Triangle Singularities and Cusps

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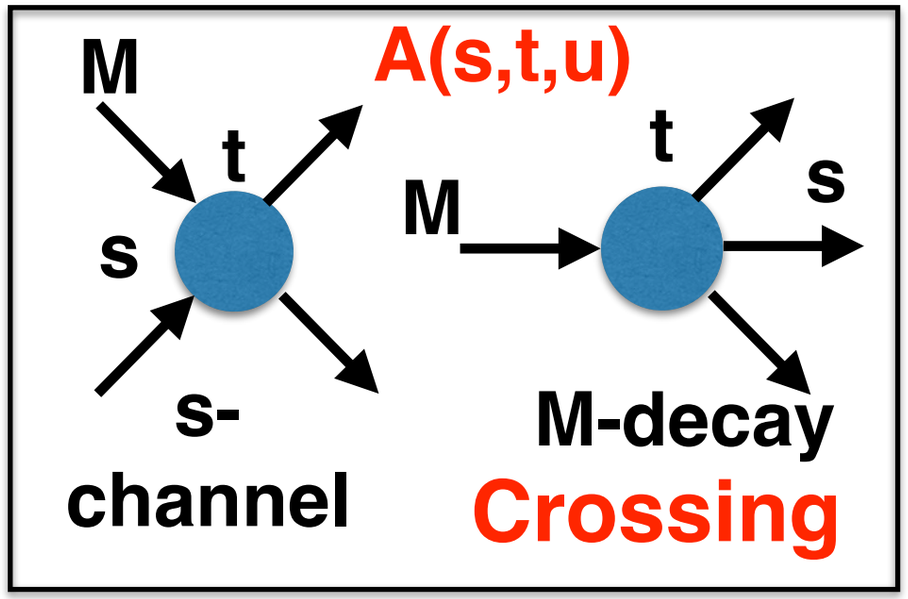
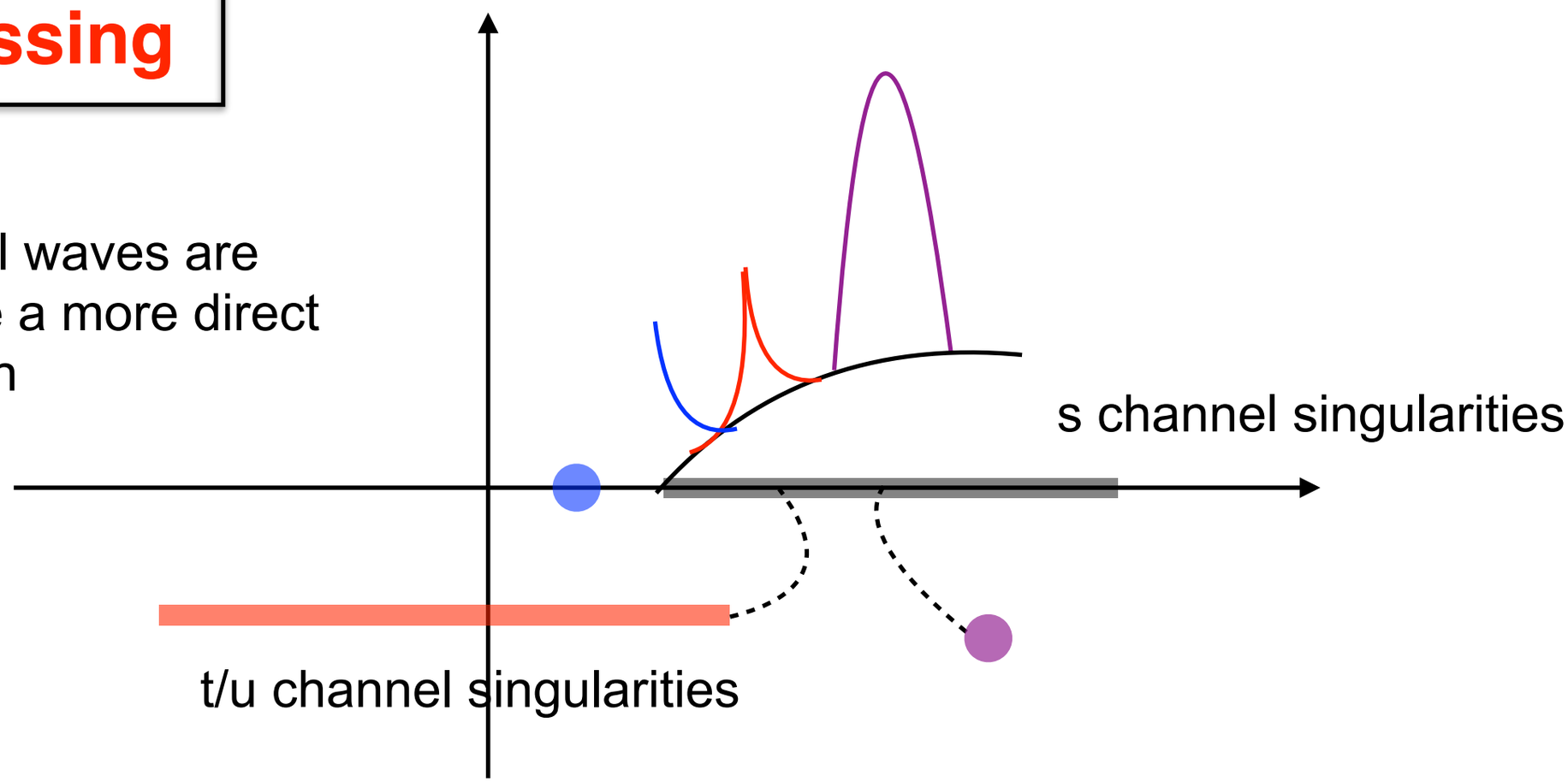
1. Amplitude analyticity
2. Where do cusps come from
3. Examples

