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Applications of Multibang Regularization in X-Ray Imaging

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Introduction

In many applications, such as cargo scanning and object identification, material properties are known a priori. With this information, we can make use of a recently developed regularizer known as Multibang regularization [1]. This technique aims to improve image reconstruction in such cases, as well as produce good results when less data is available. In particular we focus on the material property attentuation. We denote the known attenuation values as A and refer to this as the admissable set.

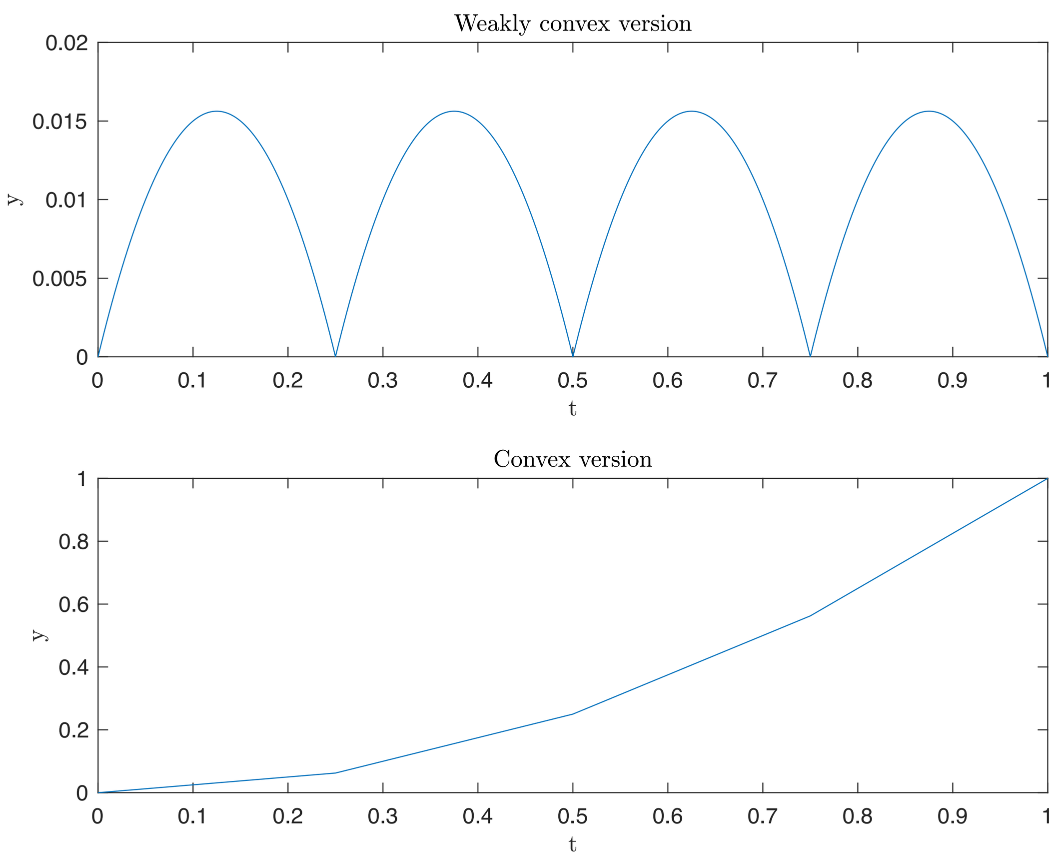
The Multibang Regularizer

Suppose that the Admissable set $A = \{a_1, a_2, a_3, ..., a_k\}$ is known. If we consider the following function

$$m(t) = \begin{cases} (t-a_i)(a_{i+1}-t) & t \in [a_i, a_{i+1}] \text{ for } i \in 1,2,3,...,n-1 \\ \infty & \text{otherwise} \end{cases} \quad (1)$$

then we see that minimas occur when $t \in A$ which is the desired property. This function is weakly convex and work by [2] allows us to extend the definition of sub differentiability to this interesting case, which allows us to extend methods designed for convex functions. An example of this pointwise multibang function is given below as well as $m(t)+t^2$ which is the convex version[1].

Example of Multibang regularizer for $A = \{0, 0.25, 0.75, 1\}$.



Given the pointwise Multibang penalty $m(t)$ we then define the Multibang penalty via

$$\mathcal{M}(a) = \int_\Omega m(a(x))\mathrm{d}x \quad (2)$$

where Ω is the domain of interest.

A major advantage of this weakly convex penalty over the original convex version is that when solving an a update, values in the admissable set get sent to themselves, which allows us to use proximal point methods more readily.

