Beam Transfer Lines

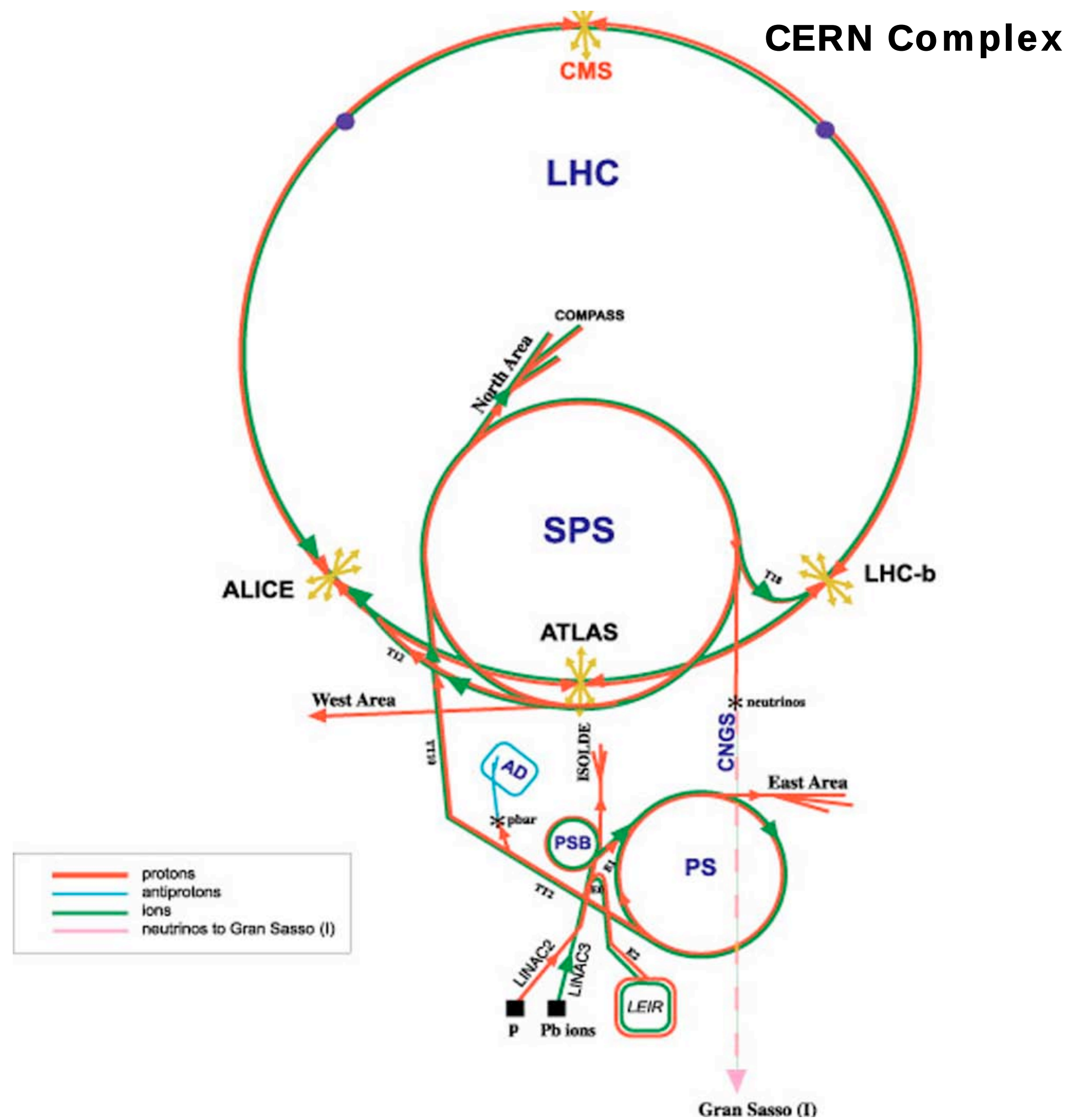
- Distinctions between transfer lines and circular machines
- Linking machines together
- Trajectory correction
- Emittance and mismatch measurement
- Blow-up from steering errors, optics mismatch and thin screens
- Phase-plane exchange

Brennan Goddard

CERN

Injection, extraction and transfer

- An accelerator has limited dynamic range.
- Chain of stages needed to reach high energy
- Periodic re-filling of storage rings, like LHC
- External experiments, like CNGS



Transfer lines transport the beam between accelerators, and onto targets, dumps, instruments etc.

LHC:	Large Hadron Collider
SPS:	Super Proton Synchrotron
AD:	Antiproton Decelerator
ISOLDE:	Isotope Separator Online Device
PSB:	Proton Synchrotron Booster
PS:	Proton Synchrotron
LINAC:	LINear Accelerator
LEIR:	Low Energy Ring
CNGS:	CERN Neutrino to Gran Sasso

Normalised phase space

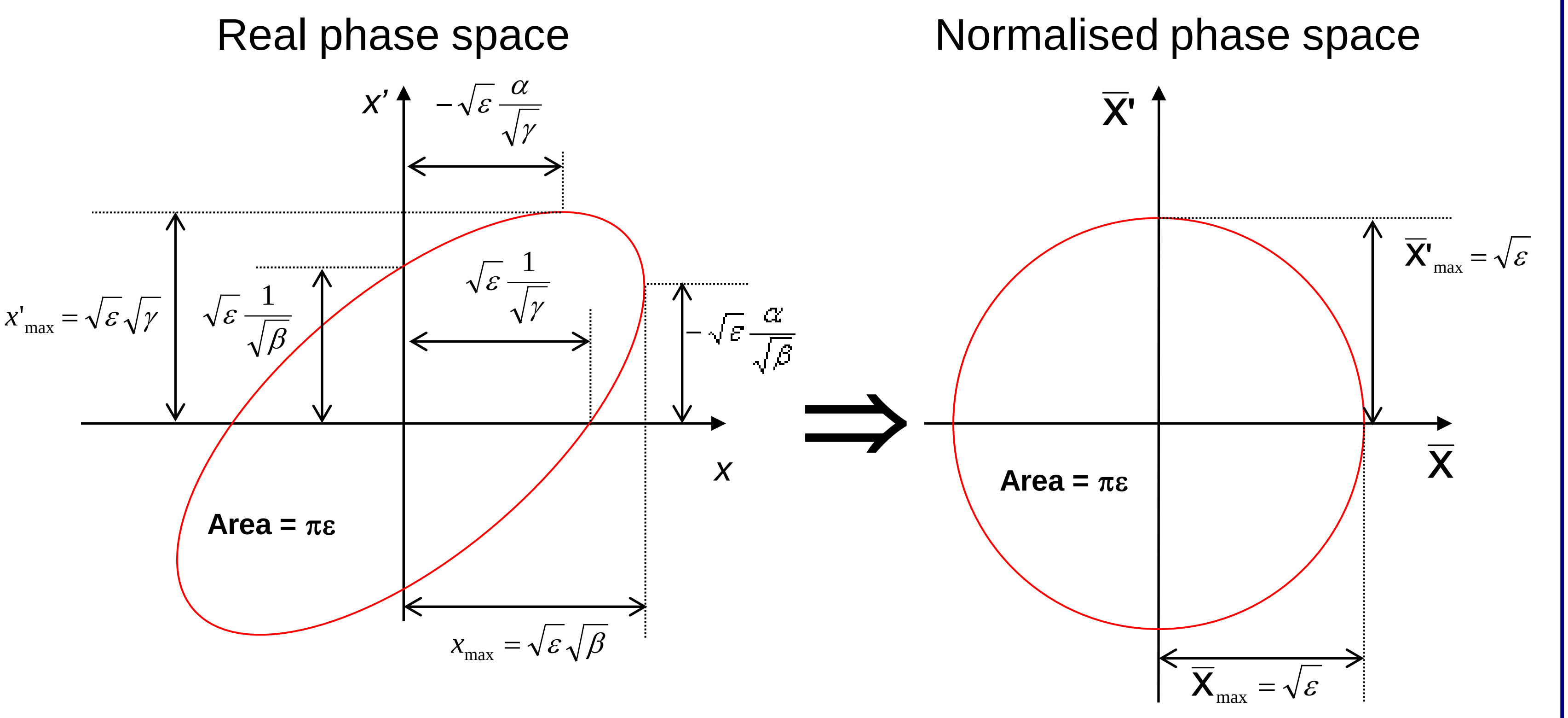
- Transform real transverse coordinates *x*, *x'* by

$$\begin{bmatrix} \overline{\mathsf{X}} \\ \overline{\mathsf{X}}' \end{bmatrix} = \mathbf{N} \cdot \begin{bmatrix} x \\ x' \end{bmatrix} = \sqrt{\frac{1}{\beta_S}} \cdot \begin{bmatrix} 1 & 0 \\ \alpha_S & \beta_S \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \end{bmatrix}$$

$$\overline{\mathsf{X}} = \sqrt{\frac{1}{\beta_S}} \cdot x$$

$$\overline{\mathsf{X}}' = \sqrt{\frac{1}{\beta_S}} \cdot \alpha_S x + \sqrt{\beta_S}\, x'$$

Normalised phase space

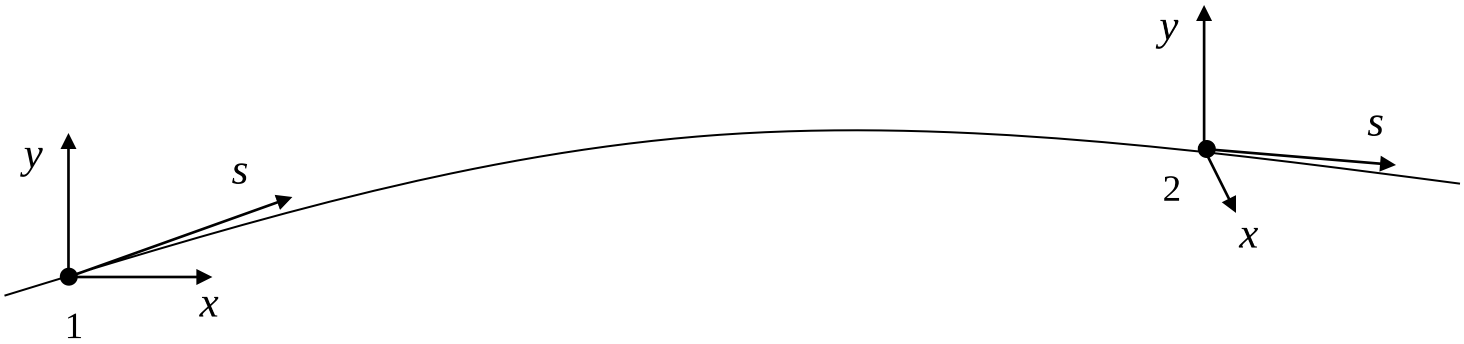


$$\varepsilon = \gamma \cdot x^2 + 2\alpha \cdot x \cdot x' + \beta \cdot x'^2$$

$$\varepsilon = \overline{\mathbf{X}}^2 + \overline{\mathbf{X}}'^2$$

General transport

Beam transport: moving from s_1 to s_2 through *n* elements, each with transfer matrix M_i

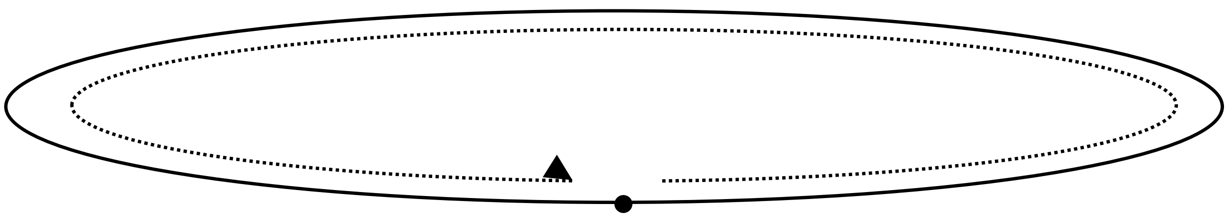


$$\begin{bmatrix} x_2 \\ x_2' \end{bmatrix} = \mathbf{M}_{1\to 2} \cdot \begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} C & S \\ C' & S' \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \end{bmatrix} \qquad \mathbf{M}_{1\to 2} = \prod_{i=1}^{n} \mathbf{M}_n$$

Twiss parameterisation

$$\mathbf{M}_{1\to 2} = \begin{bmatrix} \sqrt{\beta_2 / \beta_1}\left(\cos\Delta\mu + \alpha_1 \sin\Delta\mu\right) & \sqrt{\beta_1\beta_2}\,\sin\Delta\mu \\ \sqrt{1/\beta_1\beta_2}\left[\left(\alpha_1-\alpha_2\right)\cos\Delta\mu - \left(1+\alpha_1\alpha_2\right)\sin\Delta\mu\right] & \sqrt{\beta_1/\beta_2}\left(\cos\Delta\mu - \alpha_2\sin\Delta\mu\right) \end{bmatrix}$$