Amplitude singularities



- A(s,t,u) has simple singularity structure. Its connection to particles arises through (complicated) partial waves

$$A_l(s) = \frac{1}{2}\int_{-1}^{1} dz_s A(s, t(s,z), u(s,z))$$



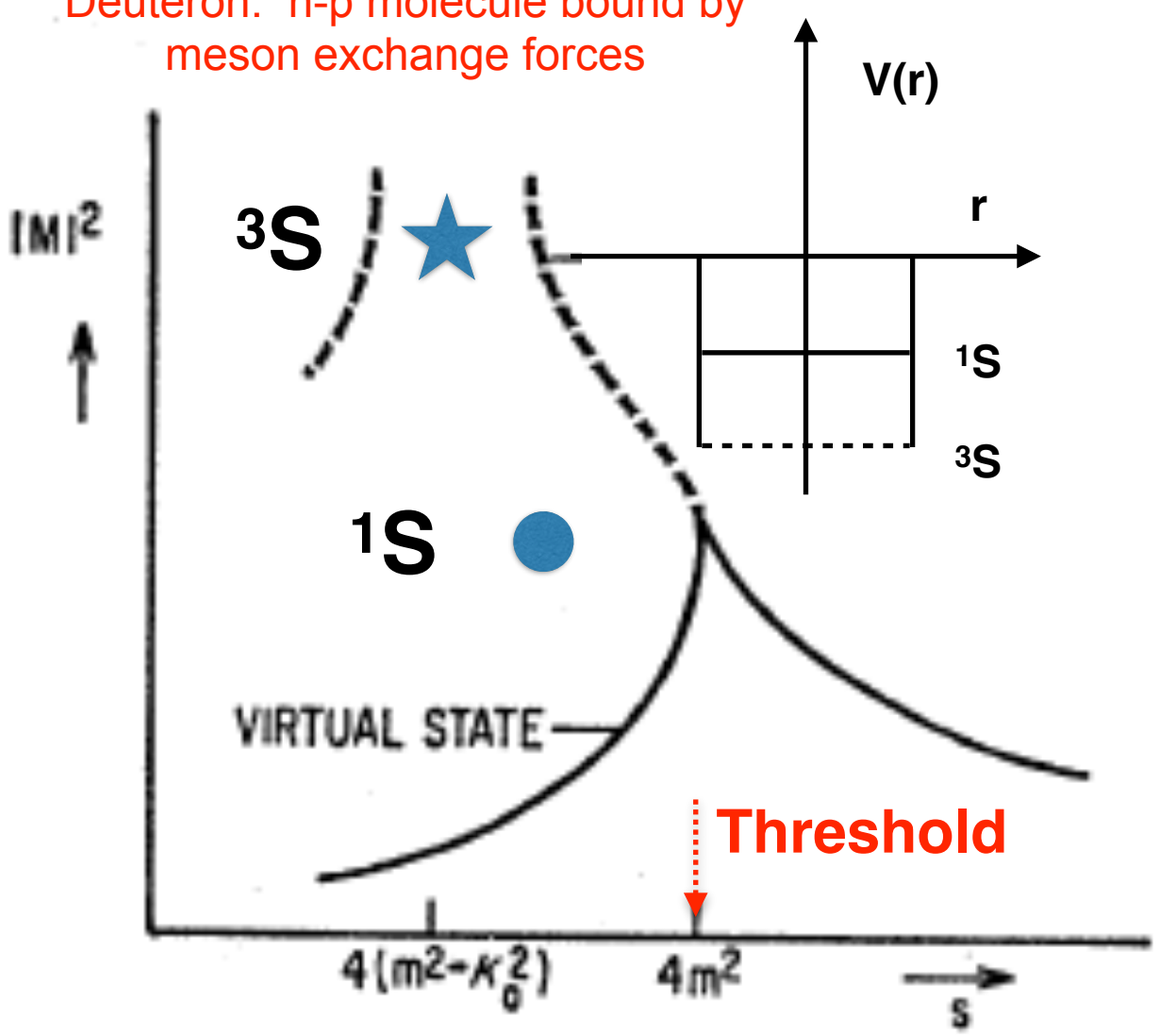
- Singularities of partial waves are complicated but have a more direct physical interpretation

- However, X-sections are given by A(s,t,u) and not by partial waves. In general "bumps" in partial waves are "washed out" and require partial wave analysis.

