# *Linear*

# *Imperfections*

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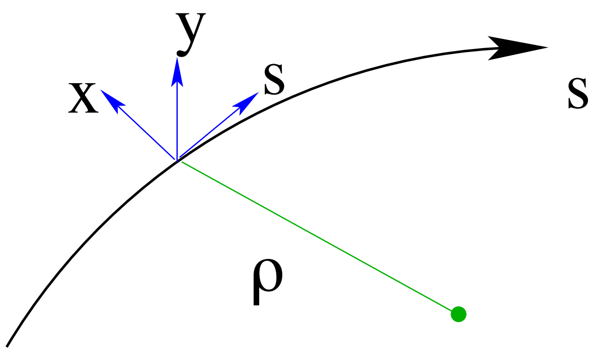
# *Linear Imperfections*

- equation of motion in an accelerator
  - → Hills equation
  - → sine and cosine like solutions
  - → closed orbit
  - → sources for closed orbit perturbations
- dipole perturbations
  - → closed orbit response
  - → dispersion orbit
  - → integer resonances
  - → BPMs & dipole correctors
- quadrupole perturbations
  - → one–turn map & tune error
  - → beta–beat
  - → half–integer resonances
- orbit correction
  - → local orbit bumps

# Variable Definition

## Variables in moving coordinate system:




$$x' = \frac{d}{ds} x$$

$$\frac{d}{dt} = \frac{ds}{dt} \cdot \frac{d}{ds} \longrightarrow x' = \frac{p_x}{p_0}$$

($\frac{ds}{dt}$ : v)

## Hill's Equation:

$$\frac{d^2x}{d s^2} + K(s) \cdot x = 0; \qquad K(s) = K(s + L);$$

$$K(s) = \begin{cases} 0 & \text{drift} \\ 1/\rho^2 & \text{dipole} \\ 0.3 \cdot \dfrac{B[T/m]}{p[GeV]} & \text{quadrupole} \end{cases}$$

## Perturbations:

$$\frac{d^2x}{d s^2} + K(s) \cdot x = G(s); \qquad G(s) = \frac{F(s)_{Lorentz}}{v \cdot p_0}$$

## *Sinelike and Cosinelike Solutions*

- system of first order linear differential equations:

$$\vec{y} = \begin{pmatrix} x \\ x' \end{pmatrix} \longrightarrow \vec{y}\,' + \begin{pmatrix} 0 & -1 \\ K & 0 \end{pmatrix} \cdot \vec{y} = 0$$

$K = \text{const} \longrightarrow$

$$\vec{Y}_1(s) = \begin{pmatrix} \sin(\sqrt{K}\cdot s) \\ \sqrt{K}\cdot\cos(\sqrt{K}\cdot s) \end{pmatrix} \qquad \vec{Y}_2(s) = \begin{pmatrix} \cos(\sqrt{K}\cdot s) \\ -\sqrt{K}\cdot\sin(\sqrt{K}\cdot s) \end{pmatrix}$$

- initial conditions:

$$\vec{Y}_1(0) = \begin{pmatrix} Y_1 \\ Y_1' \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{Y}_2(0) = \begin{pmatrix} Y_2 \\ Y_2' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- general solution: $\vec{y}(s) = a\cdot\vec{Y}_1(s) + b\cdot\vec{Y}_2(s)$

- transport map: $\vec{y}(s) = \underline{M}(s - s_0)\cdot\vec{y}(s_0)$

with: $$= \begin{pmatrix} \cos(\sqrt{K}\cdot[s-s_0]) & \sin(\sqrt{K}\cdot[s-s_0]) \\ -\sqrt{K}\cdot\sin(\sqrt{K}\cdot[s-s_0]) & \sqrt{K}\cdot\cos(\sqrt{K}\cdot[s-s_0]) \end{pmatrix}$$

# Sinelike and Cosinelike Solutions

- Floquet theorem:

$$\vec{Y}_1(s) = \begin{pmatrix} \sqrt{\beta(s)} \cdot \sin(\phi(s) + \phi_0) \\ [\cos(\phi(s) + \phi_0) + \alpha(s) \cdot \sin(\phi(s) + \phi_0)] / \sqrt{\beta(s)} \end{pmatrix}$$

$$\vec{Y}_2(s) = \begin{pmatrix} \sqrt{\beta(s)} \cdot \cos(\phi(s) + \phi_0) \\ -[\sin(\phi(s) + \phi_0) + \alpha(s) \cdot \cos(\phi(s) + \phi_0)] / \sqrt{\beta(s)} \end{pmatrix}$$

$$\beta(s) = \beta(s + L); \quad \phi(s) = \int \frac{1}{\beta}\, ds; \quad \alpha(s) = -\frac{1}{2} \beta'(s)$$

- ´sinelike´ and ´cosinelike´ solutions:

$$\vec{C}(s) = a \cdot \vec{Y}_1(s) + b \cdot \vec{Y}_2(s) \qquad \vec{S}(s) = c \cdot \vec{Y}_1(s) + d \cdot \vec{Y}_2(s)$$

with: $\vec{C}(s_0) = \begin{pmatrix} C(s_0) \\ C'(s_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{S}(s_0) = \begin{pmatrix} S(s_0) \\ S'(s_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

→ one can generate a transport matrix in analogy to the case with constant K(s)!

# *Sinelike and Cosinelike Solutions*

´sinelike´ and ´cosinelike´ solutions:

$$\vec{S}(s) = \begin{pmatrix} \sqrt{\beta(s)\beta(s_0)} \cdot \sin(\phi(s)+\phi_0) \\ \sqrt{\beta(s_0)} \cdot [\cos(\phi(s)+\phi_0) + \alpha(s) \cdot \sin(\phi(s)+\phi_0)] / \sqrt{\beta(s)} \end{pmatrix}$$

$$\vec{C}(s) = \begin{pmatrix} \sqrt{\beta(s)} \cdot [\cos(\phi(s)+\phi_0) + \alpha(s_0) \cdot \sin(\phi(s)+\phi_0)] / \sqrt{\beta(s_0)} \\ -(1+\alpha\alpha_0) \cdot [\sin(\phi(s)+\phi_0) + (\alpha_0-\alpha) \cdot \cos(\phi(s)+\phi_0)] / \sqrt{\beta\beta_0} \end{pmatrix}$$

transport map from $s_0$ to s: $\vec{y}(s) = \underline{M}(s, s_0) \cdot \vec{y}(s_0)$

with: $\underline{M} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix}$

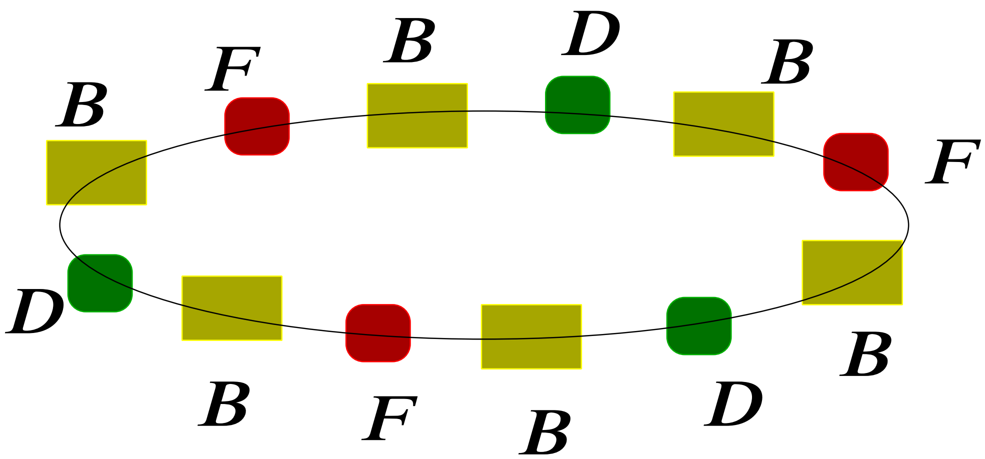
transport map for $s = s_0 + L$:

$$\underline{M} = \underline{I} \cdot \cos(2\pi Q) + \underline{J} \cdot \sin(2\pi Q)$$

$$\underline{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \underline{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}; \; \gamma = [1+\alpha^2] / \beta$$

# Closed Orbit

- *particles oscillate around an ideal orbit:*




- *additional dipole fields perturb the orbit:*

- error in dipole field

- energy error

$$\alpha = \frac{l}{\rho} = \frac{q \cdot B \cdot l}{p + \Delta p} \approx \left(1 - \frac{\Delta p}{p}\right) \cdot \frac{q \cdot B \cdot l}{p}$$

