coordinates of the geodesic circles $\{K'\}$.

Suppose that a sequence of $2m+1$ positive numbers

$$c_{-m}, \ldots, c_{-2}, c_{-1}, c_0, c_1, \ldots, c_m \tag{2.12}$$

approximate a given circle C with a sufficient accuracy, then the circles $\{K'\}$ will closely approach the circle C if it will be possible to find an even index n such that *the section of the A-chain* (2.6) of the length $2m+1$

$$a_{n-m}, a_{n-m+1}, \ldots, a_{n-1}, a_n, a_{n+1} \ldots, a_{n+m}$$

will coincide either with $c_{-m}, \ldots, c_{-2}, c_{-1}, c_0, c_1, \ldots, c_m$ or with its reverse sequence. Therefore for the quasi-ergodicity of the trajectory K, represented by the infinite A-chain (2.6), it is necessary and sufficient to have all imaginable finite sequences (2.12) of positive integers to be a section of the A-chain (2.6).

Let us briefly outline what has been achieved. The geodesic trajectory K is represented by an infinite A-chain (2.6). The points on the $\mathcal{E}$ plane which correspond to the circles $\{K'\}$ are generated by the algorithm (2.7) and (2.11). In order for the geodesic trajectory K to be quasi-ergodic it is necessary and sufficient that every imaginable finite sequence of positive integers can be found as a section in the associated A-chain (2.6). It is known that almost all numbers ξ have a quasi-ergodic continued fractions. Thus almost all geodesic trajectories on the surface $\bar{\mathcal{F}}$ are quasi-ergodic.

Let us give an example where the trajectory is not quasi-ergodic but quasi periodic. It is obvious that periodic A-chains correspond to periodic orbits and vice versa. An example of a periodic A-chain with a period 3 is given below:

$$.......,1\,,2\,,3\,,1\,,2\,,3\,,1\,,2\,,3\,,1\,,2\,,3\,,1\,,2\,,3\,,1\,,2\,,3\,,......, \qquad (3.1)$$

where $a_0 = 1$. We can find the base points of the trajectory corresponding to the chin given above in the following form:

$$\xi = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\xi}}}, \qquad -\eta = \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{1 - \eta}}}, \qquad (3.2)$$

where we used the fact that the chain has period three and that the continued fractions repeat themselves after three steps.

These are the quadratic equations on the base coordinates ξ, η of K:

$$7\xi^2 - 8\xi - 3 = 0\,,$$
$$7\eta^2 - 8\eta - 3 = 0\,.$$

In order to have the base points of the circle K in the region $(-1 < \eta < 0\,, \quad 1 < \xi\,)$ we have to choose the solutions:

$$\xi = \frac{1}{7}\left(4 + \sqrt{37}\right)\,, \qquad \eta = \frac{1}{7}\left(4 - \sqrt{37}\right)\,. \tag{3.3}$$

From $\xi = d_n \cdot \xi_n$ and and $\eta = d_n \cdot \eta_n$ it follows that

$$\xi_n = d_n^{-1} \cdot \xi \,, \qquad \eta_n = d_n^{-1} \cdot \eta \,,$$

where

$$d_n^{-1} = \begin{pmatrix} Q_{n-1} & -P_{n-1} \\ -Q_n & P_n \end{pmatrix} .$$

These are the matrices of the modular group which are defining the geodesic circles $\{K'\}$.

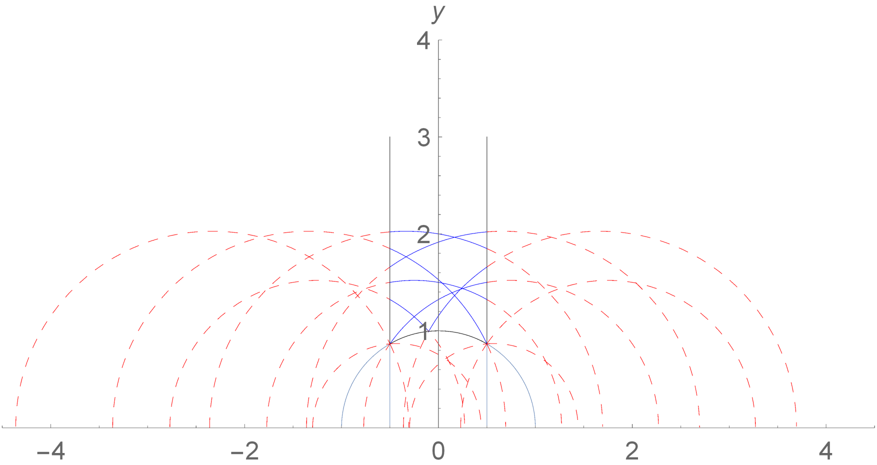
With the help of the last two expressions it is easy to find their base points describing the periodic geodesic:

$$\xi_0 = \xi = \frac{1}{7}\left(4+\sqrt{37}\right)\,, \quad \xi_1 = \frac{1}{4}\left(\sqrt{37}+3\right)\,,$$
$$\xi_2 = \frac{1}{3}\left(\sqrt{37}+5\right)\,, \quad \xi_3 = \frac{1}{7}\left(\sqrt{37}+4\right)\,, \quad \xi_4 = \frac{1}{3}\left(\sqrt{37}+3\right)$$

and

$$\eta_0 = \eta = \frac{1}{7}\left(4-\sqrt{37}\right)\,, \quad \eta_1 = \frac{1}{4}\left(3-\sqrt{37}\right)\,, \qquad (3.4)$$
$$\eta_2 = \frac{1}{3}\left(5-\sqrt{37}\right)\,, \quad \eta_3 = \frac{1}{7}\left(4-\sqrt{37}\right)\,, \quad \eta_4 = \frac{1}{4}\left(3-\sqrt{37}\right)\,.$$

These base coordinates define the full trajectory on the surface $\bar{\mathcal{F}}$


y
4
3
2
1
−4
−2
0
2
4

The dynamical system is defined on the fundamental region of the Lobachevsky plane which is obtained by the identification of points congruent with respect to the modular group, a discrete subgroup of the Lobachevsky plane isometries. The fundamental region in this case is a hyperbolic triangle. The geodesic trajectories of the non-Euclidean billiard are bounded to propagate on the fundamental hyperbolic triangle.

Almost all geodesic trajectories are quasi-ergodic meaning that all trajectories, with the exception of measure zero, during their time evolution will approach infinitely close any point and any given direction on surface $\mathcal{F}$.

We also demonstrated how one can use the Artin algorithm to construct a periodic geodesic trajectory.

THANKS