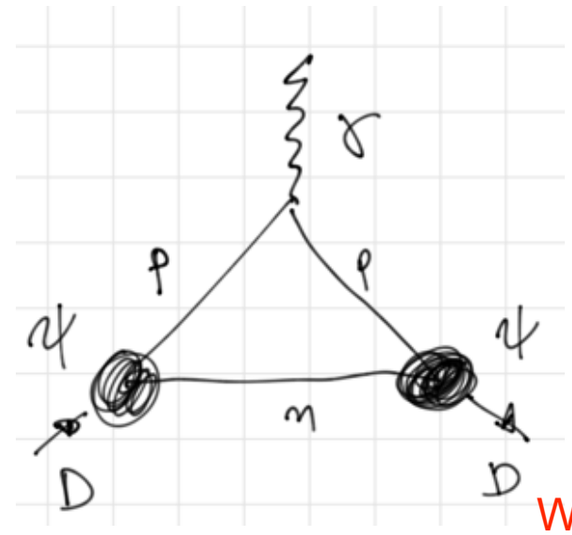
es inherit singularities of cross-

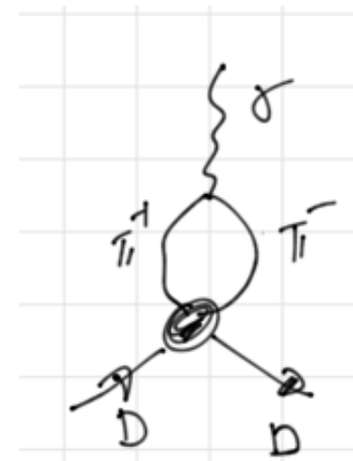
Well known examples of cusps

Deuteron: n-p molecule bound by meson exchange forces

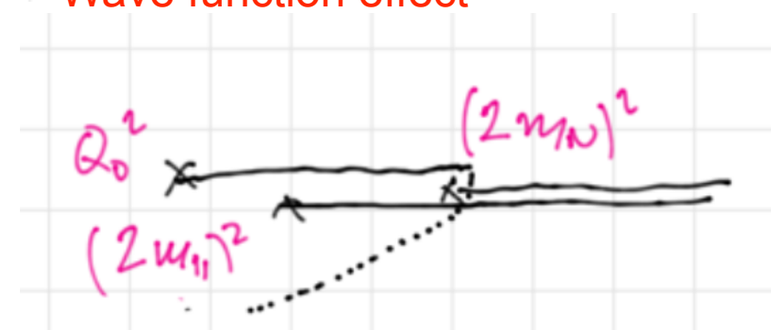


$$Q_0 \sim 100 \text{ MeV} < 2m_\pi << 2m_N$$

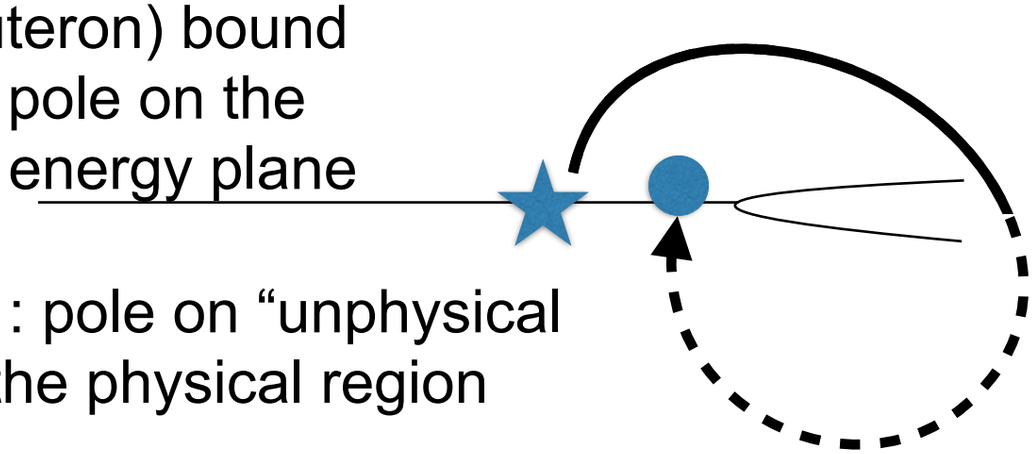




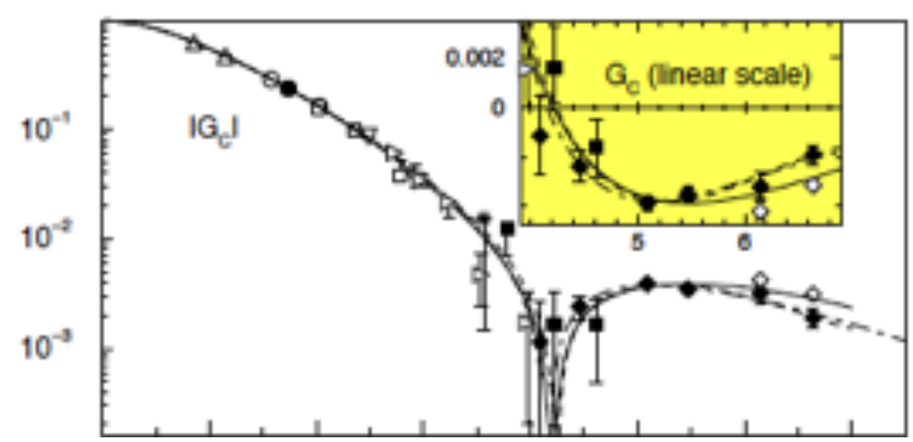
Wave function effect