- offset in quadrupole field

$$B_x = g \cdot y \qquad\qquad B_x = g \cdot \tilde{y}$$

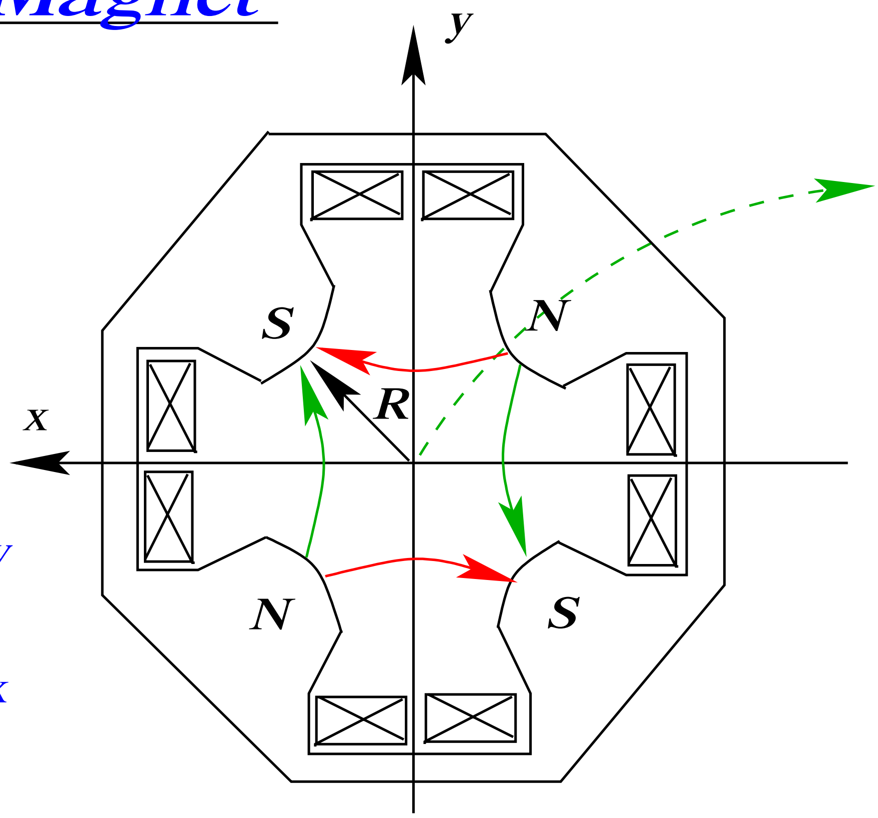
$$B_y = g \cdot x \qquad x = x_0 + \tilde{x} \rightarrow B_y = g \cdot x_0 + g \cdot \tilde{x}$$

*dipole component* ($g \cdot x_0$)

## Quadrupole Magnet

$$B_x = g \cdot y$$

$$B_y = g \cdot x$$




$$F_x = -q \cdot v \cdot B_y$$

$$F_y = q \cdot v \cdot B_x$$

$$\frac{d^2 x}{d s^2} + K(s) \cdot x = G(s); \qquad G(s) = \frac{F(s)_{Lorentz}}{v \cdot p_0}$$

## normalized fields:

dipole: $$k_0(s) = 0.3 \cdot \frac{B_0[T]}{p_0[GeV]}$$

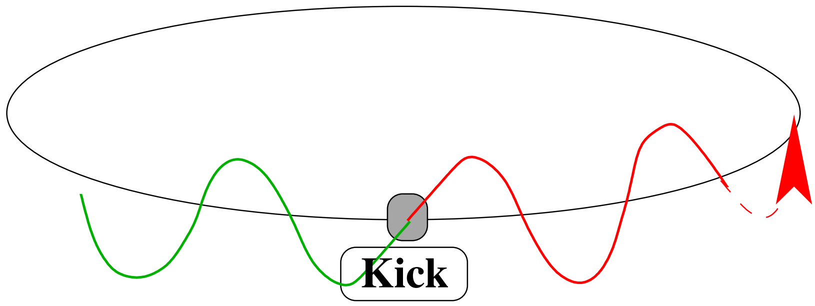
quadrupole: $$k_1(s) = 0.3 \cdot \frac{g_0[T/m]}{p_0[GeV]}$$

quadrupole misalignment: $$\Delta k_0(s) = 0.3 \cdot \frac{g[T/m]}{p[GeV]} \cdot x_0$$

# Dipole Error and Orbit Stability

- **Q:** number of β–oscillations per turn

**Q = N**

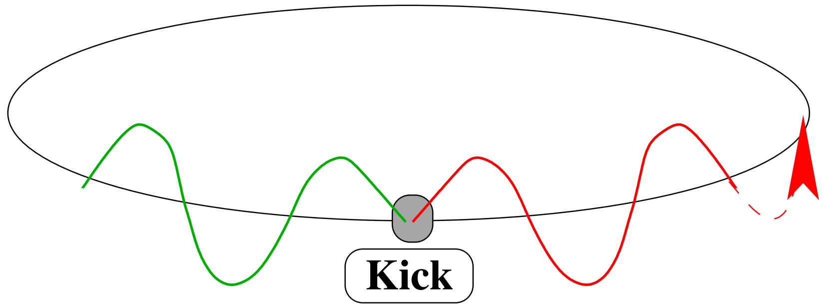



→ the perturbation adds up

→ amplitude growth and particle loss

→ **watch out for integer tunes!**

**Q = N + 0.5**




→ the perturbation cancels after each turn

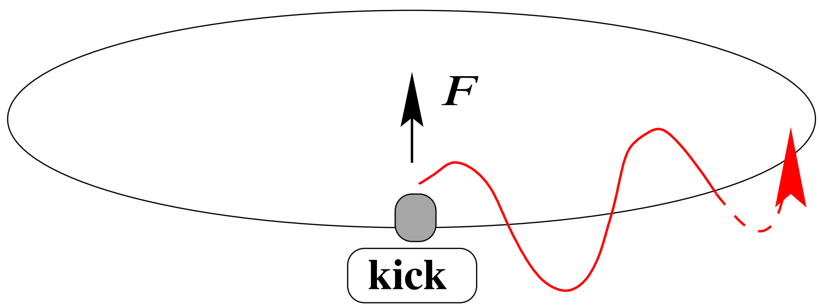
# Quadrupole Error and Orbit Stability

## Quadrupole Error:

→ orbit kick proportional to beam offset in quadrupole

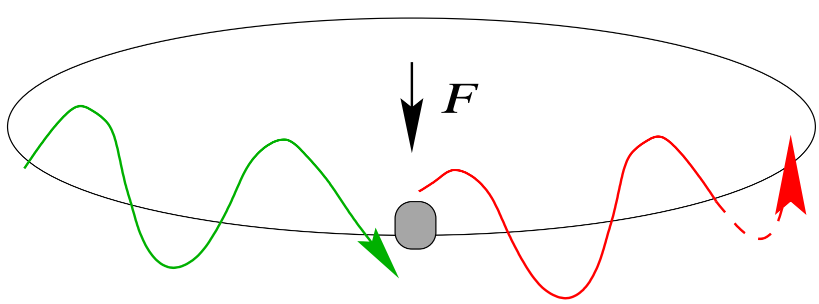
## $Q = N + 0.5$

1. Turn: $x > 0$




→ amplitude increase

2. Turn: $x < 0$




→ amplitude increase

→ **watch out for half integer tunes!**

# Sources for Orbit Errors

- *Quadrupole offset:*
  - *alignment* *+/- 0.1 mm*
  - *ground motion*
    - *slow drift*
    - *civilisation*
    - *moon*
    - *seasons*
    - *civil engineering*
- *Error in dipole strength*
  - *power supplies*
  - *calibration*
- *Energy error of particles*
  - *injection energy (RF off)*
  - *RF frequency*
  - *momentum distribtion*

# *Example Quadrupole Alignment inLEP*

| | |
|---|---|
| Transversal tilt dispersion of the 3278 dipoles | σ = ± 0.34 mrd |
| Vertical dispersion of the 784 quadrupoles (with respect to the smoothing polynomial) | σ = ± 0.65 mm |