Application to X-ray imaging

As there are only 5 or 6 different types of tissue in the human body, medical X-ray imaging seems a natural candidate for Multibang regularization. We can model the emitted radiation in the most general case via the Attenuated Radon Transform(AtRT):

$$R_a f(s,\theta) = \int_{-\infty}^{\infty} f(s\theta^\perp + t\theta)e^{-Da(s\theta^\perp + t\theta,\theta))}\mathrm{d}t, \quad (3)$$

where f is source density and

$$Da(x,\theta) = \int_0^\infty a(x+t\theta)\mathrm{d}t. \quad (4)$$

We can discretise the domain Ω and assuming piecewise constant a and f can rewrite (3) as a matrix equation.

Given data about the rays we then aim to solve the inverse problem

$$\mathrm{argmin}_{a,f}\|RAD_a f - d\|_2^2 + \alpha\mathcal{M}(a) + \lambda_a \mathrm{TV}(a) + \lambda_f \mathrm{TV}(f) \quad (5)$$

with RAD_a being the discretised AtRT and TV is the Isotropic Total Variation penalty

$$\mathrm{TV}(x) = \sum_{i,j=1}^{n} \sqrt{(x_{i+1,j}-x_{i,j})^2 + (x_{i,j+1}-x_{i,j})^2} \quad (6)$$

with $x_{i,j} = 0$ whenever i, j lies outside the range of the matrix x.