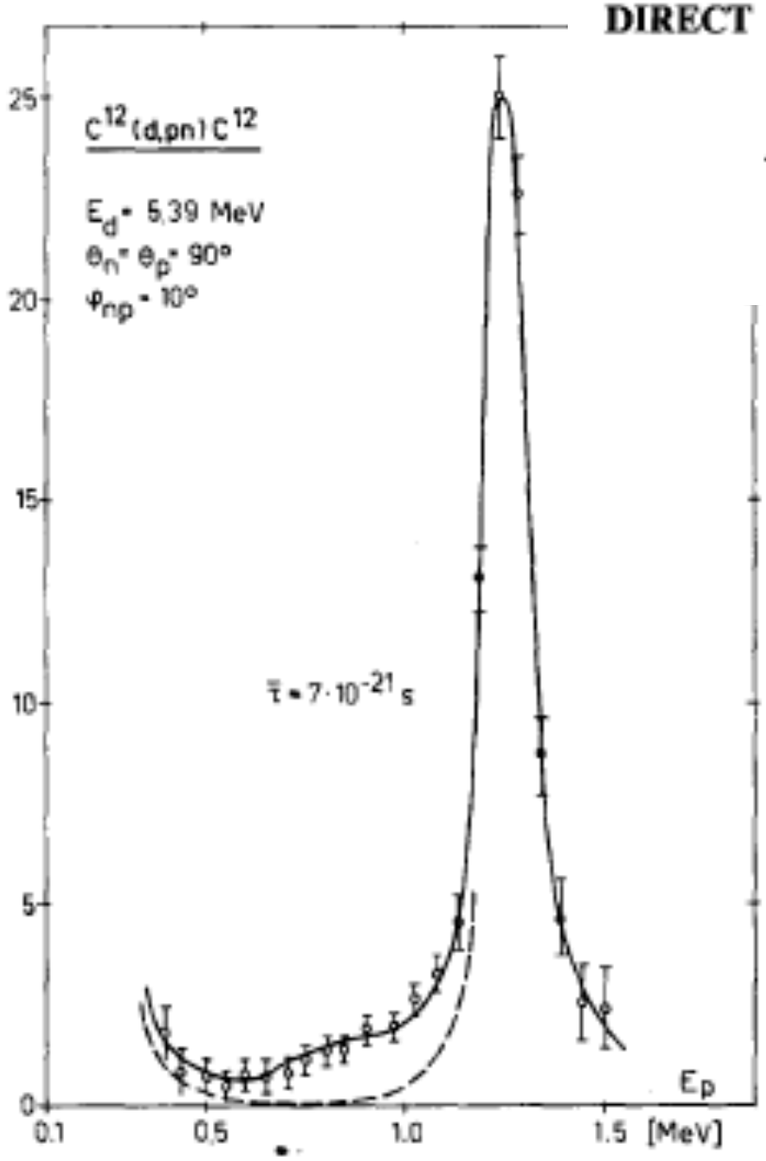


$^3S_1$ (deuteron) bound state : pole on the physical energy plane



$^1S_1$ virtual state : pole on “unphysical sheet” close the physical region





DIRECT DETERMINATION OF A SHORT NUCLEAR LIFETIME ($\approx 10^{-20}$ s) BY THE PROXIMITY SCATTERING METHOD

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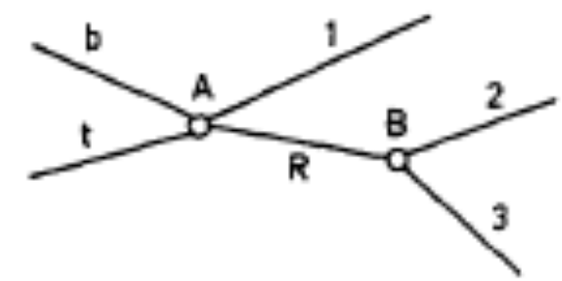