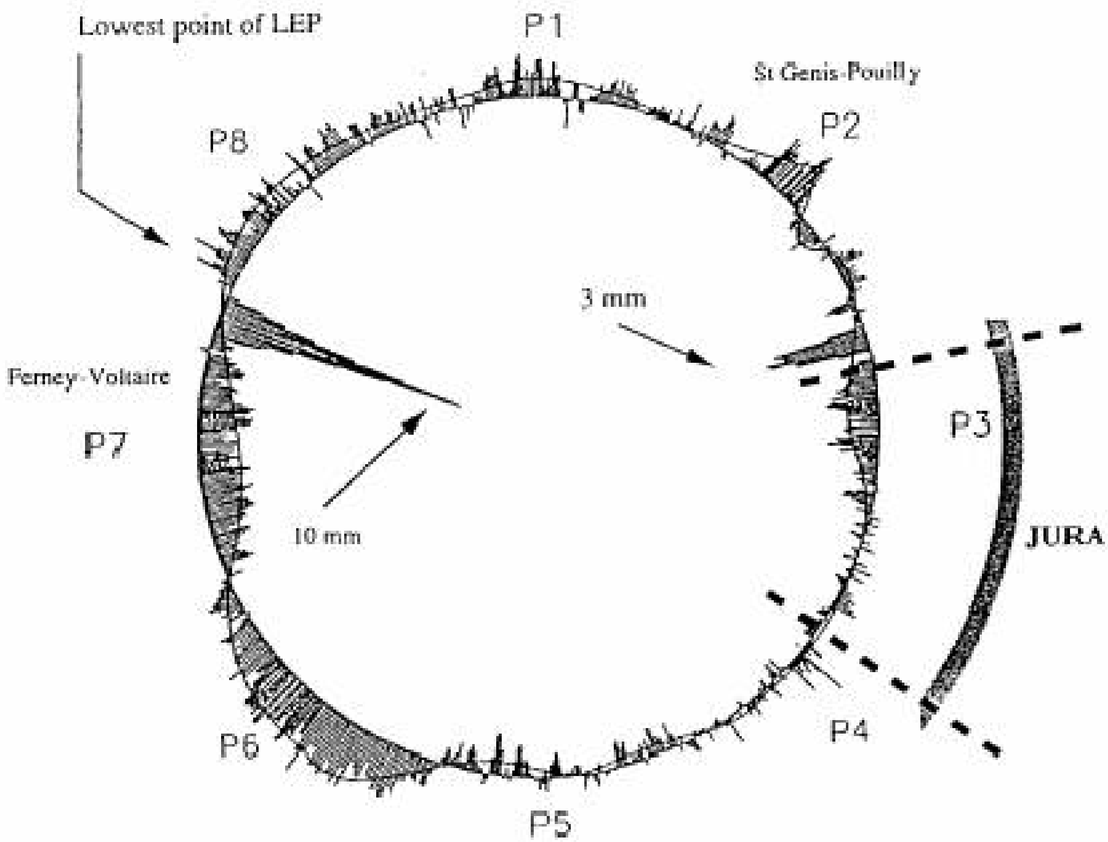


Figure 1 : observed status, end 1992

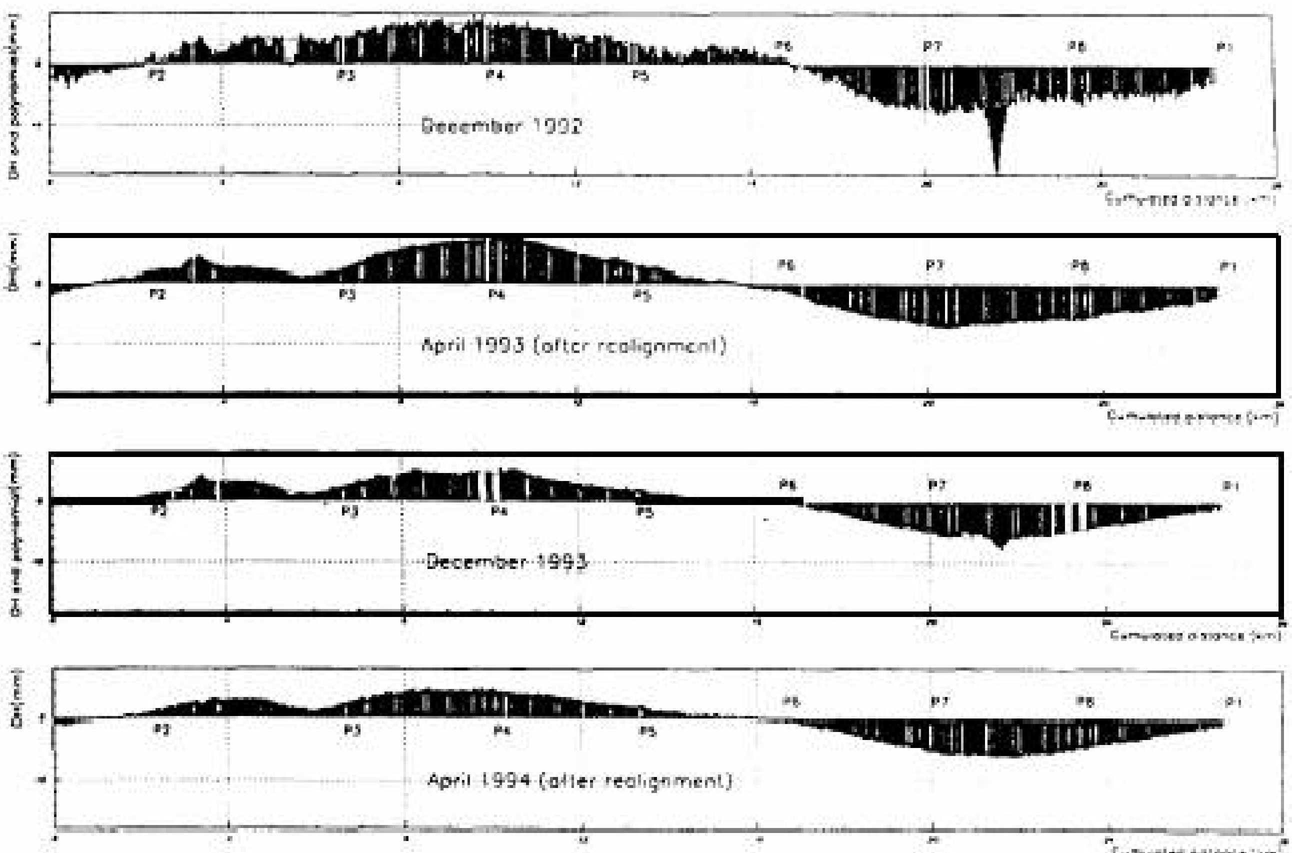


Figure 2 : progressive corrections in 1993 and 1994

# Problems Generated by Orbit Errrors

- *injection errors:*
  - aperture → beam losses
  - filamentation → beam size

- *closed orbit errors:*
  - x–y coupling
  - aperture
  - energy error
  - field imperfections
  - dispersion → beam size at IP
  - beam separation