Circular Machine



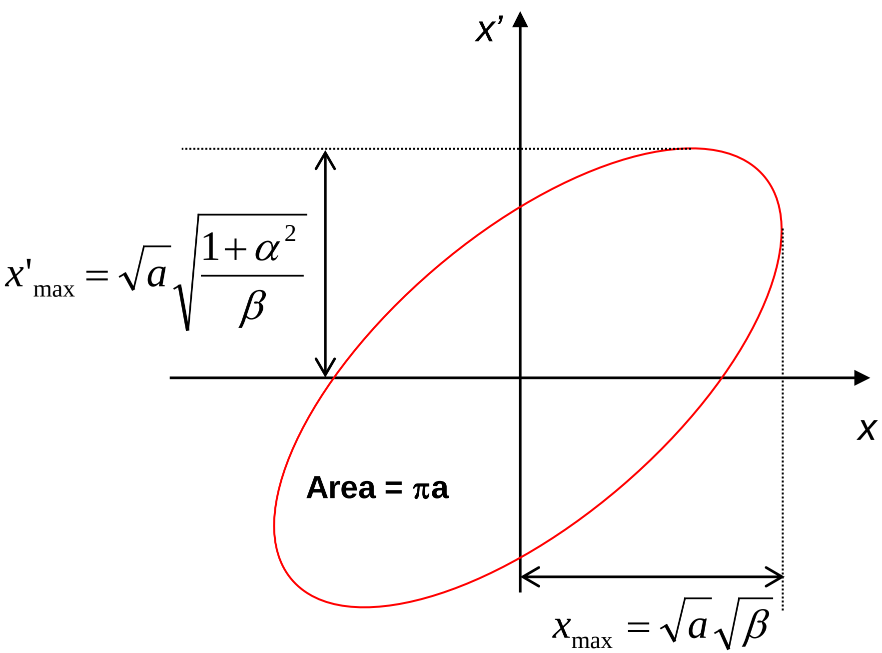
One turn

$$\mathbf{M}_{1\to 2} = \mathbf{M}_{0\to L} = \begin{bmatrix} \cos 2\pi Q + \alpha \sin 2\pi Q & \beta \sin 2\pi Q \\ -\frac{1}{\beta}\left(1+\alpha^2\right)\sin 2\pi Q & \cos 2\pi Q - \alpha \sin 2\pi Q \end{bmatrix}$$

- The solution is *periodic*
- Periodicity condition for one turn (closed ring) imposes $\alpha_1=\alpha_2$, $\beta_1=\beta_2$, $D_1=D_2$
- This condition *uniquely* determines $\alpha(s)$, $\beta(s)$, $\mu(s)$, $D(s)$ around the whole ring

Circular Machine

- Periodicity of the structure leads to regular motion
  - Map single particle coordinates on each turn at any location
  - Describes an ellipse in phase space, defined by one set of α and β values $\Rightarrow$ Matched Ellipse (for this location)

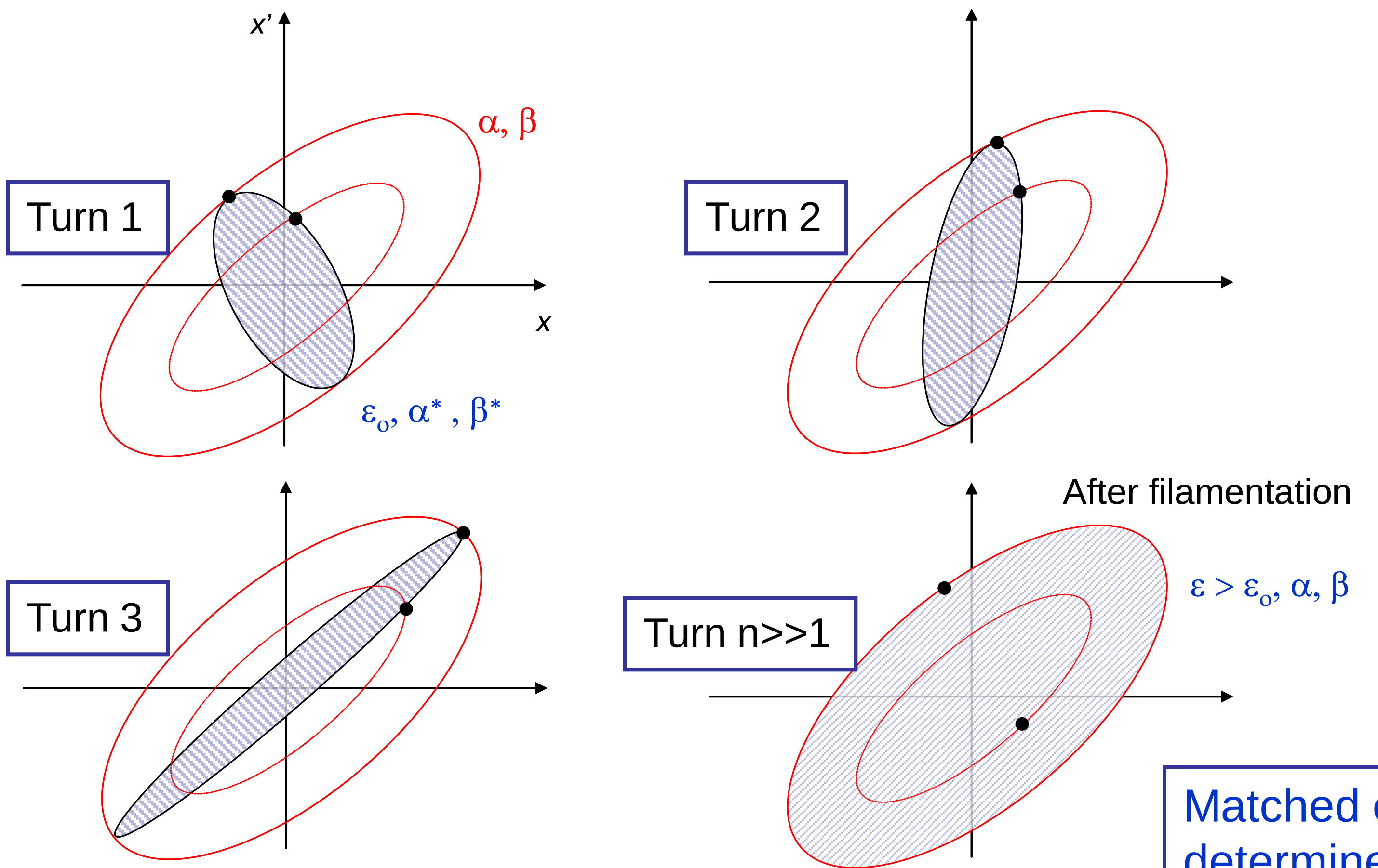


$$a = \gamma \cdot x^2 + 2\alpha \cdot x \cdot x' + \beta \cdot x'^2$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

Circular Machine

- For a location with matched ellipse (α, β), an injected beam of emittance ε, characterised by a different ellipse (α^*, β^*) generates (via filamentation) a large ellipse <u>with the original α, β, but larger ε</u>



Matched ellipse determines beam shape

Transfer line

One pass: $$\begin{bmatrix} x_2 \\ x_2' \end{bmatrix} = \mathbf{M}_{1\to 2} \cdot \begin{bmatrix} x \\ x' \end{bmatrix}$$

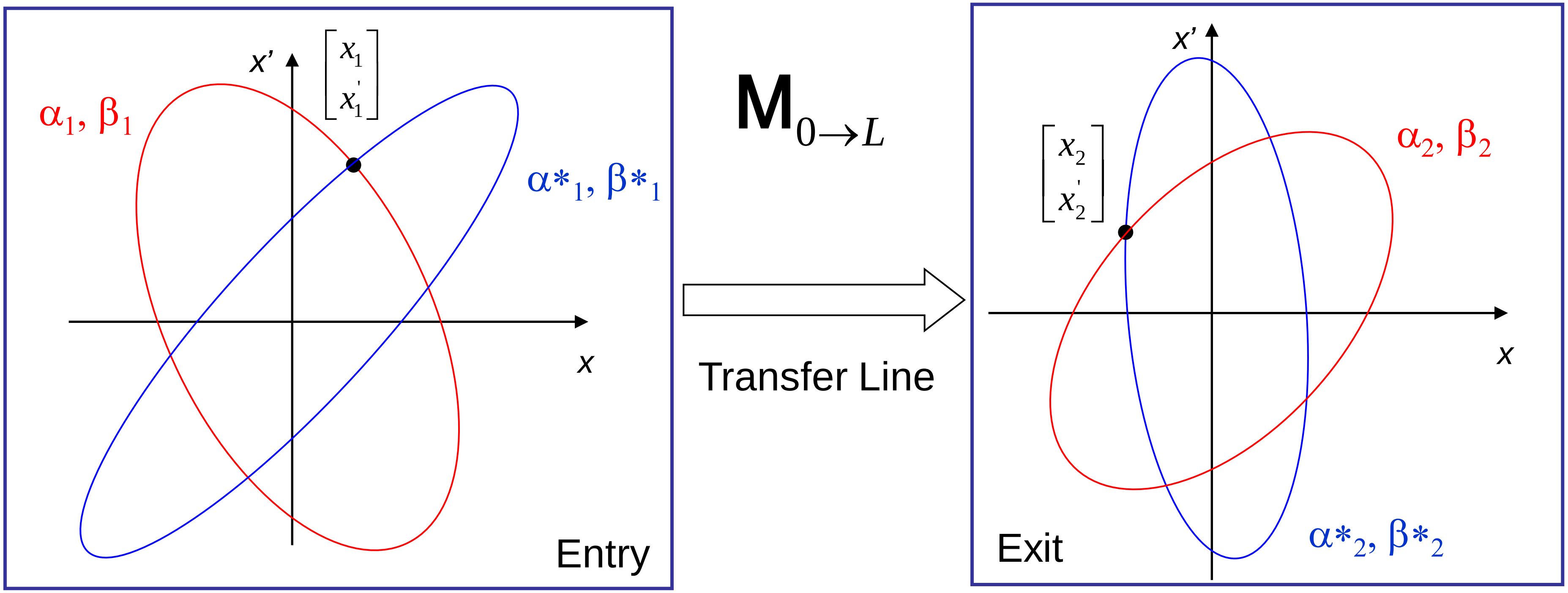
$$\begin{bmatrix} x_1 \\ x_1' \end{bmatrix} \longrightarrow \begin{bmatrix} x_2 \\ x_2' \end{bmatrix}$$

$$\mathbf{M}_{1\to 2} = \begin{bmatrix} \sqrt{\beta_2/\beta_1}\left(\cos\Delta\mu + \alpha_1 \sin\Delta\mu\right) & \sqrt{\beta_1\beta_2}\,\sin\Delta\mu \\ \sqrt{1/\beta_1\beta_2}\left[\left(\alpha_1 - \alpha_2\right)\cos\Delta\mu - \left(1 + \alpha_1\alpha_2\right)\sin\Delta\mu\right] & \sqrt{\beta_1/\beta_2}\left(\cos\Delta\mu - \alpha_2 \sin\Delta\mu\right) \end{bmatrix}$$

- No periodic condition exists
- The Twiss parameters are simply propagated from beginning to end of line
- At any point in line, $\alpha(s)$ $\beta(s)$ are functions of α_1 β_1

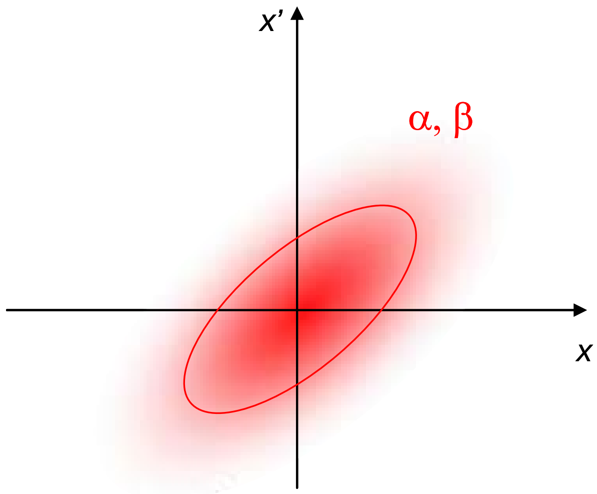
Transfer line

- On a single pass there is no regular motion
  - Map single particle coordinates at entrance and exit.
  - Infinite number of equally valid possible starting ellipses for single particle ……transported to infinite number of final ellipses…

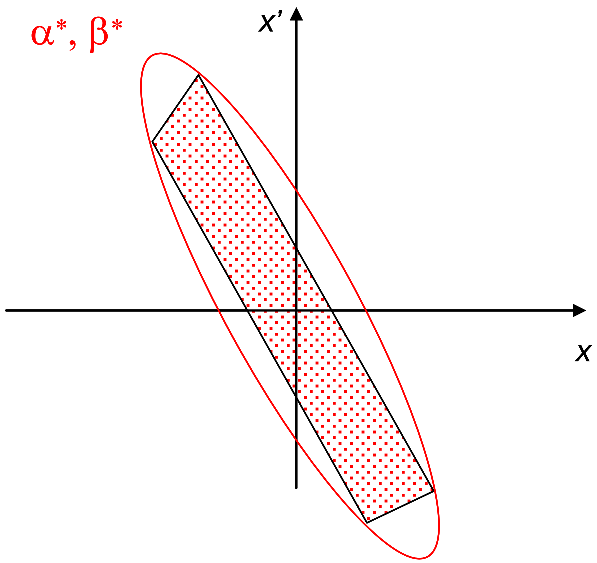


Transfer Line

- Initial α, β defined for transfer line by beam shape at entrance



Gaussian beam

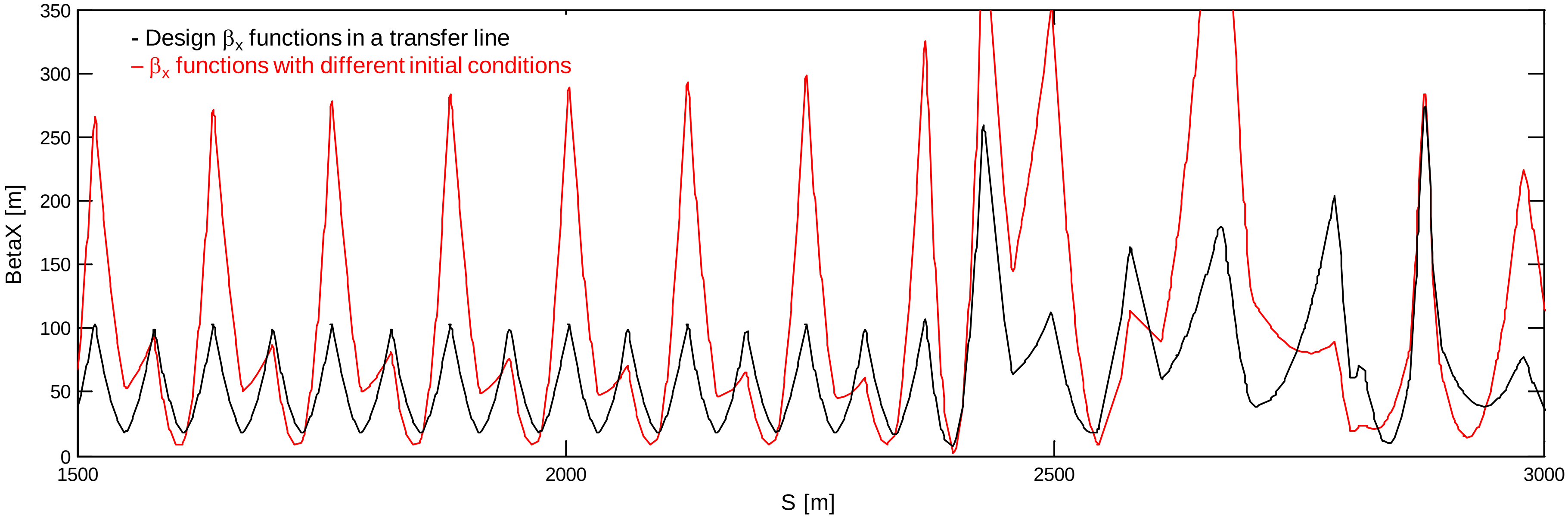


Non-Gaussian beam
(e.g. slow extracted)