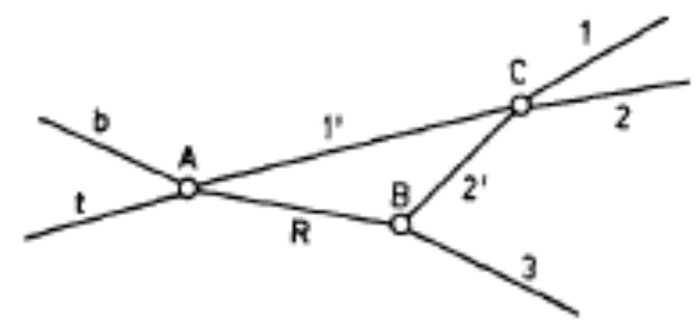
Received 4 February 1966

$$b + t \rightarrow 1 + 2 + 3 \qquad [d + {}^{12}C \rightarrow n + p + {}^{12}C],$$

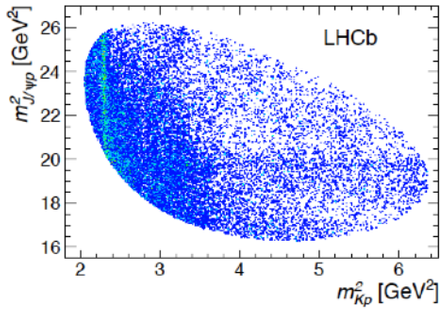




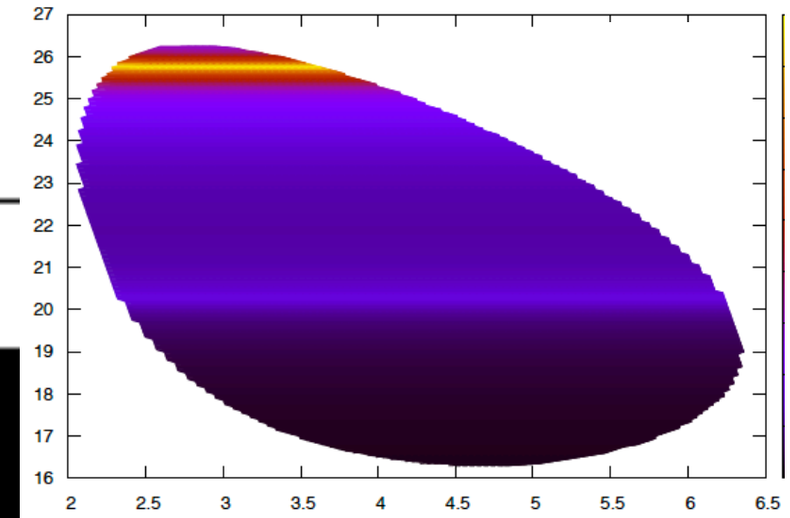
$$b + t \rightarrow 1 + R, \qquad \text{with } Q\text{-value } Q_1 \qquad [d + {}^{12}C \rightarrow n + {}^{13}N^*, \qquad Q_1 = -3.82 \text{ MeV}],$$