*Aim:*

$\Delta x,\ \Delta y < 4$ mm

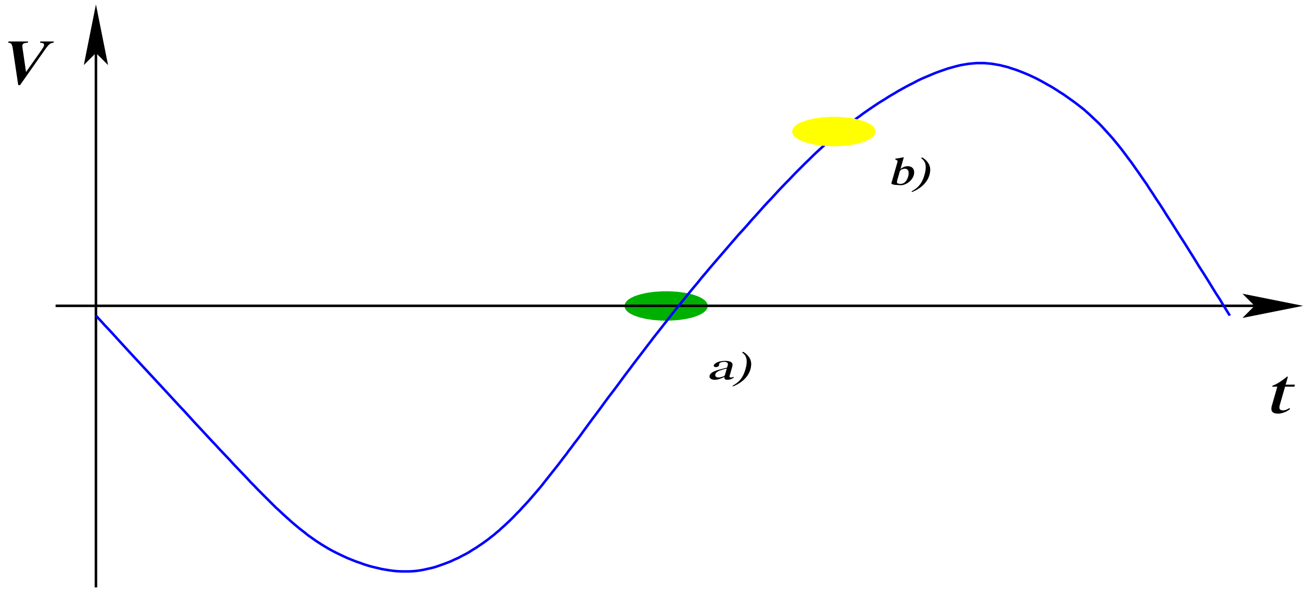
rms < 0.5 mm

→ beam monitors and orbit correctors

## Synchrotron:

→ *the orbit determines the particle energy!*

▬ *assume: L > design orbit*




→ *energy increase*

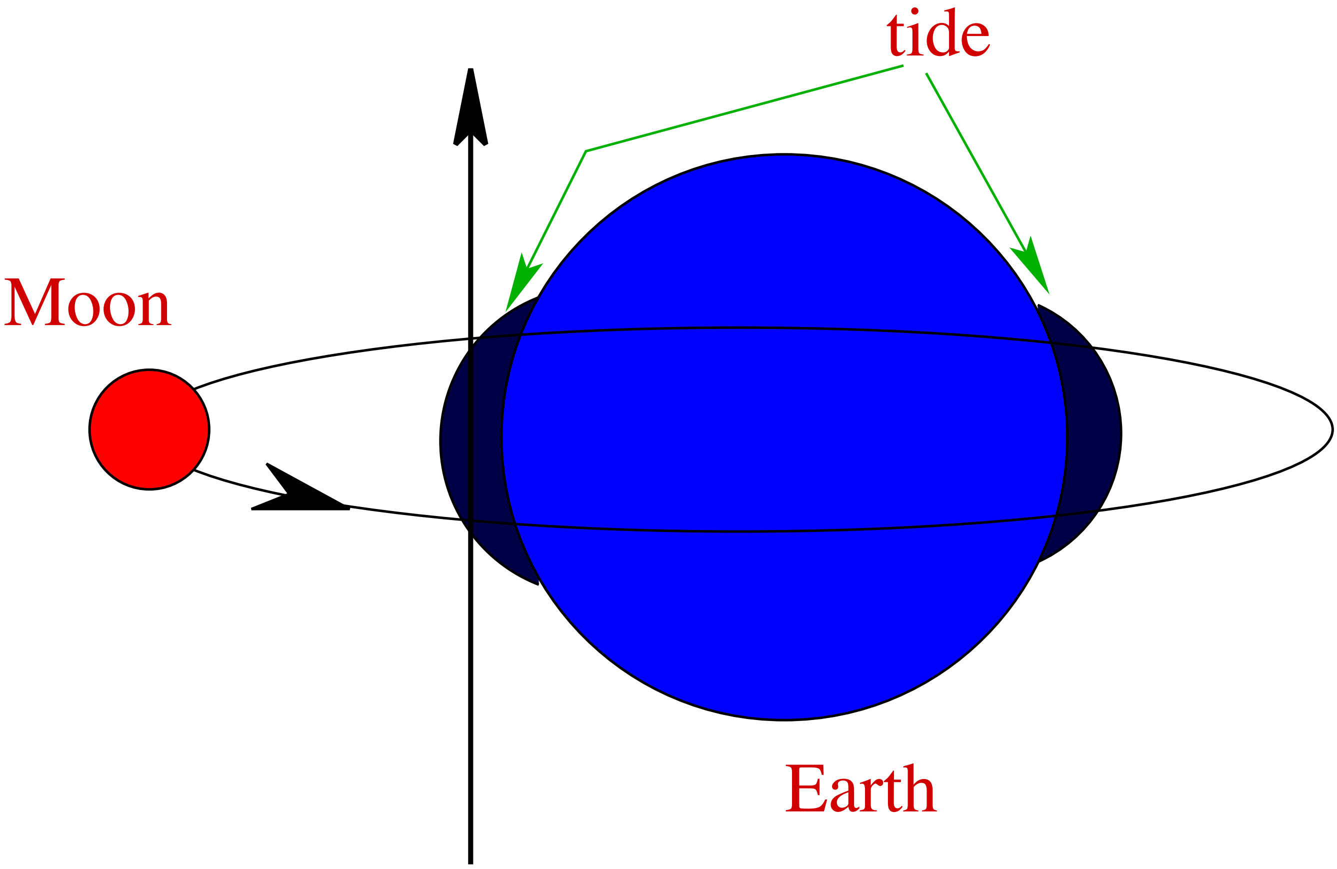
## Equilibrium:

$$f_{RF} = h \cdot f_{rev}$$

$$f_{rev} = \frac{1}{2 \cdot \pi} \cdot \frac{q}{m \cdot \gamma} \cdot B$$

→ *E depends on orbit and magnetic field!*

## *tidal motion of the earth:*




## *orbit and beam energy modulation:*

$$f_{mod} = 24\ h;\ 12\ h$$

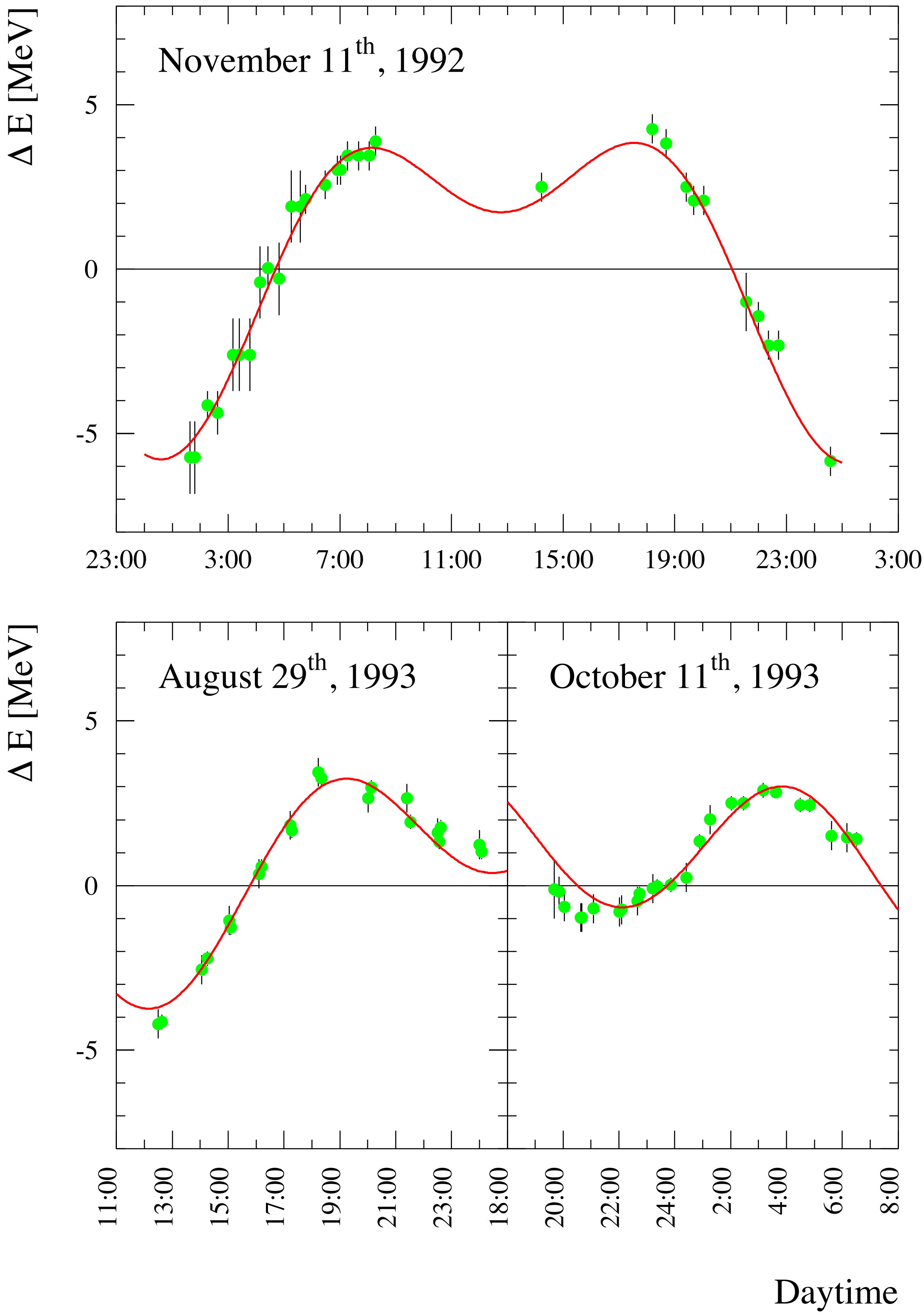
→ $\Delta E \approx 10\ MeV$

$\approx 0.02\%$

*aim:* $\Delta E \lesssim 0.003\%$

→ *requires correction!*

energy modulation due to tidal motion of earth

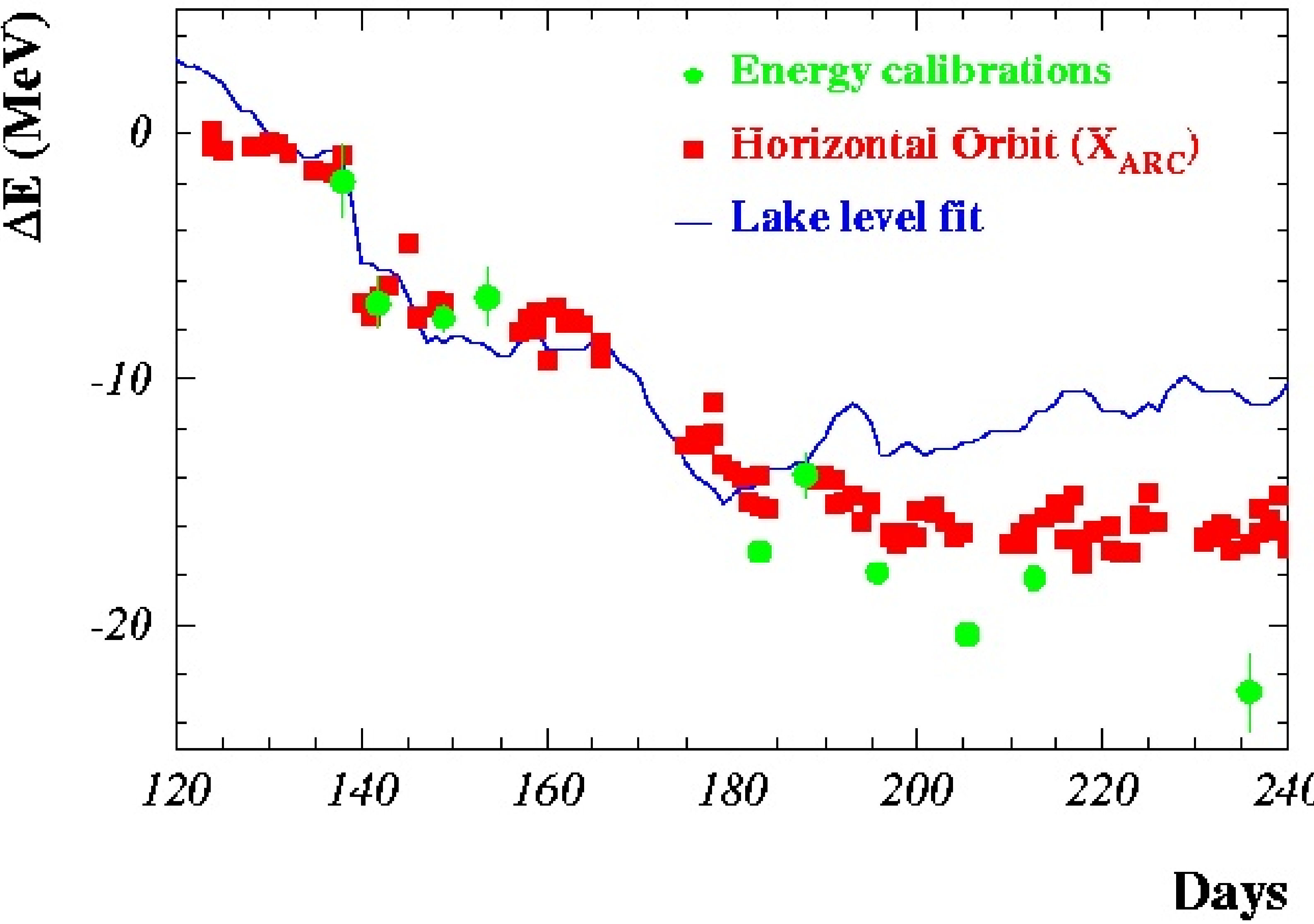



$\rightarrow$ $\Delta E \approx 10\ MeV$

energy modulation due to lake level changes

changes in the water level of lake Geneva change the position of the LEP tunnel and thus the quadrupole positions