- Propagation of this beam ellipse depends on line elements
- A transfer line optics is different for different input beams

Transfer Line

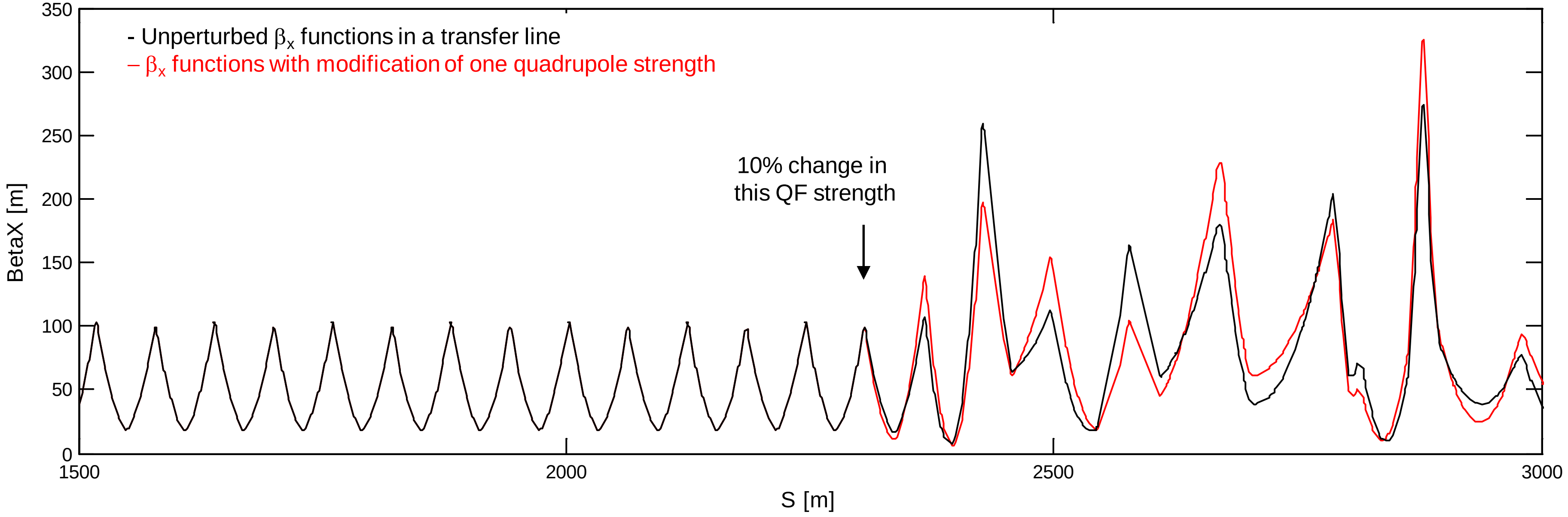
- The optics functions in the line depend on the initial values



- Same considerations are true for Dispersion function:
  - Dispersion in ring defined by periodic solution $\rightarrow$ ring elements
  - Dispersion in line defined by initial D and D' and line elements

Transfer Line

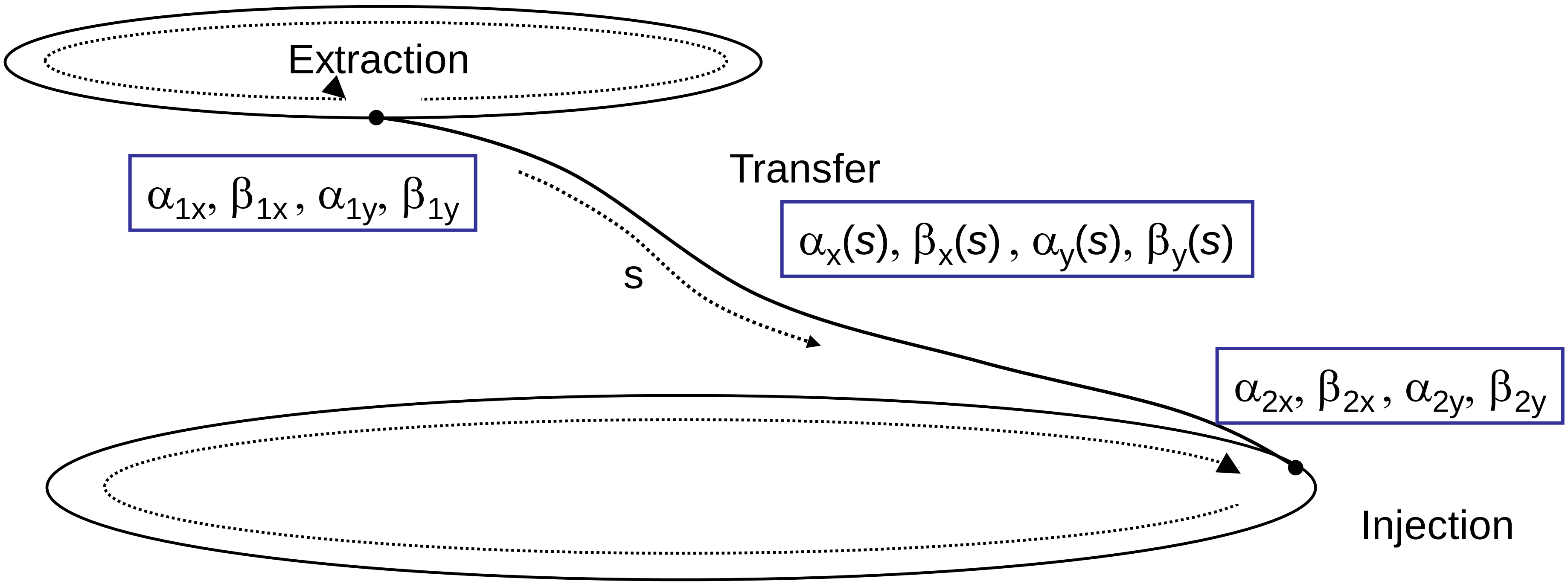
- Another difference….unlike a circular ring, a change of an element in a line affects *only* the downstream Twiss values (including dispersion)



Linking Machines

- Beams have to be transported from extraction of one machine to injection of next machine
  - Trajectories must be matched, ideally in all 6 geometric degrees of freedom (x,y,z,θ,ϕ,ψ)
- Other important constraints can include
  - Minimum bend radius, maximum quadrupole gradient, magnet aperture, cost, geology

Linking Machines



The Twiss parameters can be propagated when the transfer matrix **M** is known

$$\begin{bmatrix} x_2 \\ x_2' \end{bmatrix} = \mathbf{M}_{1\to 2} \cdot \begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} C & S \\ C' & S' \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \end{bmatrix}$$

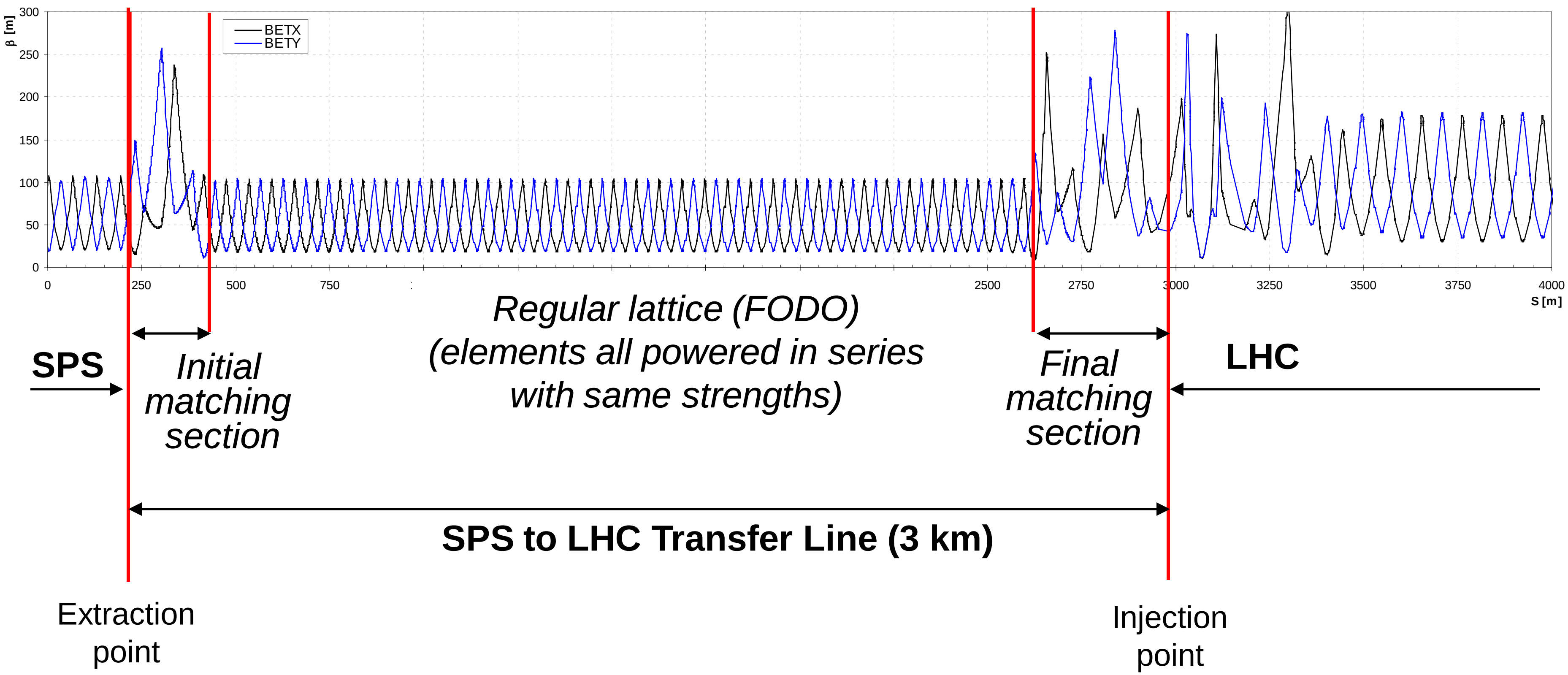
$$\begin{bmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} C^2 & -2CS & S^2 \\ -CC' & CS'+SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{bmatrix} \cdot \begin{bmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{bmatrix}$$

Linking Machines

- Linking the optics is a complicated process
  - Parameters at start of line have to be propagated to matched parameters at the end of the line
  - Need to "match" 8 variables (α_x β_x D_x D'_x and α_y β_y D_y D'_y)
  - Maximum β and D values are imposed by magnet apertures
  - Other constraints can exist
    - phase conditions for collimators,
    - insertions for special equipment like stripping foils
  - Need to use a number of independently powered ("matching") quadrupoles
  - Matching with computer codes and relying on mixture of theory, experience, intuition, trial and error, …

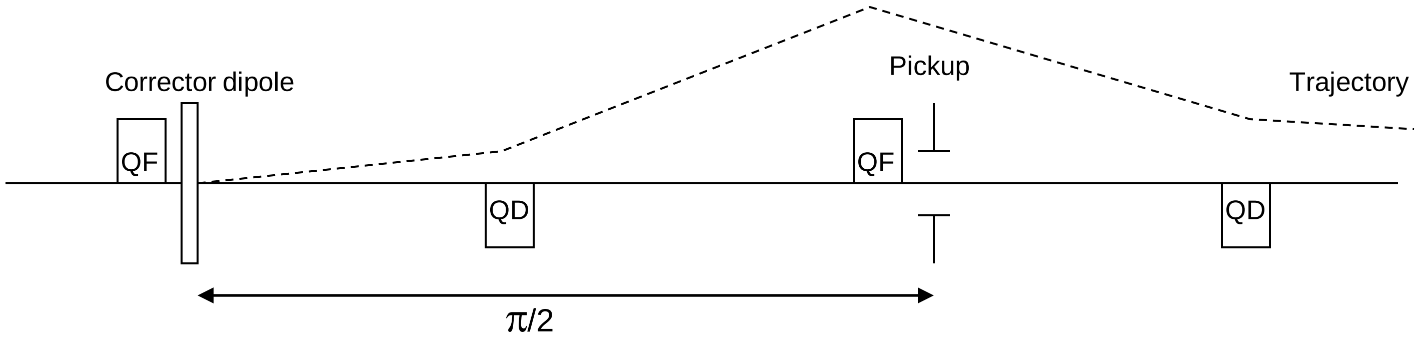
Linking Machines

- For long transfer lines we can simplify the problem by designing the line in separate sections
  - Regular central section – e.g. FODO or doublet, with quads at regular spacing, (+ bending dipoles), with magnets powered in series
  - Initial and final matching sections – independently powered quadrupoles, with sometimes irregular spacing.