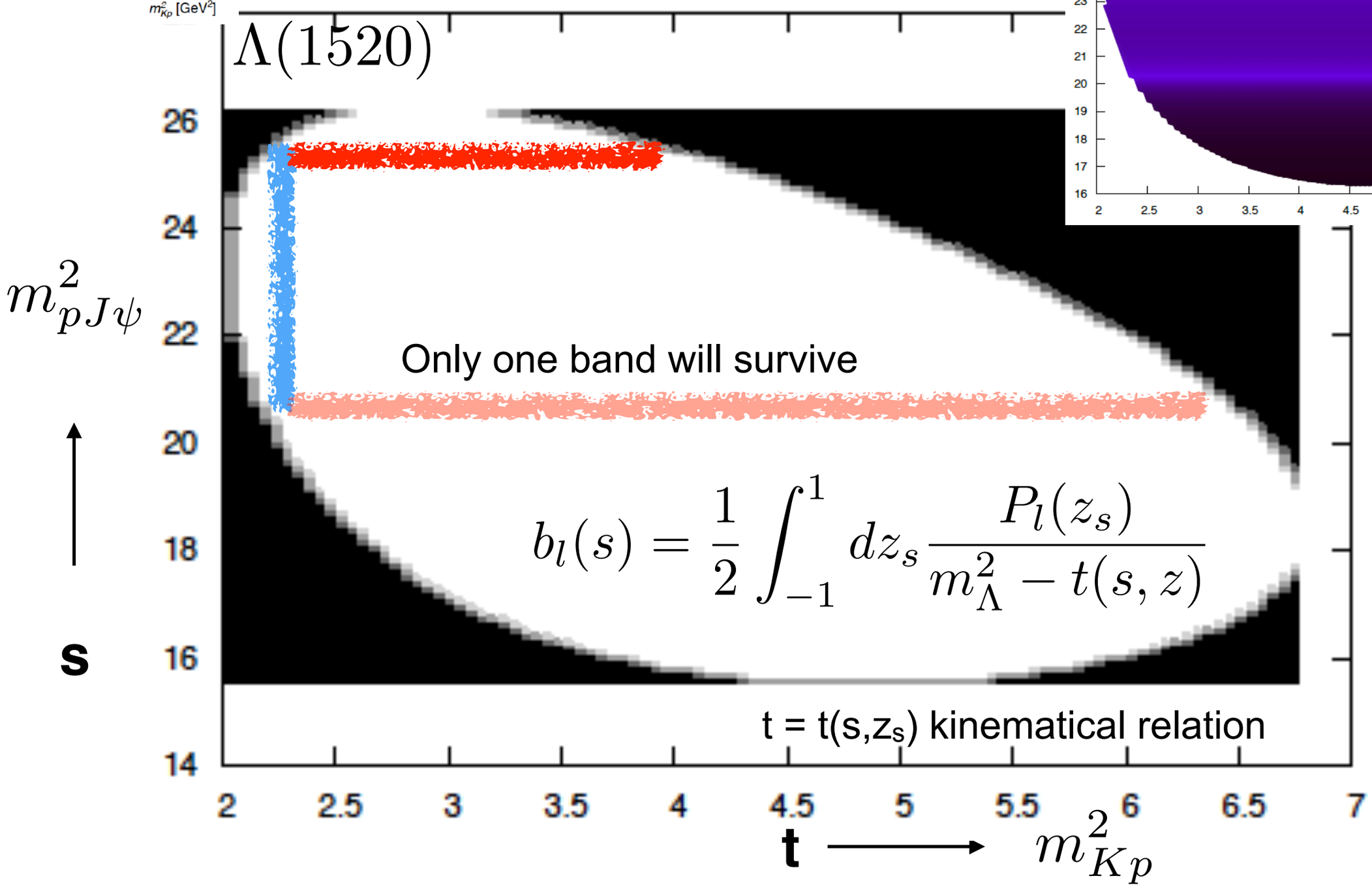
$$R \rightarrow 2 + 3 \qquad \text{with } Q\text{-value } Q_2 \qquad [{}^{13}N^* \rightarrow p + {}^{12}C, \qquad Q_2 = 1.59 \text{ MeV}],$$

s-channel band originating from a non-s-channel pole (naive)