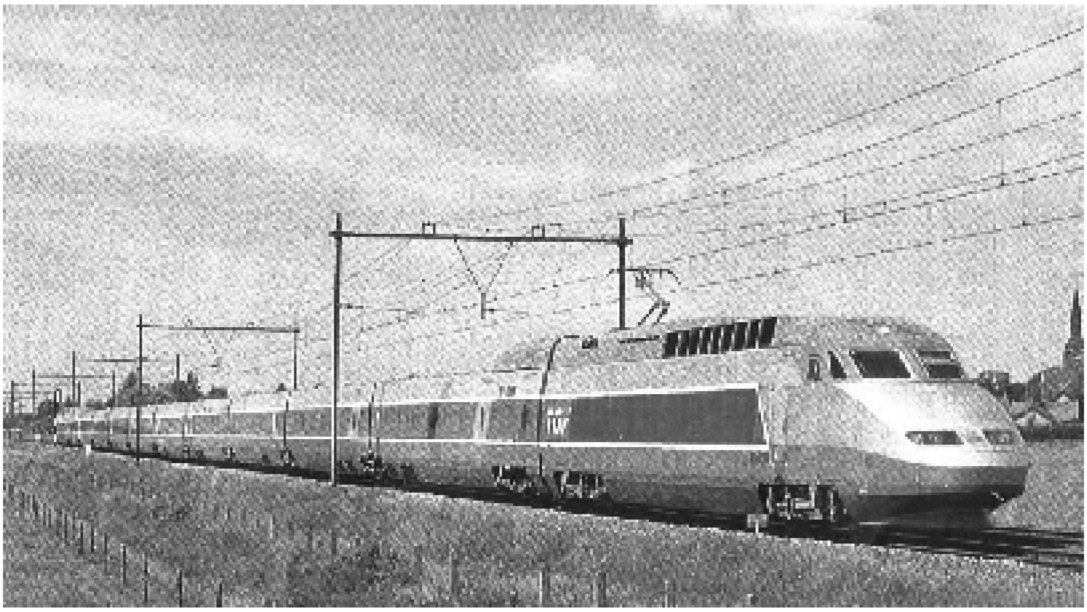
→ orbit and energy perturbations

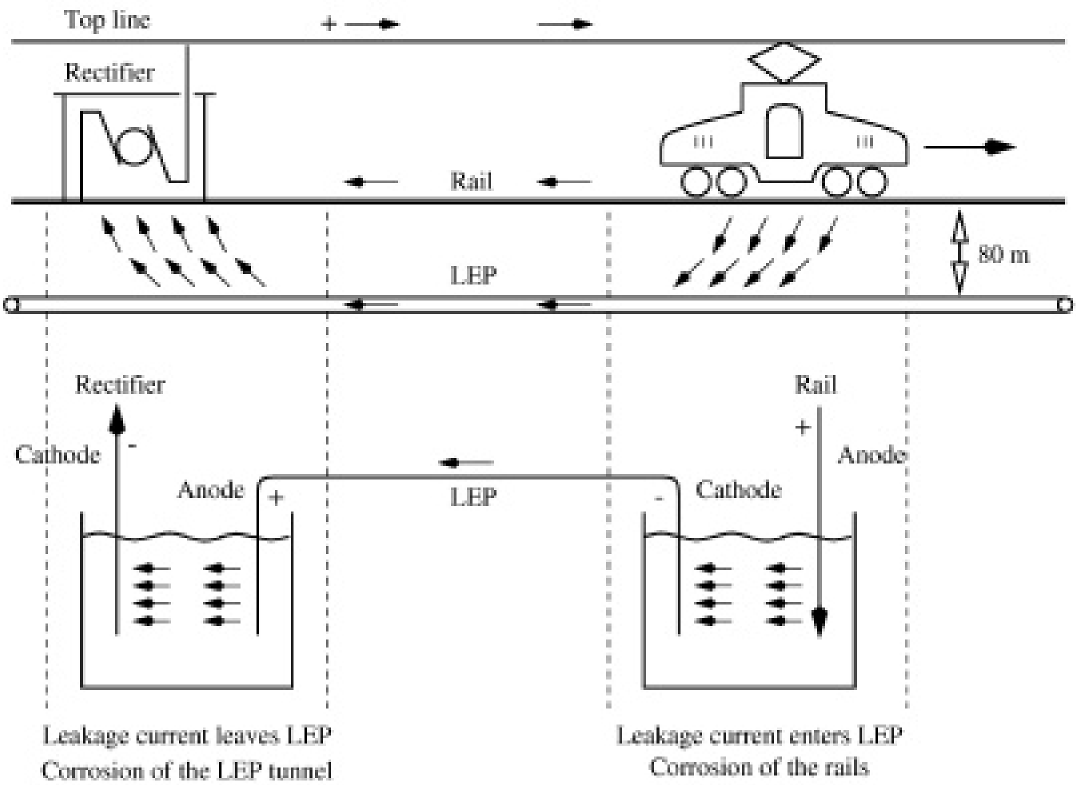



→ $\Delta E \approx 20\ MeV$

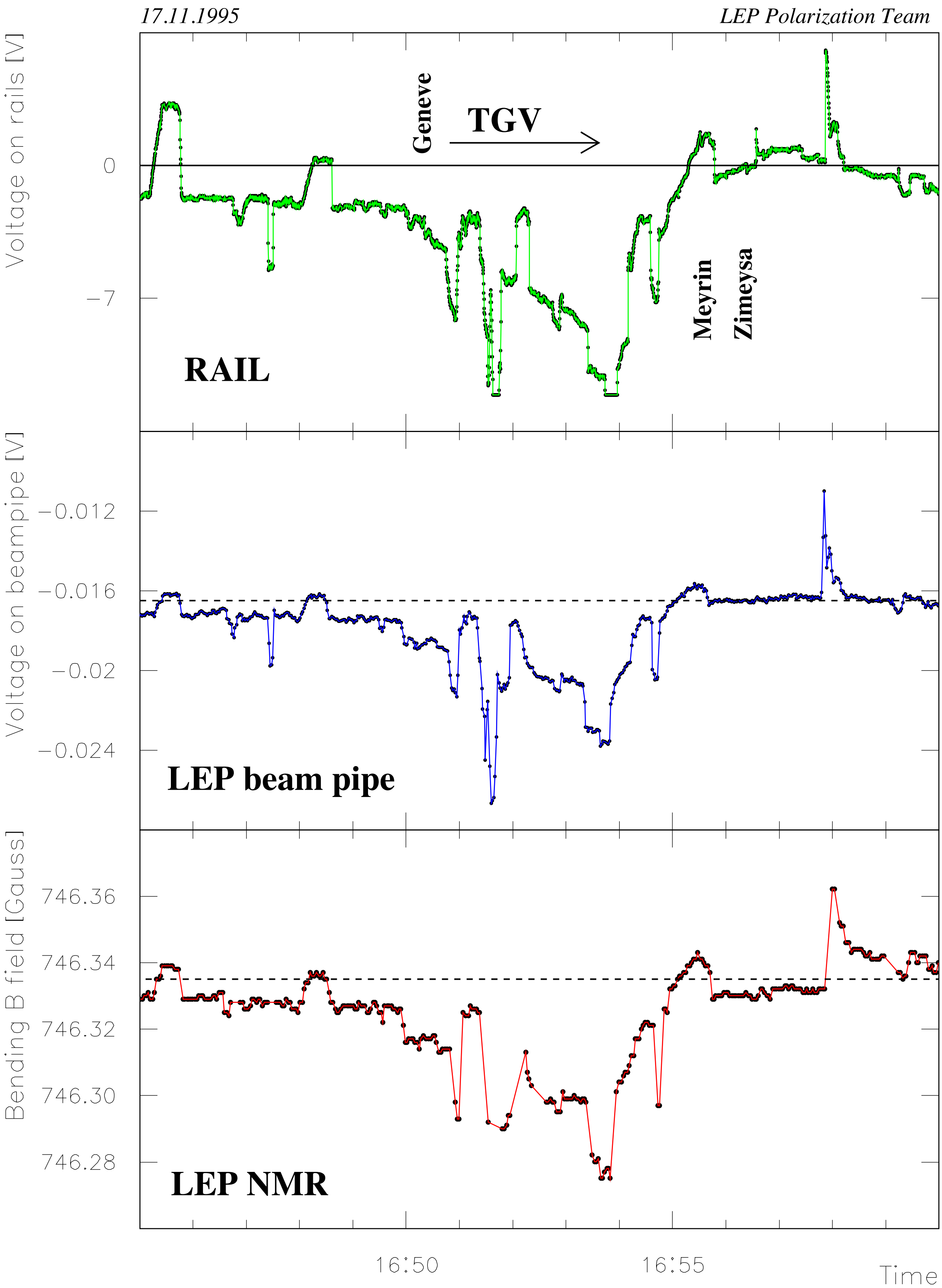
- energy modulation due current perturbations in the main dipole magnets

- TGV line between Geneva and Bellegarde





correlation of NMR dipole field measurements with the voltage on the TGC train tracks




ΔE ~ 5 MeV for LEP operation at 45 GeV

ground motion due to human activity

quadrupole motion in HERA–p (DESY Hamburg)

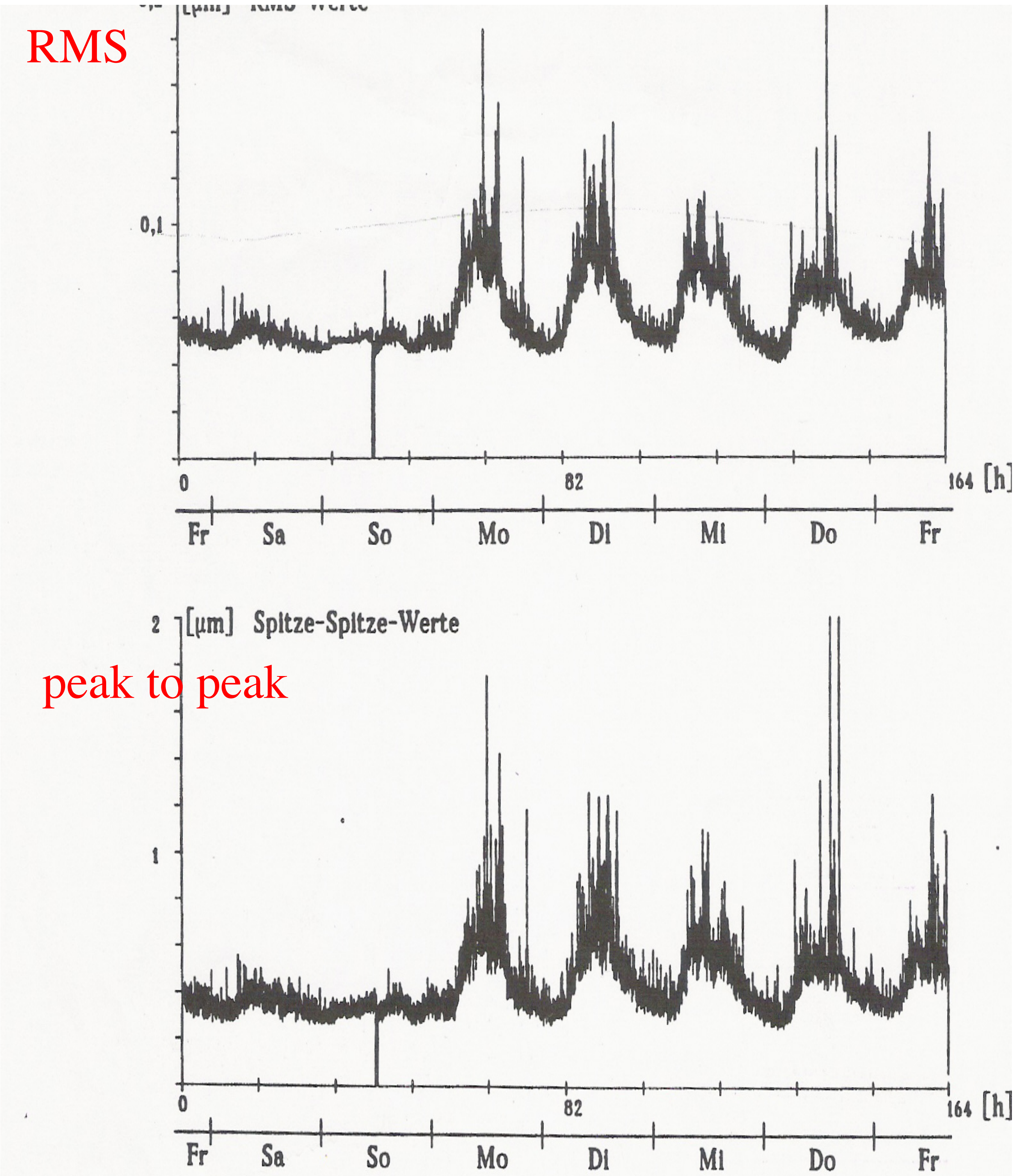



Abb. 3.13 Zeitabhängigkeit der Bodenbewegung
oben RMS-Werte
unten Spitze-Spitze-Werte