Trajectory correction

- Magnet misalignments, field and powering errors cause the trajectory to deviate from the design
- Use small independently powered dipole magnets (correctors) to steer the beam
- Measure the response using monitors (pick-ups) downstream of the corrector ($\pi/2$, $3\pi/2$, …)



- Horizontal and vertical elements are separated
- H-correctors and pick-ups located at F-quadrupoles (large β_x)
- V-correctors and pick-ups located at D-quadrupoles (large β_y)

Trajectory correction

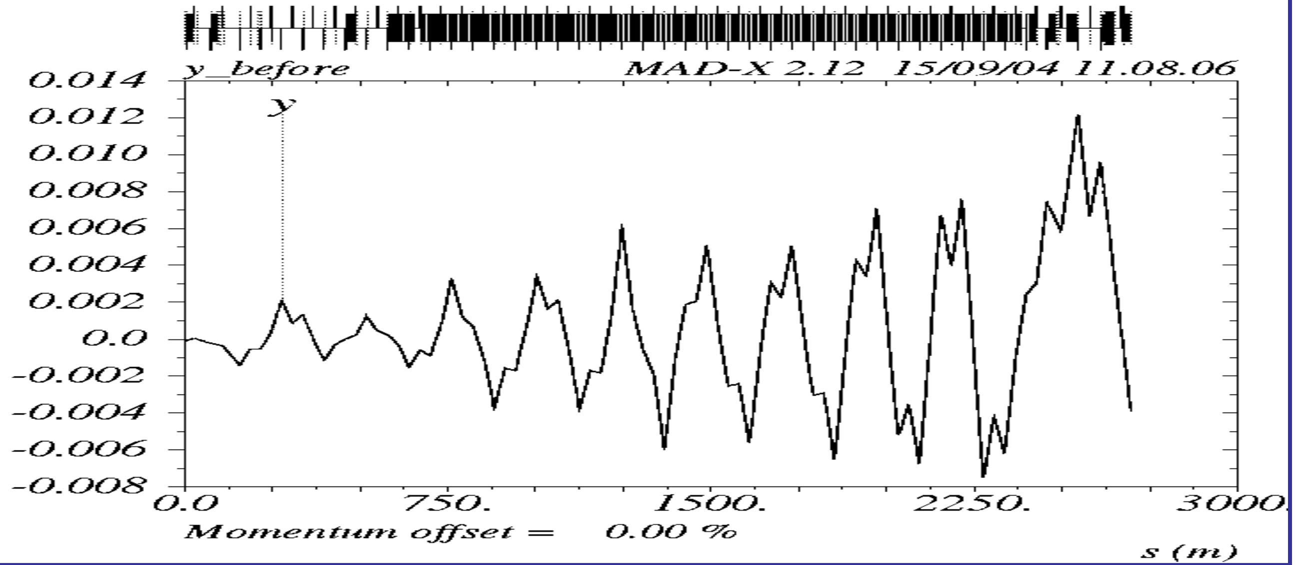
- Global correction can be used which attempts to minimise the RMS offsets at the BPMs, using all or some of the available corrector magnets.
- Steering in matching sections, extraction and injection region requires particular care
  - D and β functions can be large $\rightarrow$ bigger beam size
  - Often very limited in aperture
  - Injection offsets can be detrimental for performance

Trajectory correction

Uncorrected trajectory.

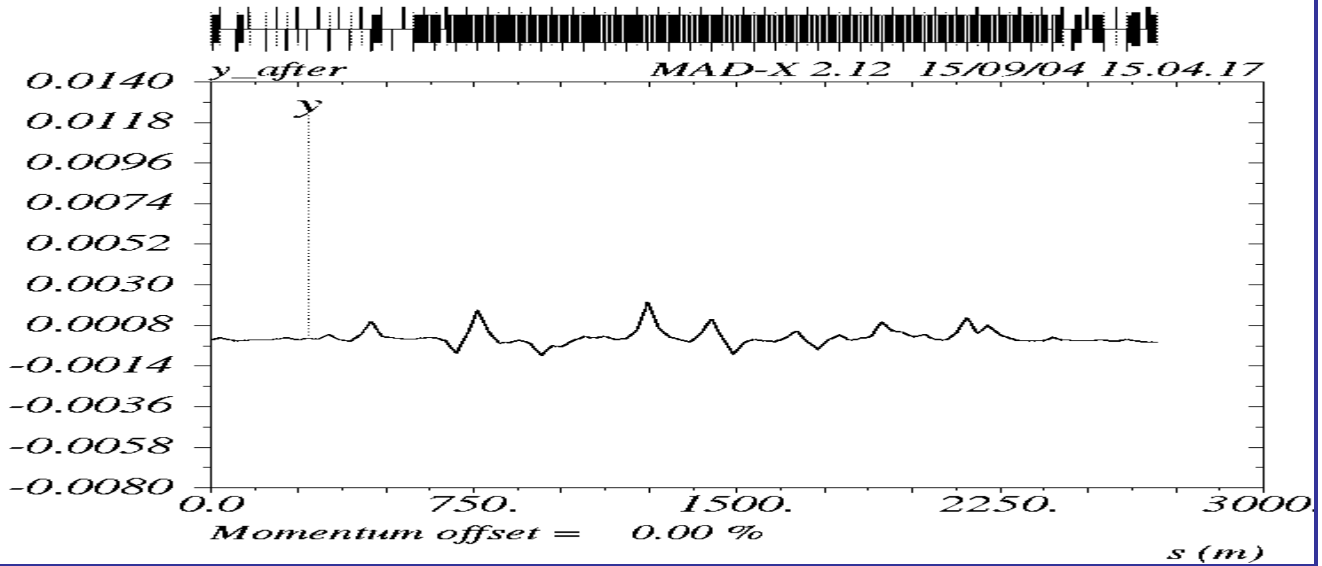
y growing as a result of random errors in the line.

The RMS at the BPMs is 3.4 mm, and y_{max} is 12.0mm



Corrected trajectory.

The RMS at the BPMs is 0.3mm and y_{max} is 1mm

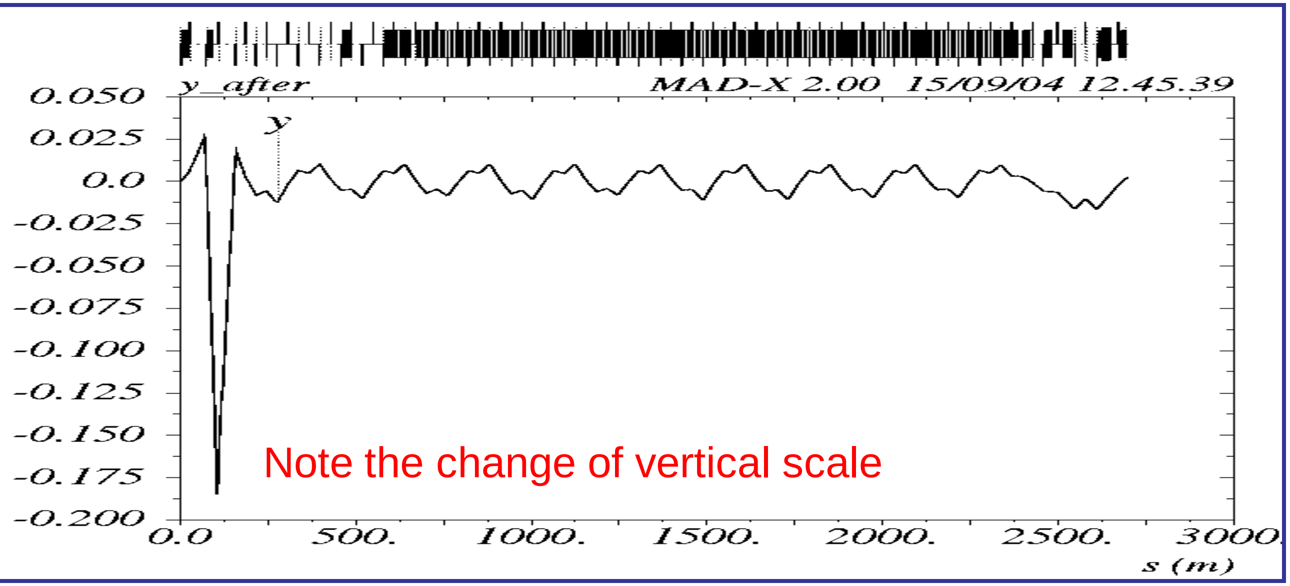


Trajectory correction

- Sensitivity to BPM errors is an important issue
  - If the BPM phase sampling is poor, the loss of a few key BPMs can allow a very bad trajectory, while all the monitor readings are ~zero

Correction with some monitors disabled

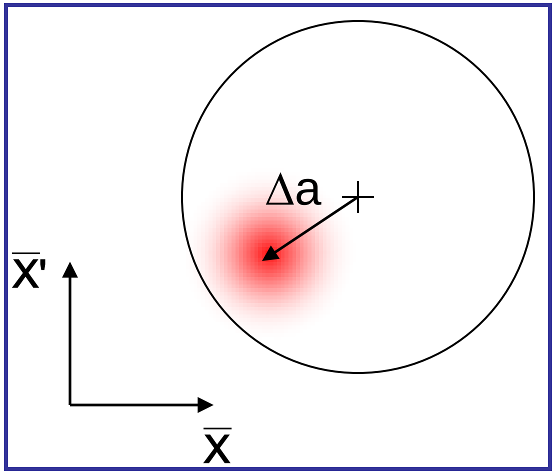
With poor BPM phase sampling the correction algorithm produces a trajectory with 185mm y_{max}

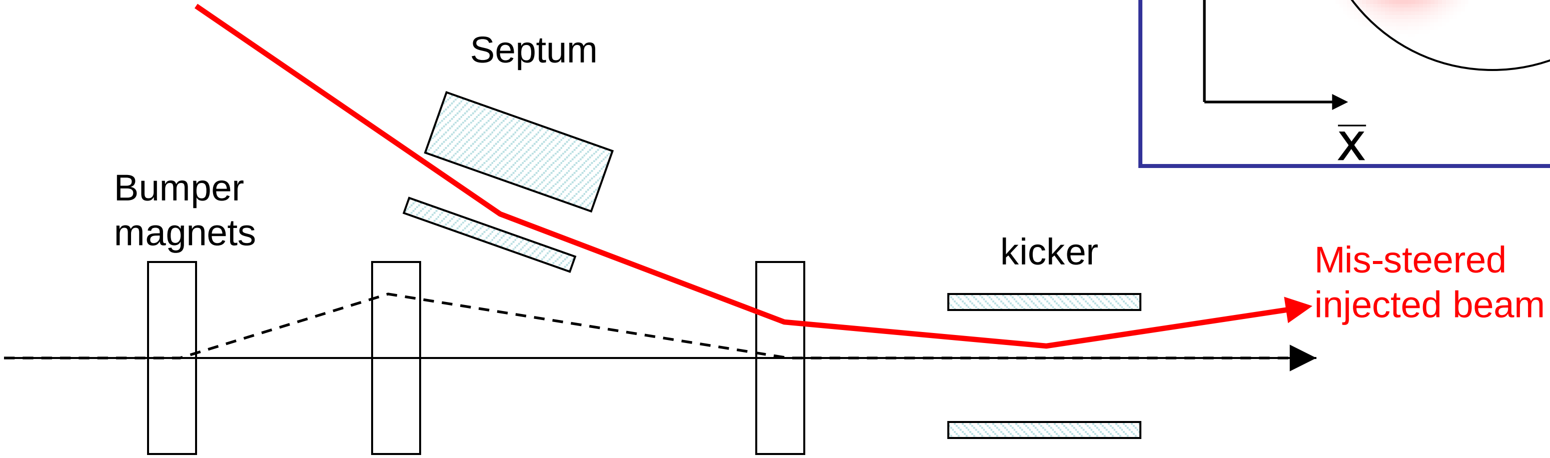


Steering (dipole) errors

- Precise delivery of the beam is important.
  - To avoid injection oscillations and emittance growth in rings
  - For stability on secondary particle production targets
- Convenient to express injection error in σ (includes x and x' errors)

$$\Delta a\ [\sigma] = \sqrt{(\mathbf{X}^2+\mathbf{X}'^2)/\varepsilon} = \sqrt{(\gamma x^2 + 2\alpha x x' + \beta x'^2)/\varepsilon}$$



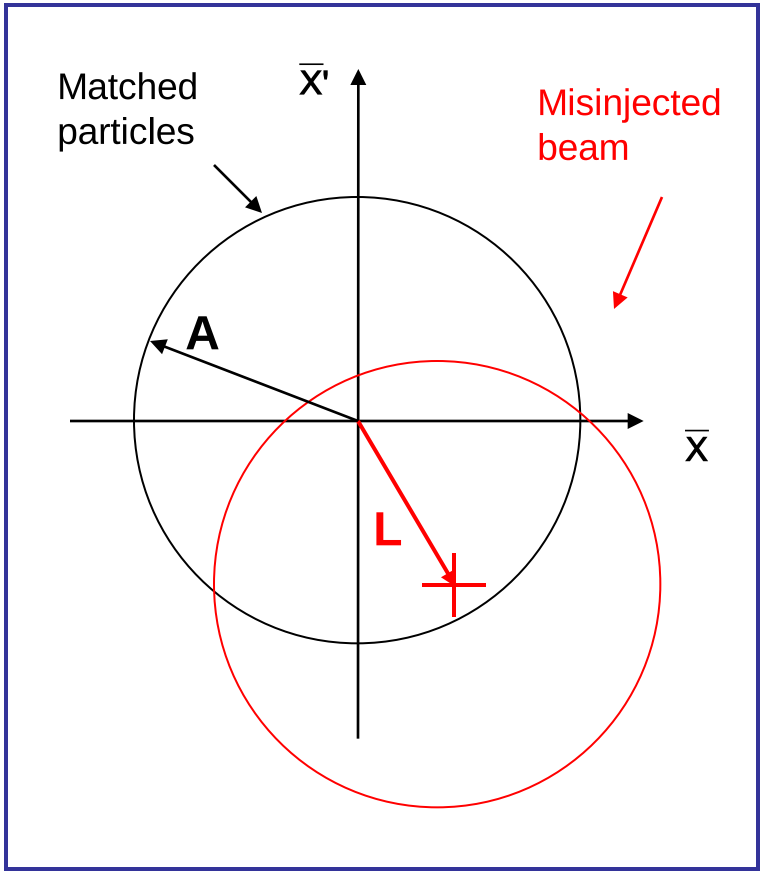


Steering (dipole) errors

- Static effects (e.g. from errors in alignment, field, calibration, …) are dealt with by trajectory correction (steering).
- But there are also dynamic effects, from:
  - Power supply ripples
  - Temperature variations
  - Non-trapezoidal kicker waveforms
- These dynamic effects produce a variable injection offset which can vary from batch to batch, or even within a batch.
- An injection damper system is used to minimise effect on emittance