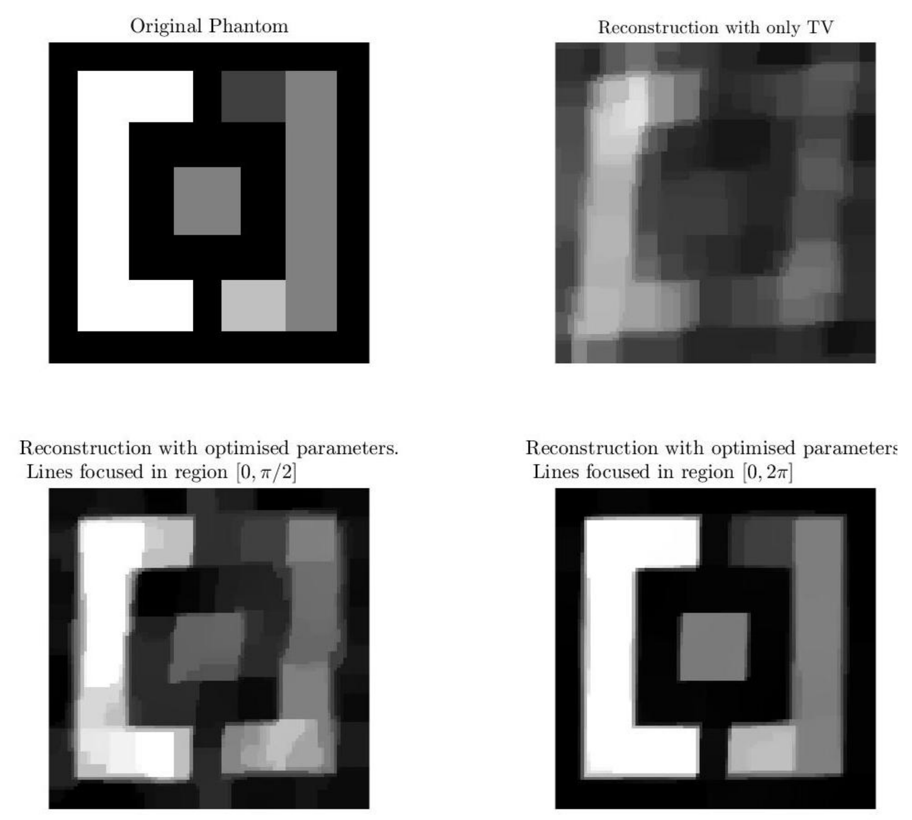
The Reconstruction Algorithm

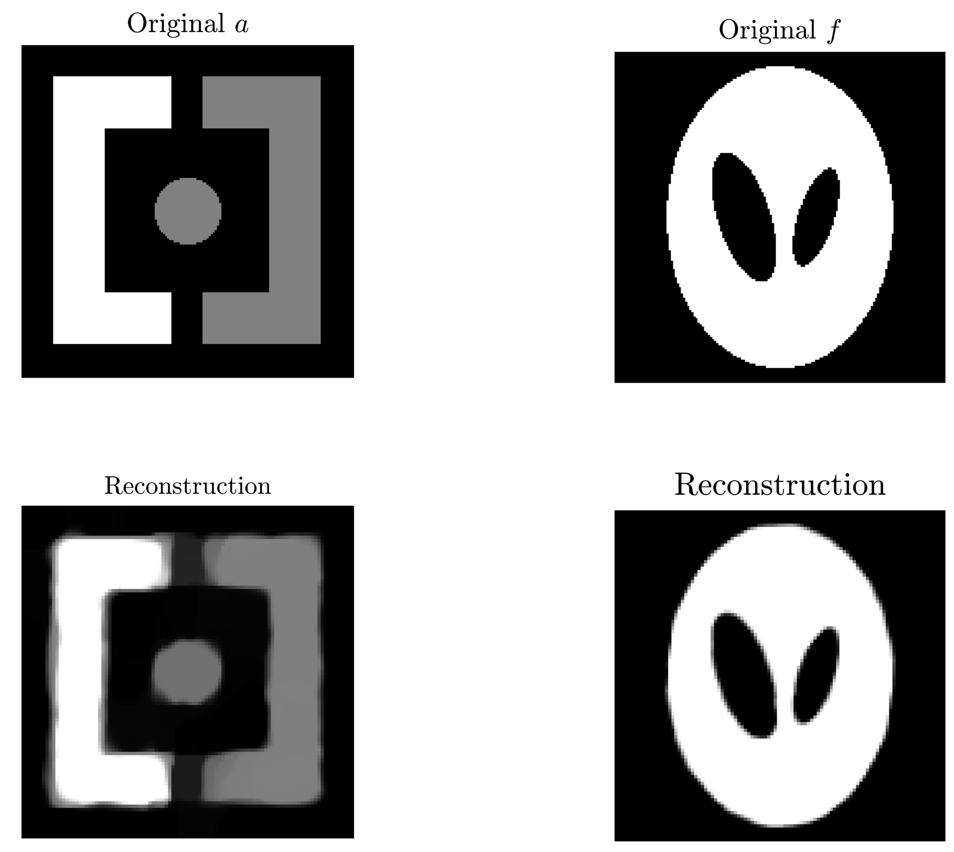
The difficulty in reconstruction comes from the non-linear dependence on a. With these conditions we can split the a part of the objective function in (5) into two parts a and y related by a matrix equation. This sort of optimisation problem can be solved using ADMM(Alternating Direction Method of Multipliers)[3] when the objective function is convex. For the weakly convex case we apply a similar algorithm with some modifications.

- Input initial guesses a_0 and f_0 as well as an admissible set for the attenuation, y_0 and λ_0 for dual variables. Then for $k \geq 1$ do the following .
- Update f via a linear least squares method.
- Update a via proximal map calculated via the subdifferential of the weakly convex Multibang penalty.
- Update y via group thresholding, an extension of soft thresholding[3].
- Update dual variable via $\lambda^{k+1} = \lambda^k + \beta(Ha - y)$.
- Terminate when primal residual $\|Ha^{k+1} - y^{k+1}\| < \epsilon_1$ and dual residual $\|\beta H^T(y^{k+1} - y^k)\| < \epsilon_2$, for suitably small $\epsilon_1, \epsilon_2 > 0$.

Reconstructions

The following reconstructions use data simulated on a 170 by 170 grid with 5% Gaussian white noise added. Each reconstruction uses 12 projections, and is produced on a 100 by 100 grid. In the following $A = \{0, 0.25, 0.5, 0.75, 1\}$. The first set of reconstructions compare regular TV with the Multibang and TV approach for when we only aim to reconstruct a. The second set show a simultaneous reconstruction with optimized parameters.





Conclusion/ Further work

Weakly convex Multibang regularisation with ADMM for weighted X-Ray imaging has been implemented with good results so far. Main further points of research are

- Proof of convergence of algorithm.
- Further limited ray reconstruction work.
- Analysis of simultaneous reconstruction of a and f.

Acknowledgements

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References

[1] C.Clason, F. Kruse, and K.Kunisch, *Total Variation Regularization of the Multi-Material Topology Optimization*, ESAIM Mathematical modelling and Numerical Analysis 2017. DOI:10.1051/m2an/2017061

[2] Ilker Bayram, *On the Convergence of the Iterative Shrinkage/Thresholding Algorithm With a Weakly Convex Penalty*, IEEE Trans. Signal Process., vol. 64, no.6, pp.1597-1608 2015. DOI:10.1109/TSP.2015.2502551

[3] S.Boyd, et.al, *Distributed Optimisation and Statistical Learning via the Alternating Direction Method of Multipliers*, Foundations and Trends in Machine Learning, **Vol 4**,No.1 (2010)