Example: $\Lambda_b \to K^- p J/\psi$

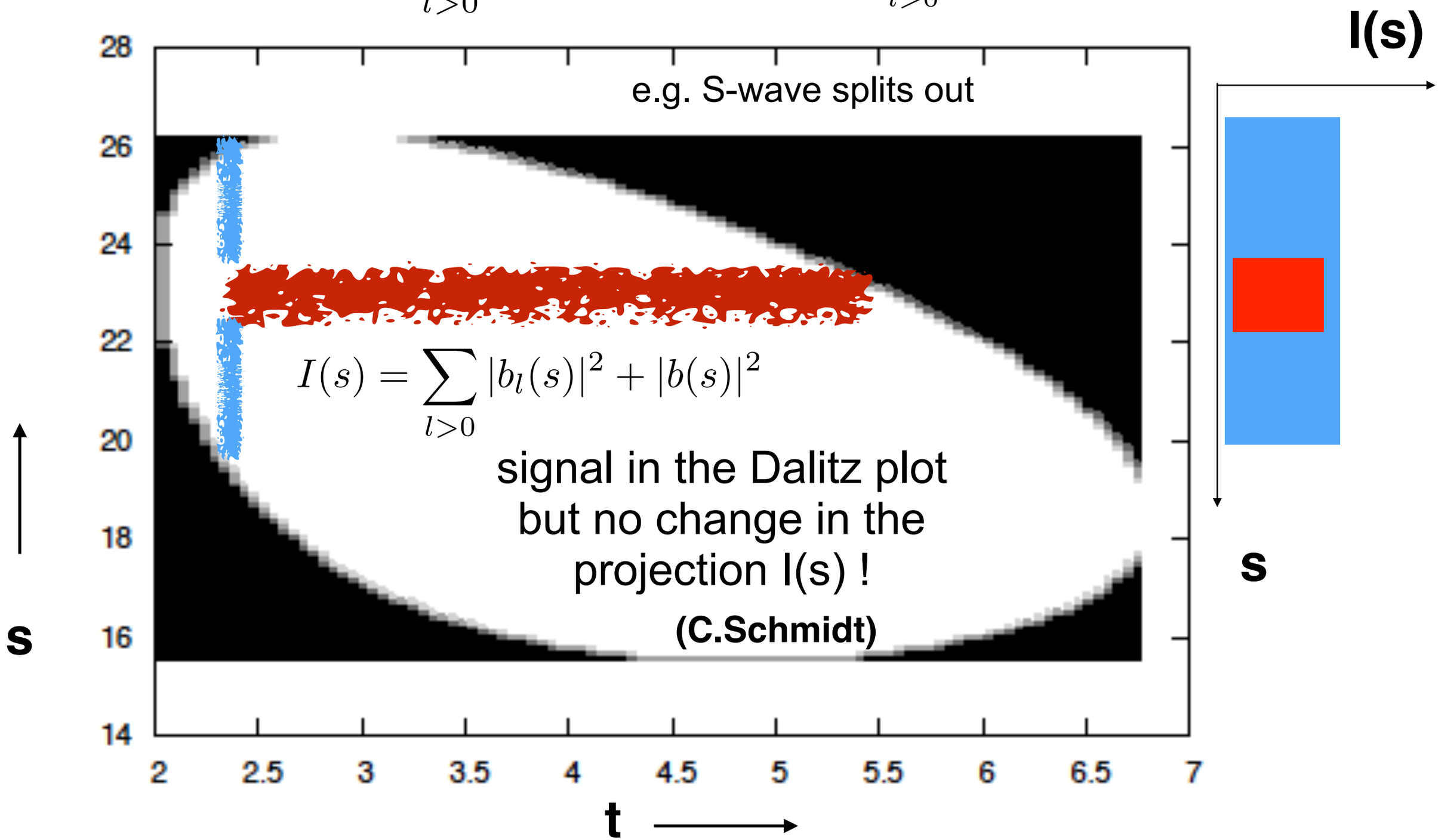




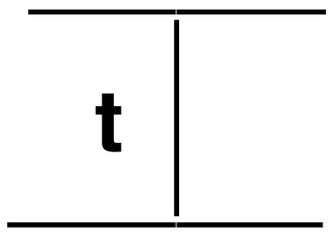
$$b_l(s) = \frac{1}{2}\int_{-1}^{1} dz_s \frac{P_l(z_s)}{m_\Lambda^2 - t(s,z)}$$

Suppose coherence is (somehow) broken

$$|A(t)|^2 = |b(s) + \sum_{l>0} \cdots|^2 \qquad |A(t)|^2 \to |\sum_{l>0} \cdots|^2 + |b(s)|^2 = |A(s,t)|^2$$