Blow-up from steering error

- Consider a collection of particles with amplitudes A
- The beam can be injected with a error in angle and position.
- For an injection error Δa_y (in units of sigma = $\sqrt{\beta\varepsilon}$) the mis-injected beam is offset in normalised phase space by $L = \Delta a_y \sqrt{\varepsilon}$

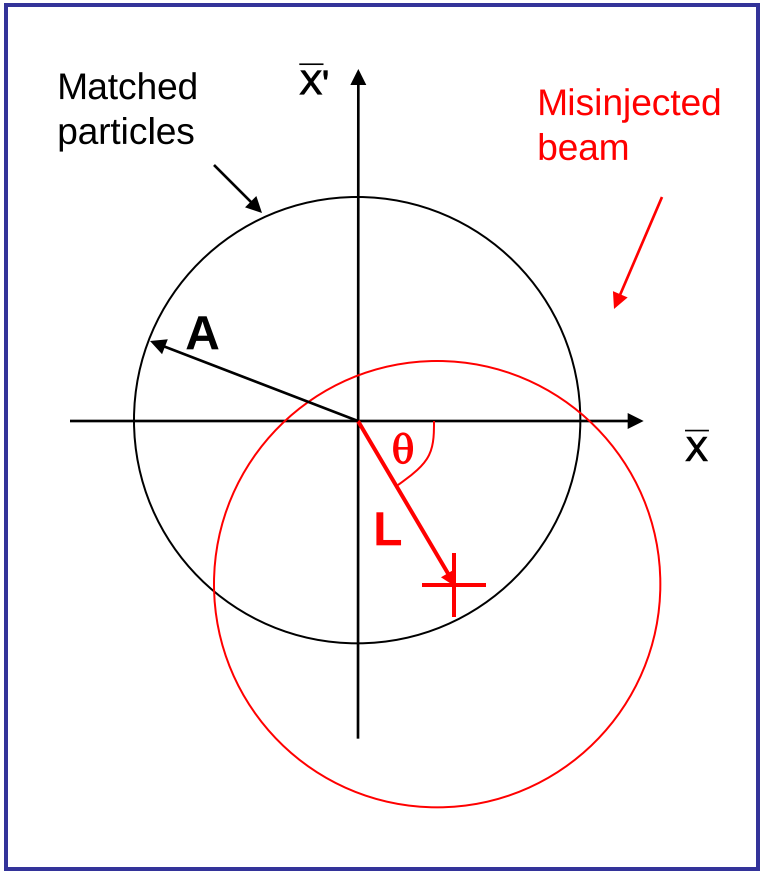


Blow-up from steering error

- The new particle coordinates in normalised phase space are

$$\overline{X}_{new} = \overline{X}_0 + L\,cos\theta$$

$$\overline{X}'_{new} = \overline{X}'_0 + L\,sin\theta$$



- For a general particle distribution, where A denotes amplitude in normalised phase space

$$A^2 = \overline{X}^2 + \overline{X}'^2$$

$$\varepsilon = \left\langle A^2 \right\rangle / 2$$

Blow-up from steering error

- So if we plug in the new coordinates….

$$\mathbf{A}^2_{new} = \overline{\mathbf{X}}^2_{new} + \overline{\mathbf{X}}'^2_{new} = \left(\overline{\mathbf{X}}_0 + \mathbf{L}cos\theta\right)^2 + \left(\overline{\mathbf{X}}'_0 + \mathbf{L}sin\theta\right)^2$$

$$= \overline{\mathbf{X}}_0^2 + \overline{\mathbf{X}}_0'^2 + 2\mathbf{L}\left(\overline{\mathbf{X}}_0 cos\theta + \overline{\mathbf{X}}'_0 sin\theta\right) + \mathbf{L}^2$$

$$\left\langle \mathbf{A}^2_{new} \right\rangle = \left\langle \overline{\mathbf{X}}_0^2 \right\rangle + \left\langle \overline{\mathbf{X}}_0'^2 \right\rangle + \left\langle 2\mathbf{L}\left(\overline{\mathbf{X}}_0 cos\theta + \overline{\mathbf{X}}'_0 sin\theta\right)\right\rangle + \left\langle \mathbf{L}^2 \right\rangle$$

$$= 2\varepsilon_0 + 2\mathbf{L}\left(cos\theta \overset{0}{\cancel{\left\langle \overline{\mathbf{X}}_0 \right\rangle}} + sin\theta \overset{0}{\cancel{\left\langle \overline{\mathbf{X}}'_0 \right\rangle}}\right) + \mathbf{L}^2$$

$$= 2\varepsilon_0 + \mathbf{L}^2$$

- Giving for the emittance increase

$$\varepsilon_{new} = \left\langle \mathbf{A}_{new}{}^2 \right\rangle / 2 = \varepsilon_0 + \mathbf{L}^2 / 2$$

$$= \varepsilon_0 \left(1 + \Delta \mathbf{a}^2 / 2\right)$$

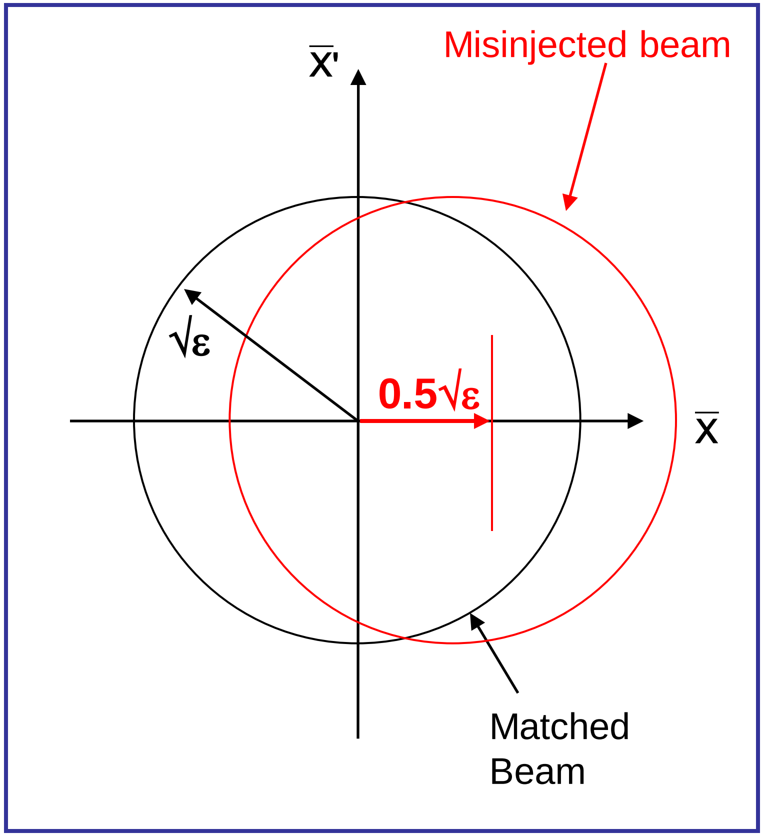
Blow-up from steering error

A numerical example….

Consider an offset Δa of 0.5 sigma for injected beam

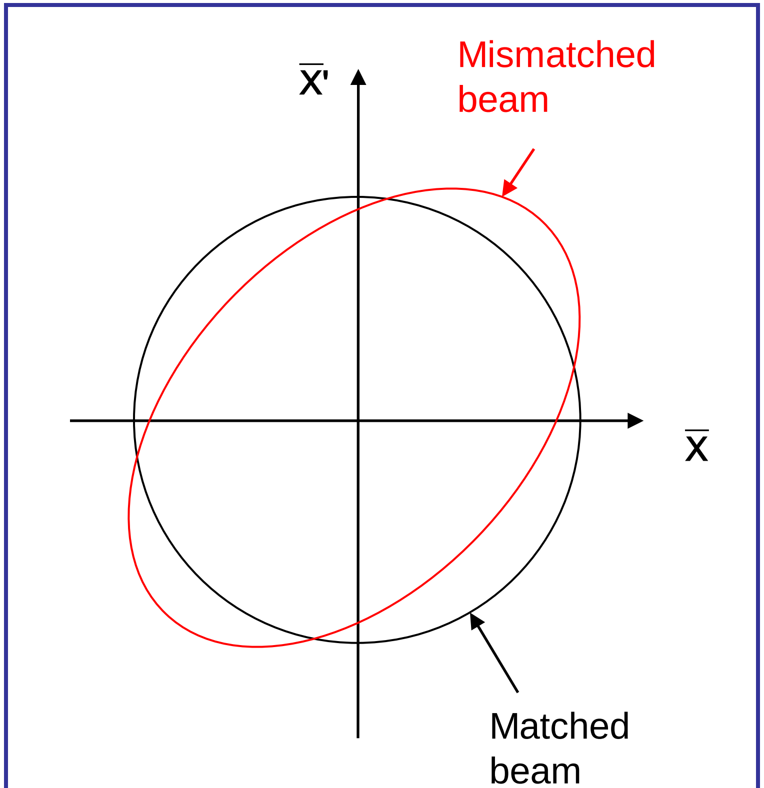
$$\varepsilon_{new} = \varepsilon_0 \left(1 + \Delta \mathbf{a}^2 / 2\right)$$

$$= 1.125\varepsilon_0$$



Blow-up from betatron mismatch

- Optical errors occur in transfer line and ring, such that the beam can be injected with a mismatch.
- Filamentation will produce an emittance increase.
- In normalised phase space, consider the matched beam as a circle, and the mismatched beam as an ellipse.



Blow-up from betatron mismatch

General betatron motion

$$x_2 = \sqrt{\varepsilon_2 \beta_2}\ \sin(\phi + \phi_o), \qquad x'_2 = \sqrt{\varepsilon_2 / \beta_2}\left[\cos(\phi + \phi_o) - \alpha_2 \sin(\phi + \phi_o)\right]$$

applying the normalising transformation for the matched beam

$$\begin{bmatrix} \overline{X}_2 \\ \overline{X}'_2 \end{bmatrix} = \sqrt{\frac{1}{\beta_1}} \cdot \begin{bmatrix} 1 & 0 \\ \alpha_1 & \beta_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ x'_2 \end{bmatrix}$$

an ellipse is obtained in normalised phase space

$$A^2 = \overline{X}_2^2\left[\frac{\beta_1}{\beta_2} + \frac{\beta_2}{\beta_1}\left(\alpha_1 - \alpha_2\frac{\beta_1}{\beta_2}\right)^2\right] + \overline{X}'^2_2\frac{\beta_2}{\beta_1} - 2\overline{X}_2\overline{X}'_2\left[\frac{\beta_2}{\beta_1}\left(\alpha_1 - \alpha_2\frac{\beta_1}{\beta_2}\right)\right]$$

characterised by γ_{new}, β_{new} *and* α_{new}, where

$$\alpha_{new} = \frac{-\beta_2}{\beta_1}\left(\alpha_1 - \alpha_2\frac{\beta_1}{\beta_2}\right), \qquad \beta_{new} = \frac{\beta_2}{\beta_1}, \quad \gamma_{new} = \frac{\beta_1}{\beta_2} + \frac{\beta_2}{\beta_1}\left(\alpha_1 - \alpha_2\frac{\beta_1}{\beta_2}\right)^2$$

Blow-up from betatron mismatch

From the general ellipse properties

$$a = \frac{A}{\sqrt{2}}\left(\sqrt{H+1}+\sqrt{H-1}\right), \qquad b = \frac{A}{\sqrt{2}}\left(\sqrt{H+1}-\sqrt{H-1}\right)$$

where

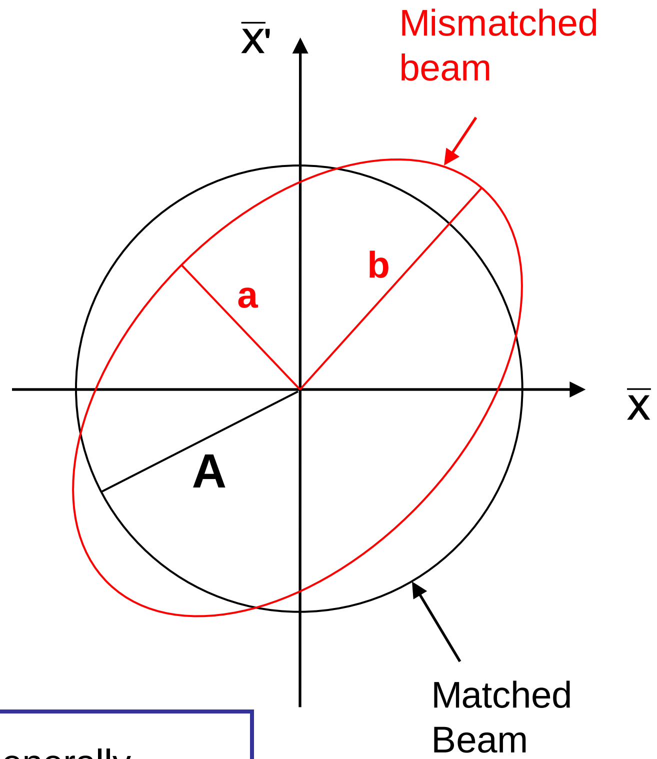
$$H = \frac{1}{2}\left(\gamma_{new}+\beta_{new}\right)$$

$$= \frac{1}{2}\left(\frac{\beta_1}{\beta_2}+\frac{\beta_2}{\beta_1}\left(\alpha_1-\alpha_2\frac{\beta_1}{\beta_2}\right)^2+\frac{\beta_2}{\beta_1}\right)$$

giving

$$\lambda = \frac{1}{\sqrt{2}}\left(\sqrt{H+1}+\sqrt{H-1}\right), \qquad \frac{1}{\lambda} = \frac{1}{\sqrt{2}}\left(\sqrt{H+1}-\sqrt{H-1}\right)$$

$$\overline{\mathbf{X}}_{new} = \lambda\cdot\mathbf{A}\sin(\phi+\phi_1), \qquad \overline{\mathbf{X}}'_{new} = \frac{1}{\lambda}\mathbf{A}\cos(\phi+\phi_1)$$



generally

$$a = \mathrm{A}/\lambda$$
$$b = \mathrm{A}\cdot\lambda$$

Blow-up from betatron mismatch

We can evaluate the square of the distance of a particle from the origin as

$$\mathrm{A}^2_{new} = \overline{\mathrm{X}}^2_{new} + \overline{\mathrm{X}}'^2_{new} = \lambda^2 \cdot \mathrm{A}_0^2 \sin^2(\phi+\phi_1) + \frac{1}{\lambda^2}\mathrm{A}_0^2 \cos^2(\phi+\phi_1)$$