# Closed Orbit Response

- *inhomogeneous equation:*

$$\frac{d^2x}{ds^2} + K(s)\cdot x = G(s); \qquad G(s) = \Delta k_0(s)$$

$$\longrightarrow \quad \vec{y}^{\,\prime} + \begin{pmatrix} 0 & 1 \\ K & 0 \end{pmatrix} \cdot \vec{y} = \vec{G}; \quad \vec{G} = \begin{pmatrix} 0 \\ G \end{pmatrix}$$

$$\longrightarrow \quad \vec{y}(s) = a \cdot \vec{S}(s) + b \cdot \vec{C}(s) + \vec{\psi}(s)$$

→ *we need to find only one solution!*

- *variation of the constant:*

$$\vec{\psi}(s) = c(s) \cdot \vec{S}(s) + d(s) \cdot \vec{C}(s)$$

# *Closed Orbit Response*

- *variation of the constant in matrix form:*

$$\vec{\psi}(s) = \underline{\phi}(s) \cdot \vec{u}(s); \quad \text{with}$$

$$\underline{\phi}(s) = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix}$$

- *substitute into differential equation:*

$$\longrightarrow \quad \underline{\phi}(s) \cdot \vec{u}'(s) = \vec{G}(s)$$

$$\longrightarrow \quad \vec{u}(s) = \int_{s_0}^{s} \underline{\phi}(t)^{-1} \cdot \vec{G}(t)\, dt$$

$$\longrightarrow \quad \vec{y}(s) = a \cdot \vec{S}(s) + b \cdot \vec{C}(s) + \underline{\phi}(s) \cdot \int_{s_0}^{s} \underline{\phi}(t)^{-1} \cdot \vec{G}(t)\, dt$$

# *Closed Orbit Response*

***periodic boundary conditions:***

$$\vec{y}(s) = a \cdot \vec{S}(s) + b \cdot \vec{C}(s) + \underline{\phi}(s) \cdot \int_{s0}^{s} \underline{\phi}(t)^{-1} \cdot \vec{G}(t)\, dt$$

with

$$\vec{y}(s) = \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}; \quad x(s) = x(s+L); \quad x'(s) = x'(s+L)$$

$\longrightarrow$

***periodic boundary conditions determine coefficients*** a ***and*** b

$\longrightarrow$

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi \cdot Q)} \cdot \int_{s0}^{s0+circ} \sqrt{\beta(t)} \cdot G(t)\, \cos[\phi(t) - \phi(s) - \pi Q]\, dt$$

# Closed Orbit Response

**Example:** particle momentum error

normalized dipole strength: $$k_0(s) = 0.3 \cdot \frac{B[T]}{p[GeV]}$$

$$k_0(s) = \frac{1}{\rho(t)} - \frac{1}{\rho(t)} \cdot \frac{\Delta p}{p_0} \longrightarrow G(t) = \frac{1}{\rho(t)} \cdot \frac{\Delta p}{p_0}$$

$$x(s) = \frac{\sqrt{\beta(s)}}{2 \sin(\pi \cdot Q)} \cdot \oint \sqrt{\beta(t)} \cdot G(t) \cos\left[\left|\phi(t) - \phi(s)\right| - \pi Q\right] dt$$

$$\longrightarrow \quad x(s) = D(s) \cdot \frac{\Delta p}{p}$$

with

$$D(s) = \frac{\sqrt{\beta(s)}}{2 \sin(\pi \cdot Q)} \cdot \oint \frac{\sqrt{\beta(t)}}{\rho(t)} \cdot \cos\left[\left|\phi(t) - \phi(s)\right| - \pi Q\right] dt$$

→ ***Dispersion Orbit***

# *Orbit Correction*

- the orbit error in a storage ring with conventional magnets is dominated by the contributions from the quadrupole alignment errors

- orbit perturbation is proportional to the local $\beta$-functions at the location of the dipole error

→ alignment errors at QF cause mainly horizontal orbit errors

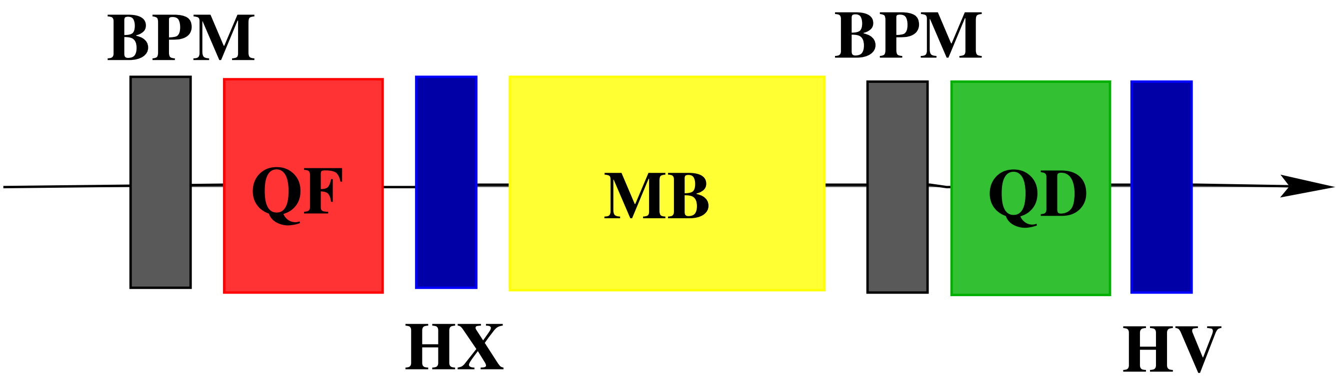
→ alignment errors at QD causes mainly vertical orbit errors

# *Orbit Correction*

- aim at a local correction of the dipole error due to the quadrupole alignment errors

→ place orbit corrector and BPM next to the main quadrupoles

→ horizontal BPM and corrector next to QF
vertical BPM and corrector next to QD




→ **orbit in the opposite plane?**

**relative alignment of BPM and quadrupole?**

# LEP Orbit

## Horizontal Orbit:

- beam offset in quadrupoles:
  - → Lake Geneva
  - → moon

→ energy error

## Vertical Orbit:

- beam offset in quadrupoles
- beam separation

→ orbit deflection depends on particle energy

→ vertical dispersion [D(s)]

$$\sigma_y = \sqrt{\varepsilon \cdot \beta_y + \delta_y^2 \cdot D^2}$$

→ small vertical beam size relies on good orbit

- 1994: 13000 vertical orbit corrections in physics

# *Quadrupole Gradient Error*

## *one turn map:*

can be generated by matrix multiplication:

→ $\vec{z}_{n+1} = \underline{M} \cdot \vec{z}_n \qquad \vec{z} = \begin{pmatrix} x \\ x' \end{pmatrix}$

and can be expressed in terms of the C and S solutions

$$\underline{M} = \underline{I} \cdot \cos(2\pi\ Q) + \underline{J} \cdot \sin(2\pi\ Q)$$

$$\underline{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \underline{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}; \ \gamma = [1 + \alpha^2\ ] / \beta$$

remember: $\cos(2\pi\ Q) = \frac{1}{2} \text{ trace } \underline{M}$

→ the coefficients of: $\dfrac{\underline{M} - \underline{I} \cdot \cos(2\pi\ Q)}{\sin(2\pi\ Q)}$

provide the optic functions at $s_0$

# *Quadrupole Gradient Error*

- *transfer matrix for single quadrupole:*

$$m_0 = \begin{pmatrix} 1 & 0 \\ -k_1 \cdot l & 1 \end{pmatrix}$$

- *matrix for single quadrupole with error:*

$$m = \begin{pmatrix} 1 & 0 \\ -[k_1 + \Delta k_1] \cdot l & 1 \end{pmatrix}$$

- *one turn matrix with quadrupole error:*

$$M = m \cdot m_0^{-1} \cdot M_0$$

trace M

→

$$\cos(2\pi Q) = \cos(2\pi Q_0) - \frac{1}{2} \beta \cdot \Delta k_1 l \cdot \sin(2\pi Q_0)$$

# *Quadrupole Gradient Error*

## *distributed perturbation:*

$$\cos(2\pi Q) = \cos(2\pi Q_0) - \frac{\sin(2\pi Q_0)}{2} \cdot \oint \beta \cdot \Delta k_1 ds$$

$$\longrightarrow \quad \boxed{\Delta Q = \frac{1}{4\pi} \cdot \oint \beta \cdot \Delta k_1 ds}$$

## *chromaticity:*

$$k_1 = \frac{e \cdot g}{p}$$

momentum error $\longrightarrow$ $\Delta k_1 = -k_1 \cdot \frac{\Delta p}{p}$

$$\Delta Q = -\frac{1}{4\pi} \cdot \oint \beta \cdot k_1 ds \cdot \frac{\Delta p}{p}$$

$$= \xi \cdot \frac{\Delta p}{p}$$

# $\beta$ – *Beat*

***quadrupole error:***

$$\rightarrow \quad \vec{z}_{n+1} = \underline{M} \cdot \vec{z}_n \qquad \underline{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

with

$$\underline{M} = \underline{I} \cdot \cos(2\pi\, Q) + \underline{J} \cdot \sin(2\pi\, Q)$$

$$\underline{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \underline{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}; \; \gamma = [1 + \alpha^2\,] / \beta$$

$\rightarrow$ calculate: $\dfrac{m_{12}}{\sin(2\pi Q)}$

$$\Delta\beta(s) = \frac{\beta(s)}{2\sin(2\pi \cdot Q)} \cdot \int_{s0}^{s0+circ} \beta(t) \cdot \Delta k(t) \; \cos[2[\phi(t) - \phi(s)] - 2\pi\, Q]\, dt$$

$\rightarrow$

$\beta$ – beat oscillates with twice the betatron frequency

# *Local Orbit Bumps I*

## *deflection angle:*

$$\theta_i = \int_{\text{dipole}} G_i(t)\, dt = \frac{0.3 \cdot B_i[T] \cdot l}{p[GeV]}$$

## *trajectory response:*

[no periodic boundary conditions]

$$\longrightarrow \quad x(s) = \sqrt{\beta_i\, \beta(s)} \cdot \theta_i \cdot \sin[\phi(s) - \phi_i]$$

$$\longrightarrow \quad x'(s) = \sqrt{\beta_i / \beta(s)} \cdot \theta_i \cdot \cos[\phi(s) - \phi_i]$$

# *Local Orbit Bumps II*

## *closed orbit bump:*

compensate the trajectory perturbation with

additional corrector kicks further down stream

→ closure of the perturbation within one turn

→ local orbit excursion

→ possibility to correct orbit errors locally

→ closure with one additional corrector magnet

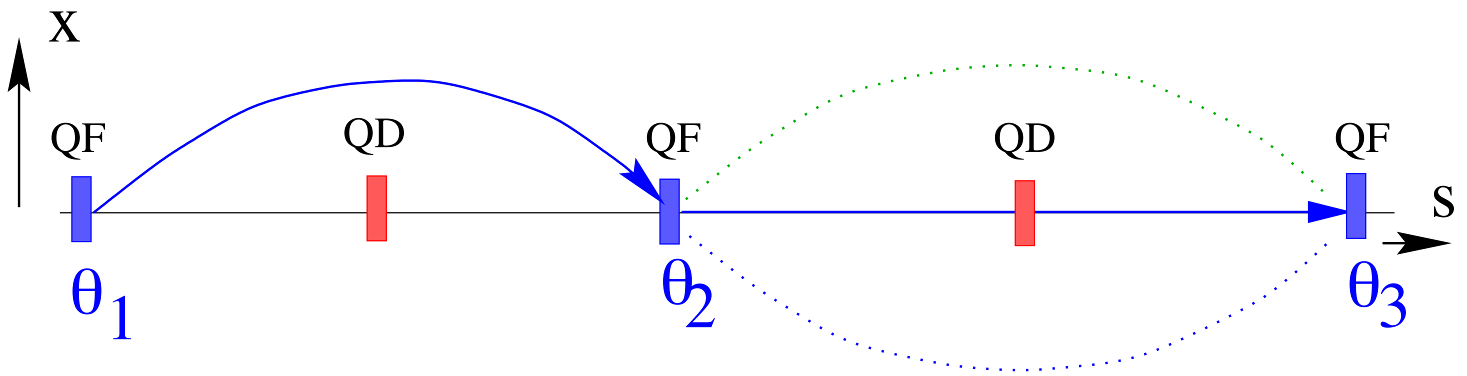
→ $\pi$ - bump

→ closure with two additional corrector magnets

→ three corrector bump

# *Local Orbit Bumps III*

**π - bump:** (quasi local correction of error)



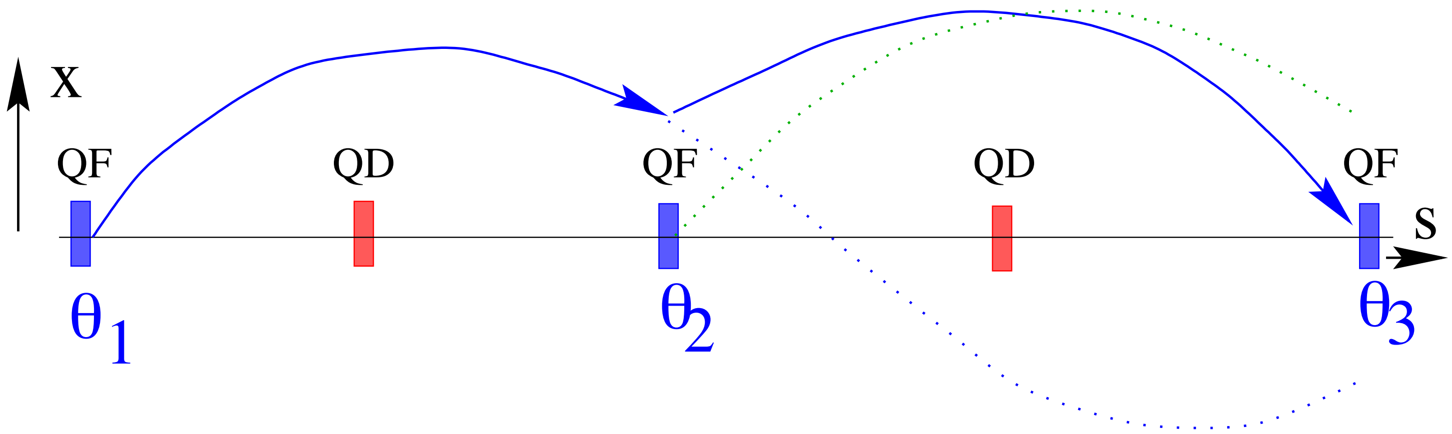

$$\theta_2 = \frac{\sqrt{\beta_1}}{\sqrt{\beta_2}} \cdot \theta_1$$

**limits / problems:**

- closure depends on lattice phase advance
  - requires 90° lattice
  - sensitive to lattice errors
- requires horizontal BPMs at QF and QD
- sensitive to BPM errors
- requires large number of correctors

# *Local Orbit Bumps IV*

- 3 corrector bump: (quasi local correction of error)




$$\theta_2 = -\frac{\sqrt{\beta_1}}{\sqrt{\beta_2}} \cdot \frac{\sin(\Delta\phi_{3\text{-}1})}{\sin(\Delta\phi_{3\text{-}2})} \cdot \theta_1$$

$$\theta_3 = \left( \frac{\sin(\Delta\phi_{3\text{-}1})}{\tan(\Delta\phi_{3\text{-}2})} - \cos(\Delta\phi_{3\text{-}1}) \right) \cdot \frac{\sqrt{\beta_1}}{\sqrt{\beta_3}} \cdot \theta_1$$

→ works for any lattice phase advance

→ requires only horizontal BPMs at QF

- limits / problems:

→ sensitive to BPM errors

→ large number of correctors

→ can not control $x'$

# *Summary Linear Imperfections*

- avoid machine tunes near integer resonances:
  - → they amplify the response to dipole field errors
  - → a closed orbit perturbation propagates with the betatron phase around the storage ring
  - → discontinuities in the derivative of the closed orbit response at the location of the perturbation
- avoid storage ring tunes near half–integer resonances:
  - → they amplify the response to quadrupole field errors
  - → betafunction perturbations propagate with twice the betatron phase advance around the storage ring
- integral expressions are mainly used for estimates numerical programs mainly rely on maps
  - → closed orbit = fixed point of ´1–turn´ map
  - → dispersion = eigenvector of extended ´1–turn´ map
  - → tune is given by the trace of the ´1–turn´ map
  - → twiss functions are given by the matrix elements