The new emittance is the average over all phases

$$\varepsilon_{new} = \frac{1}{2}\left\langle \mathrm{A}^2_{new} \right\rangle = \frac{1}{2}\left(\lambda^2 \left\langle \mathrm{A}_0^2 \sin^2(\phi+\phi_1) \right\rangle + \frac{1}{\lambda^2}\left\langle \mathrm{A}_0^2 \cos^2(\phi+\phi_1) \right\rangle \right)$$

$$= \frac{1}{2}\left\langle \mathrm{A}_0^2 \right\rangle \left(\lambda^2 \left\langle \cancelto{0.5}{\sin^2(\phi+\phi_1)} \right\rangle + \frac{1}{\lambda^2}\left\langle \cancelto{0.5}{\cos^2(\phi+\phi_1)} \right\rangle \right)$$

$$= \frac{1}{2}\varepsilon_0 \left(\lambda^2 + \frac{1}{\lambda^2} \right)$$

If we’re feeling diligent, we can substitute back for λ to give

$$\varepsilon_{new} = \frac{1}{2}\varepsilon_0 \left(\lambda^2 + \frac{1}{\lambda^2} \right) = H\varepsilon_0 = \frac{1}{2}\varepsilon_0 \left(\frac{\beta_1}{\beta_2} + \frac{\beta_2}{\beta_1}\left(\alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right)^2 + \frac{\beta_2}{\beta_1} \right)$$

where subscript 1 refers to matched ellipse, 2 to mismatched ellipse.

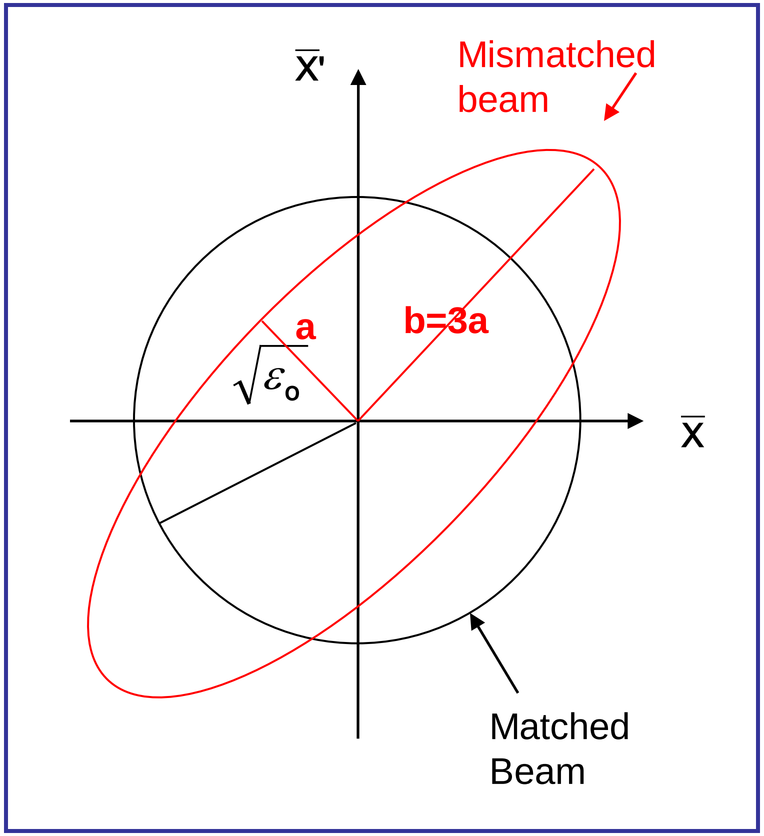
Blow-up from betatron mismatch

A numerical example….consider *b* = *3a* for the mismatched ellipse

$$\lambda = \sqrt{b/a} = \sqrt{3}$$

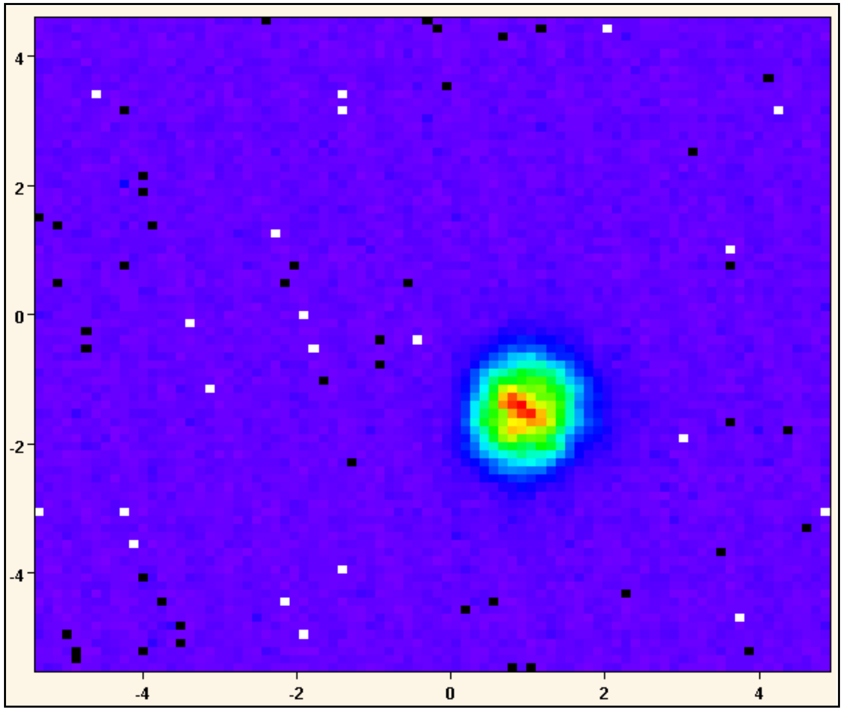
Then

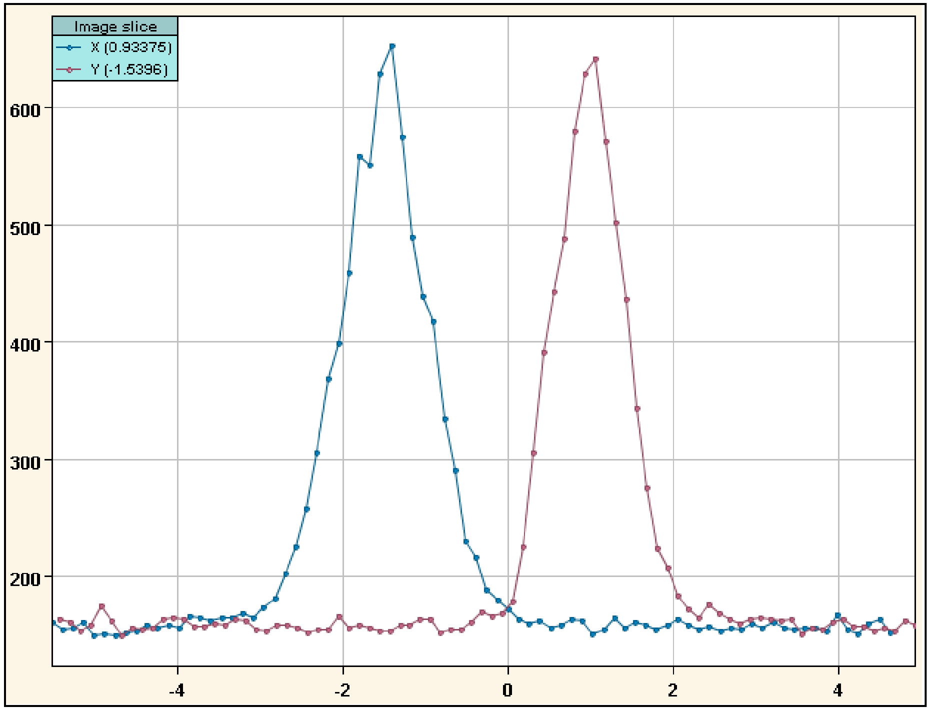
$$\varepsilon_{new} = \frac{1}{2}\varepsilon_0\left(\lambda^2 + 1/\lambda^2\right)$$
$$= 1.67\varepsilon_0$$



Emittance and mismatch measurement

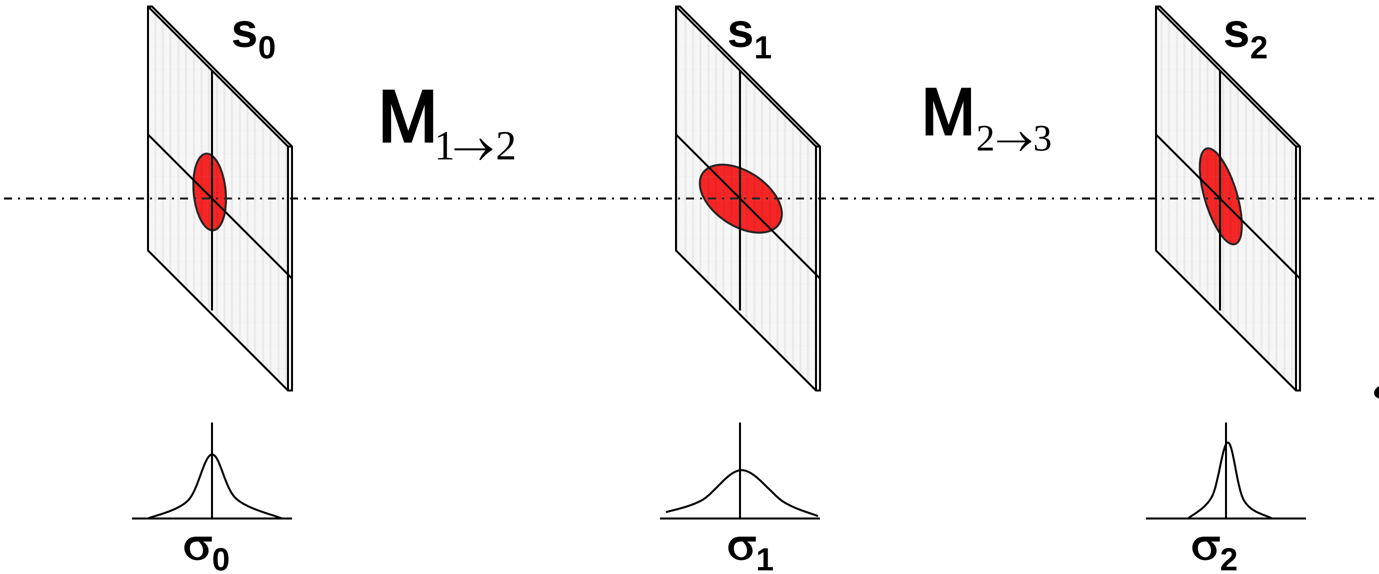
- A profile monitor is need to measure the beam size
  - E.g. beam screen (luminescent) provides 2D density profile of the beam
- Profile fit gives transverse beam sizes σ.
- In a ring, β is ‘known’ so ε can be calculated from a single screen





Emittance and mismatch measurement

- Emittance measurement in a line needs 3 profile measurements in a dispersion-free region
- Measurements of σ_0,σ_1,σ_2, plus the two transfer matrices M_{01} and M_{12} allows determination of ε, α and β



$$\varepsilon = \frac{\sigma_0^2}{\beta_0} = \frac{\sigma_1^2}{\beta_1} = \frac{\sigma_2^2}{\beta_2}$$

Emittance and mismatch measurement

We have
$$\begin{bmatrix}\beta_1\\ \alpha_1\\ \gamma_1\end{bmatrix}=\begin{bmatrix}C_1^2 & -2C_1S_1 & S_1^2\\ -C_1C_1' & C_1S_1'+S_1C_1' & -S_1S_1'\\ C_1'^2 & -2C_1'S_1' & S_1'^2\end{bmatrix}\cdot\begin{bmatrix}\beta_0\\ \alpha_0\\ \gamma_0\end{bmatrix}$$

where
$$\begin{bmatrix}C_1 & S_1\\ C_1' & S_1'\end{bmatrix}=\begin{bmatrix}\sqrt{\beta_2/\beta_1}\left(\cos\Delta\mu+\alpha_1\sin\Delta\mu\right) & \sqrt{\beta_1\beta_2}\sin\Delta\mu\\ \sqrt{1/\beta_1\beta_2}\left[(\alpha_1-\alpha_2)\cos\Delta\mu-(1+\alpha_1\alpha_2)\sin\Delta\mu\right] & \sqrt{\beta_1/\beta_2}\left(\cos\Delta\mu-\alpha_2\sin\Delta\mu\right)\end{bmatrix}$$

so that
$$\beta_1=C_1^2\beta_0-2C_1S_1\alpha_0+\frac{S_1^2}{\beta_0}\left(1+\alpha_0^2\right),\qquad \beta_2=C_2^2\beta_0-2C_2S_2\alpha_0+\frac{S_2^2}{\beta_0}\left(1+\alpha_0^2\right)$$

Using
$$\beta_0=\frac{\sigma_0^2}{\varepsilon},\qquad \beta_1=\left(\frac{\sigma_1}{\sigma_0}\right)^2\beta_0,\qquad \beta_2=\left(\frac{\sigma_2}{\sigma_0}\right)^2\beta_0$$

we find
$$\alpha_0=\frac{1}{2}\beta_0 W$$

where
$$W=\frac{(\sigma_2/\sigma_0)^2/S_2^2-(\sigma_1/\sigma_0)^2/S_1^2-(C_2/S_2)^2+(C_1/S_1)^2}{(C_1/S_1)-(C_2/S_2)}$$

Emittance and mismatch measurement

Some algebra with above equations gives

$$\beta_0 = 1 \Big/ \left| \sqrt{(\sigma_2 / \sigma_0)^2 / S_2^2 - (C_2 / S_2)^2 + \mathbf{W}(C_2 / S_2)^2 - \mathbf{W^2} / 4} \right|$$

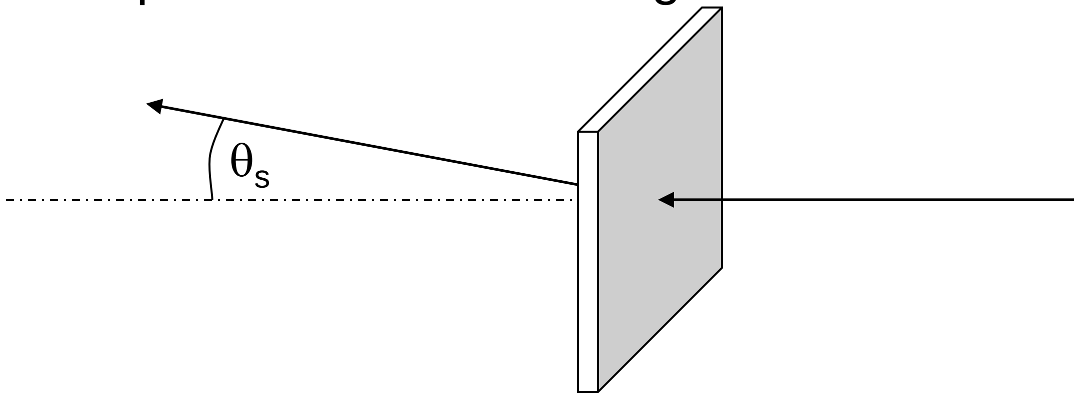
And finally we are in a position to evaluate ε and α_0

$$\varepsilon = \sigma_{\mathbf{0}}^{\mathbf{2}} \beta_{\mathbf{0}} \qquad \alpha_0 = \frac{1}{2} \beta_0 \mathbf{W}$$

Comparing measured α_0, β_0 with expected values gives measurement of mismatch

Blow-up from thin scatterer

- Scattering elements are sometimes required in the beam
  - Thin beam screens (Al_2O_3,Ti) used to generate profiles.
  - Metal windows also used to separate vacuum of transfer lines from vacuum in circular machines.
  - Foils are used to strip electrons to change charge state
- The emittance of the beam increases when it passes through, due to multiple Coulomb scattering.



rms angle increase: $\sqrt{\left\langle \theta_s^2 \right\rangle}[mrad] = \frac{14.1}{\beta_c p[MeV/c]} Z_{inc} \sqrt{\frac{L}{L_{rad}}} \left(1 + 0.11 \cdot \log_{10} \frac{L}{L_{rad}} \right)$

β_c = v/c, p = momentum, Z_{inc} = particle charge /e, L = target length, L_{rad} = radiation length

Blow-up from thin scatterer

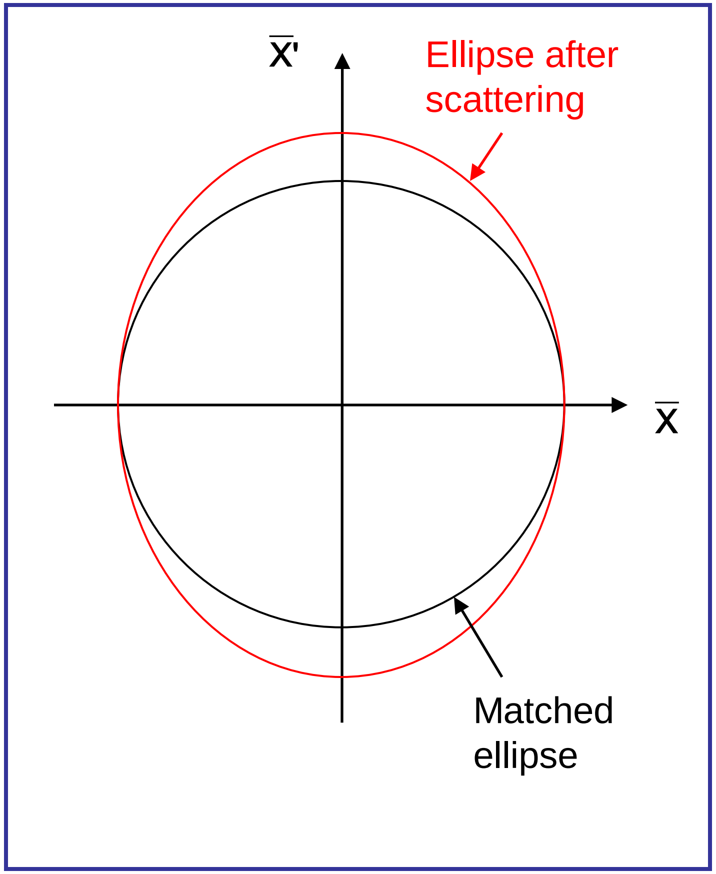
Each particles gets a random angle change θ_s but there is no effect on the positions at the scatterer

$$\overline{\mathbf{X}}_{new} = \overline{\mathbf{X}}_0$$

$$\overline{\mathbf{X}}'_{new} = \overline{\mathbf{X}}'_0 + \sqrt{\beta}\theta_s$$

After filamentation the particles have different amplitudes and the beam has a larger emittance

$$\varepsilon = \left\langle \mathbf{A}^2_{new} \right\rangle / 2$$



Blow-up from thin scatterer

$$\mathbf{A}^2_{new} = \overline{\mathbf{X}}^2_{new} + \overline{\mathbf{X}}'^2_{new}$$

$$= \overline{\mathbf{X}}^2_0 + \left(\overline{\mathbf{X}}'_0 + \sqrt{\beta}\theta_s\right)^2$$

$$= \overline{\mathbf{X}}^2_0 + \overline{\mathbf{X}}'^2_0 + 2\sqrt{\beta}\left(\overline{\mathbf{X}}'_0\theta_s\right) + \beta\theta_s^2$$

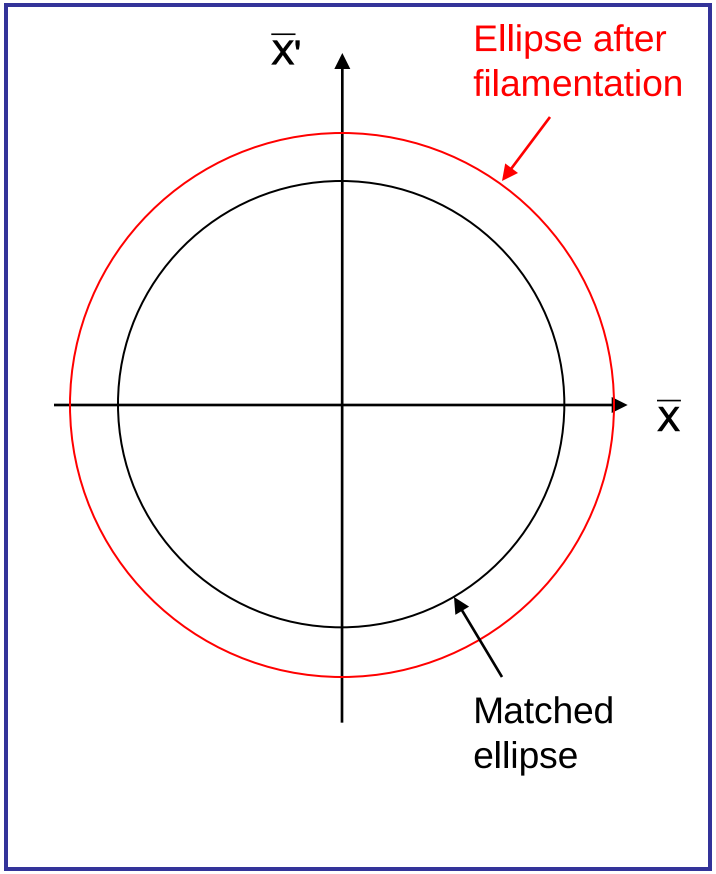
uncorrelated

$$\left\langle \mathbf{A}^2_{new} \right\rangle = \left\langle \overline{\mathbf{X}}^2_0 \right\rangle + \left\langle \overline{\mathbf{X}}'^2_0 \right\rangle + 2\sqrt{\beta}\left\langle \overline{\mathbf{X}}'_0\theta_s \right\rangle + \beta\left\langle \theta_s^2 \right\rangle$$

$$= 2\varepsilon_0 + 2\sqrt{\beta}\underbrace{\cancel{\left\langle \overline{\mathbf{X}}'_0 \right\rangle}}_{0}\left\langle \theta_s \right\rangle + \beta\left\langle \theta_s^2 \right\rangle$$

$$= 2\varepsilon_0 + \beta\left\langle \theta_s^2 \right\rangle$$

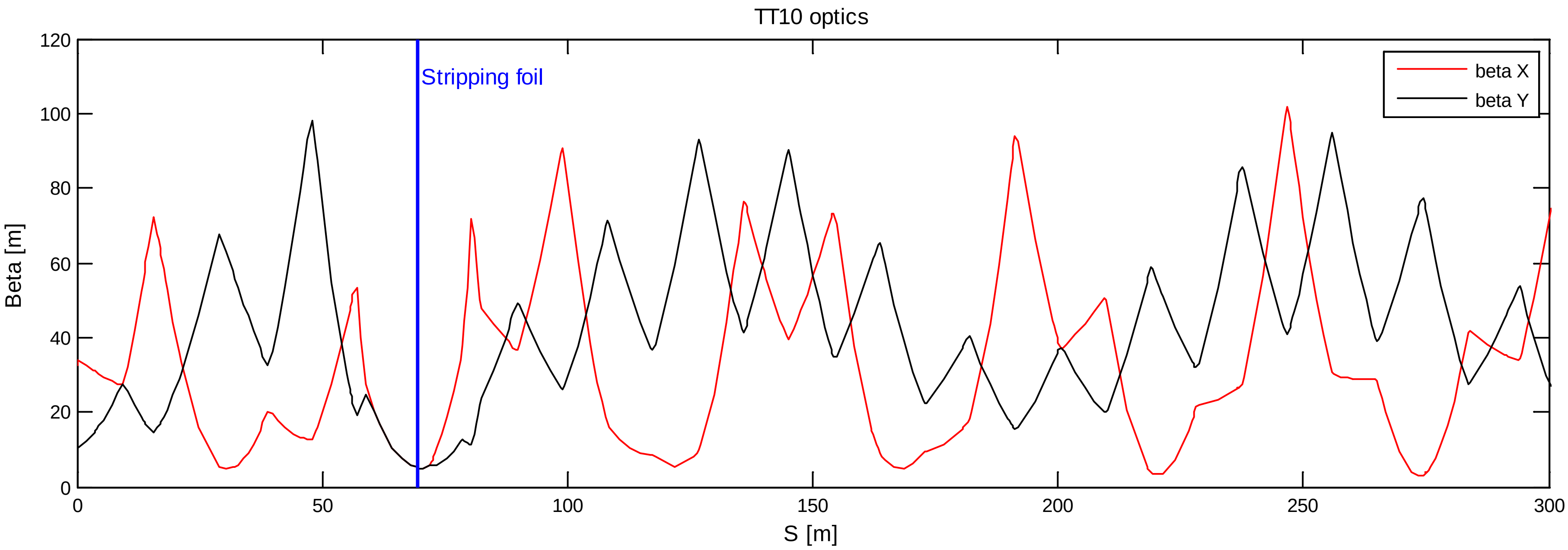
$$\boxed{\varepsilon_{new} = \varepsilon_0 + \frac{\beta}{2}\left\langle \theta_s^2 \right\rangle}$$



Need to keep β small to minimise blow-up (small β means large spread in angles in beam distribution, so additional angle has small effect on distn.)

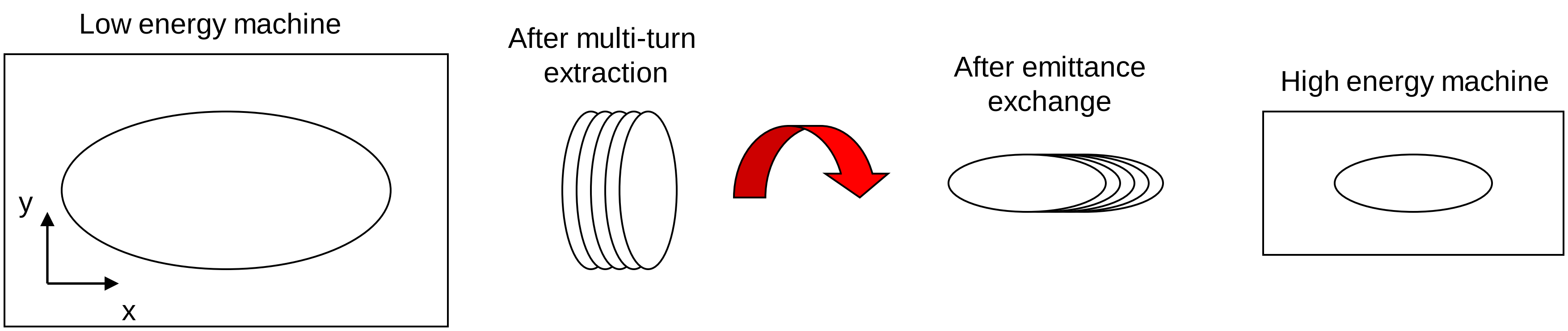
Blow-up from charge stripping foil

- For LHC heavy ions, Pb^{53+} is stripped to Pb^{82+} at 4.25GeV/u using a 0.8mm thick Al foil, in the PS to SPS line
- $\Delta\varepsilon$ is minimised with low-β insertion (β_{xy} ~5 m) in the transfer line
- Emittance increase expected is about 8%



Emittance exchange insertion

- Acceptances of circular accelerators tend to be larger in horizontal plane (bending dipole gap height small as possible)
- Several multiturn extraction process produce beams which have emittances which are larger in the *vertical* plane $\rightarrow$ larger losses
- We can overcome this by exchanging the H and V phase planes (emittance exchange)



In the following, remember that the matrix is our friend…

Emittance exchange

Phase-plane exchange requires a transformation of the form:

$$\begin{pmatrix} x_1 \\ x'_1 \\ y_1 \\ y'_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_{13} & m_{14} \\ 0 & 0 & m_{23} & m_{24} \\ m_{31} & m_{32} & 0 & 0 \\ m_{41} & m_{42} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

A skew quadrupole is a normal quadrupole rotated by an angle θ.

The transfer matrix **S** obtained by a rotation of the normal transfer matrix $\mathbf{M}_q$:

$$\mathbf{S} = \mathbf{R}^{-1}\mathbf{M}_q\mathbf{R}$$

where **R** is the rotation matrix $\begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}$

(you can convince yourself of what **R** does by checking that x_0 is transformed to $x_1 = x_0\cos\theta + y_0\sin\theta$, y_0 into $-x_0\sin\theta + y_0\cos\theta$, etc.)

Emittance exchange

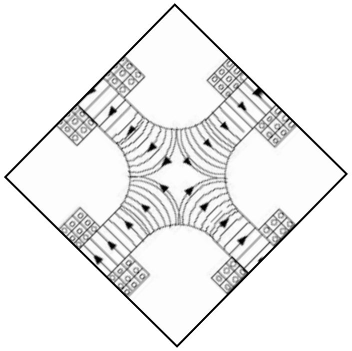
For a thin-lens approximation $\mathbf{M}_q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \delta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\delta & 1 \end{pmatrix}$ (where δ = kl = 1/f is the quadrupole strength)

So that $$\mathbf{S} = \mathbf{R}^{-1}\mathbf{M}_q\mathbf{R} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \delta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\delta & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

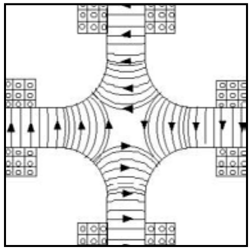
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \delta\cos 2\theta & 1 & \delta\sin 2\theta & 0 \\ 0 & 0 & 1 & 0 \\ \delta\sin 2\theta & 0 & -\delta\cos 2\theta & 1 \end{pmatrix}$$

For the case of θ = 45º, this reduces to $$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \delta & 0 \\ 0 & 0 & 1 & 0 \\ \delta & 0 & 0 & 1 \end{pmatrix}$$

45º skew quad

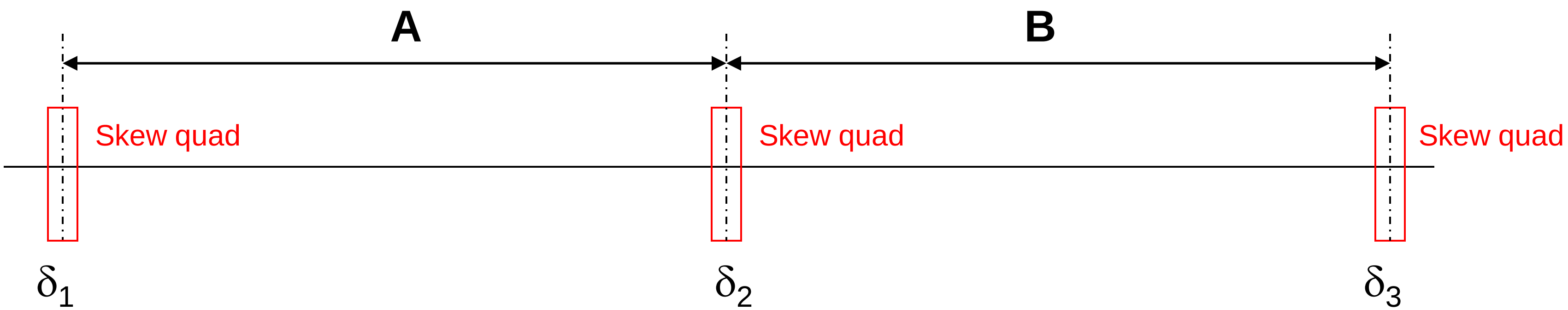


Normal quad



Emittance exchange

The transformation required can be achieved with 3 such skew quads in a lattice, of strengths δ_1, δ_2, δ_3, with transfer matrices $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$



The transfer matrix without the skew quads is $\mathbf{C} = \mathbf{B}\ \mathbf{A}$.

$$\mathbf{C} = \left(\begin{array}{c:cc} \mathbf{C}_x & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \hdashline \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \mathbf{C}_y \end{array} \right)$$

$$\mathbf{C}_x = \begin{pmatrix} \sqrt{\beta_{x2}/\beta_{x1}}\left[\cos\Delta\phi_x + \alpha_{x1}\sin\Delta\phi_x\right] & \sqrt{\beta_{x1}\beta_{x2}}\sin\Delta\phi_x \\ \dfrac{(\alpha_{x2}-\alpha_{x1})\cos\Delta\phi_x - (1+\alpha_{x1}\alpha_{x2})\sin\Delta\phi_x}{\sqrt{\beta_{x1}\beta_{x2}}} & \sqrt{\beta_{x1}/\beta_{x2}}\left[\cos\Delta\phi_x - \alpha_{x2}\sin\Delta\phi_x\right] \end{pmatrix}$$ and similar for $\mathbf{C}_y$

Emittance exchange

With the skew quads the overall matrix is $\mathbf{M} = \mathbf{S}_3\mathbf{B}\ \mathbf{S}_2\mathbf{A}\ \mathbf{S}_1$

$$
\mathbf{M} = \left(\begin{array}{cc:cc}
c_{11} + b_{12}a_{34}\delta_1\delta_2 & c_{12} & c_{12}\delta_1 + b_{12}a_{33}\delta_2 & b_{12}a_{34}\delta_2 \\
\begin{bmatrix} c_{21} + b_{22}a_{34}\delta_1\delta_2 \\ + \delta_3\left(c_{34}\delta_1 + b_{34}a_{11}\delta_2\right) \end{bmatrix} & c_{22} + b_{34}a_{12}\delta_2\delta_3 & \begin{bmatrix} c_{22}\delta_1 + b_{22}a_{33}\delta_2 \\ + \delta_3\left(c_{33} + b_{34}a_{12}\delta_1\delta_2\right) \end{bmatrix} & b_{22}a_{34}\delta_2 + c_{34}\delta_3 \\
\hdashline
c_{34}\delta_1 + b_{34}a_{11}\delta_2 & a_{12}b_{34}\delta_2 & c_{33} + b_{34}a_{12}\delta_1\delta_2 & c_{34} \\
\begin{bmatrix} \delta_3\left(c_{11} + b_{12}a_{34}\delta_1\delta_2\right) \\ + c_{44}\delta_1 + b_{22}a_{34}\delta_1\delta_2 \end{bmatrix} & c_{12}\delta_3 + b_{44}a_{12}\delta_2 & \begin{bmatrix} \delta_3\left(c_{12}\delta_1 + b_{12}a_{33}\delta_2\right) \\ + c_{43} + b_{44}a_{12}\delta_1\delta_2 \end{bmatrix} & c_{44} + b_{12}a_{34}\delta_2\delta_3
\end{array}\right)
$$

Equating the terms with our target matrix form
$$
\begin{pmatrix}
\mathbf{0} & \mathbf{0} & m_{13} & m_{14} \\
\mathbf{0} & \mathbf{0} & m_{23} & m_{24} \\
m_{31} & m_{32} & \mathbf{0} & \mathbf{0} \\
m_{41} & m_{42} & \mathbf{0} & \mathbf{0}
\end{pmatrix}
$$

a list of conditions result which must be met for phase-plane exchange.

Emittance exchange

$$
\begin{aligned}
\mathbf{0} &= c_{12} \\
\mathbf{0} &= c_{34} \\
\mathbf{0} &= c_{11} + b_{12} a_{34} \delta_1 \delta_2 \\
\mathbf{0} &= c_{22} + b_{34} a_{12} \delta_2 \delta_3 \\
\mathbf{0} &= c_{33} + b_{34} a_{12} \delta_1 \delta_2 \\
\mathbf{0} &= c_{44} + b_{12} a_{34} \delta_2 \delta_3 \\
\mathbf{0} &= c_{21} + b_{22} a_{34} \delta_1 \delta_2 + \delta_3 \left(c_{34} \delta_1 + b_{34} a_{11} \delta_2 \right) \\
\mathbf{0} &= c_{43} + b_{44} a_{12} \delta_1 \delta_2 + \delta_3 \left(c_{12} \delta_1 + b_{12} a_{33} \delta_2 \right)
\end{aligned}
$$

The simplest conditions are $c_{12} = c_{34} = 0$.

Looking back at the matrix **C**, this means that $\Delta\phi_x$ and $\Delta\phi_y$ need to be integer multiples of π (i.e. the phase advance from first to last skew quad should be 180º, 360º, …)

We also have for the strength of the skew quads

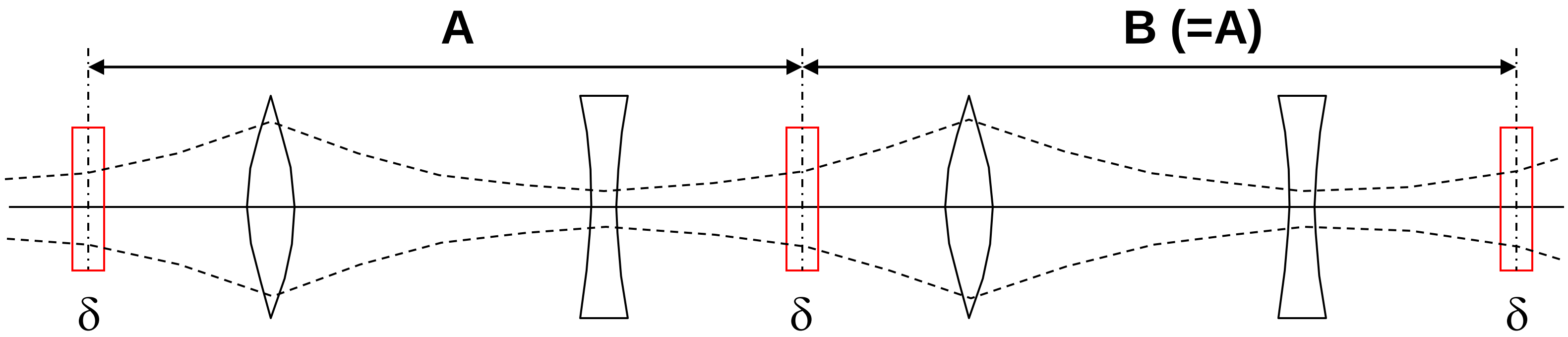
$$
\delta_1 \delta_2 = -\frac{c_{11}}{b_{12} a_{34}} = -\frac{c_{33}}{b_{34} a_{12}}
$$

$$
\delta_2 \delta_3 = -\frac{c_{22}}{b_{34} a_{12}} = -\frac{c_{44}}{b_{12} a_{34}}
$$

Emittance exchange

Several solutions exist which give **M** the target form.

One of the simplest is obtained by setting all the skew quadrupole strengths the same, and putting the skew quads at symmetric locations in a 90º FODO lattice



From symmetry **A** = **B**, and the values of α and β at all skew quads are identical.

Therefore $\mathbf{A}_x = \mathbf{B}_x = \begin{pmatrix} (\cos\Delta\phi_x + \alpha_x \sin\Delta\phi_x) & \beta_x \sin\Delta\phi_x \\ -\dfrac{(1-\alpha_x^2)\sin\Delta\phi_x}{\beta_x} & (\cos\Delta\phi_x - \alpha_x \sin\Delta\phi_x) \end{pmatrix}$ with the same form for y

The matrix **C** is similar, but with phase advances of 2Δφ

Emittance exchange

Since we have chose a 90º FODO phase advance, $\Delta\phi_x = \Delta\phi_y = \pi/2$, and $2\Delta\phi_x = 2\Delta\phi_y = \pi$ which means we can now write down **A**,**B** and **C**:

$$\mathbf{A} = \mathbf{B} = \begin{pmatrix} \alpha_x & \beta_x & \mathbf{0} & \mathbf{0} \\ -\frac{(\mathbf{1}-\alpha_x^{\mathbf{2}})}{\beta_x} & -\alpha_x & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha_y & \beta_y \\ \mathbf{0} & \mathbf{0} & -\frac{(\mathbf{1}-\alpha_y^{\mathbf{2}})}{\beta_y} & -\alpha_y \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1} \end{pmatrix}$$

i.e. 180º across the insertion in both planes

we can then write down the skew lens strength as $\delta_{\mathbf{1}} = \delta_{\mathbf{2}} = \delta_{\mathbf{3}} = \delta_s = \frac{\mathbf{1}}{\sqrt{\beta_x \beta_y}}$

For the 90º FODO with half-cell length *L*, $\boxed{\delta_F = -\delta_D = \frac{\sqrt{\mathbf{2}}}{L}, \qquad \delta_s = \frac{\mathbf{1}}{L\sqrt{\mathbf{2}}}}$

Summary

- Transfer lines present interesting challenges and differences from circular machines
  - No periodic condition mean optics is defined by transfer line element strengths and by initial beam ellipse
  - Matching at the extremes is subject to many constraints
  - Trajectory correction is rather simple compared to circular machine
  - Emittance blow-up is an important consideration, and arises from several sources
  - Phase-plane rotation is sometimes required - skew quads

Keywords for related topics

- Transfer lines
  - Achromat bends
  - Algorithms for optics matching
  - The effect of alignment and gradient errors on the trajectory and optics
  - Trajectory correction algorithms
  - SVD trajectory analysis
  - Kick-response optics measurement techniques in transfer lines
  - Optics measurements including dispersion and $\delta p/p$ with >3 screens
  - Different phase-plane exchange insertion solutions