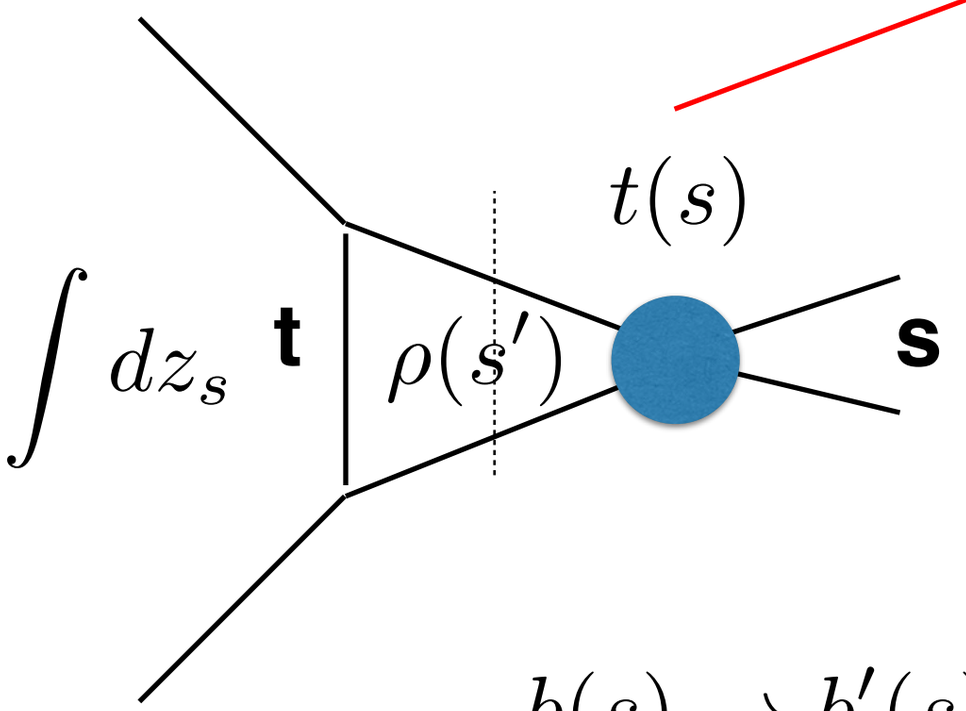


Coherence is broken by final state interactions



$$A(t) = b(s) + \sum_{l>0}(2l+1)b_l(s)P_l(z_s)$$

$$+t(s)\left[\frac{1}{\pi}\int_{s_{tr}} ds'\rho(s')\frac{b(s')}{s'-s}\right]$$



- Determined from unitarity fixing the discontinuity across the s-channel threshold

- The effect is to replace b(s) by b'(s) given by

$$b(s) \to b'(s) = b(s) + t(s)\left[\frac{1}{\pi}\int_{s_{tr}} ds'\rho(s')\frac{b(s')}{s'-s}\right]$$

The net effect is

$$A(t) \to A(s,t) = [A(t) - b(s)] + b'(s)$$
$$= \sum_{l>0} (2l+1) b_l(s) P_l(z_s) + b'(s)$$

- **Dalitz plot distribution changes**

$$I(s,t) = |A(t)|^2 \neq |A(s,t)|^2$$

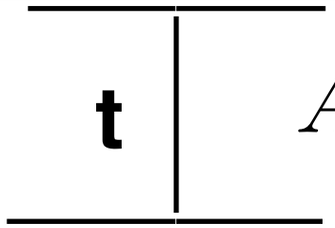
- **Projection changes if** $|b'(s)|^2 \neq |b(s)|^2$

  e.g. inelastic scattering

$$I(s) = \int_{P.S.(s)} dt |A(t)|^2 = \sum_{l>0} |b_l(s)|^2 + |b(s)|^2$$

$$\to \sum_{l>0} |b_l(s)|^2 + |b'(s)|^2$$

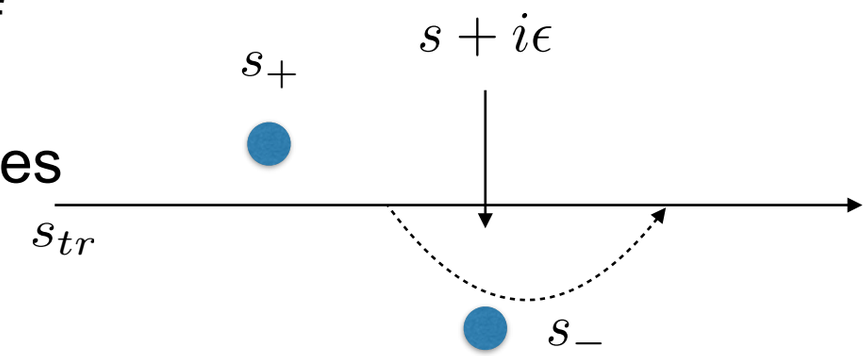
Λ* exchange

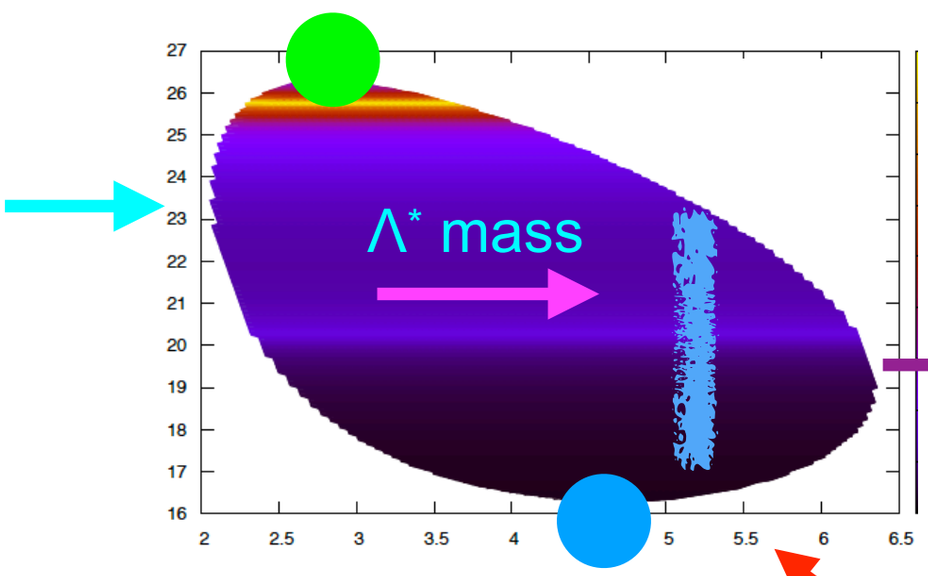


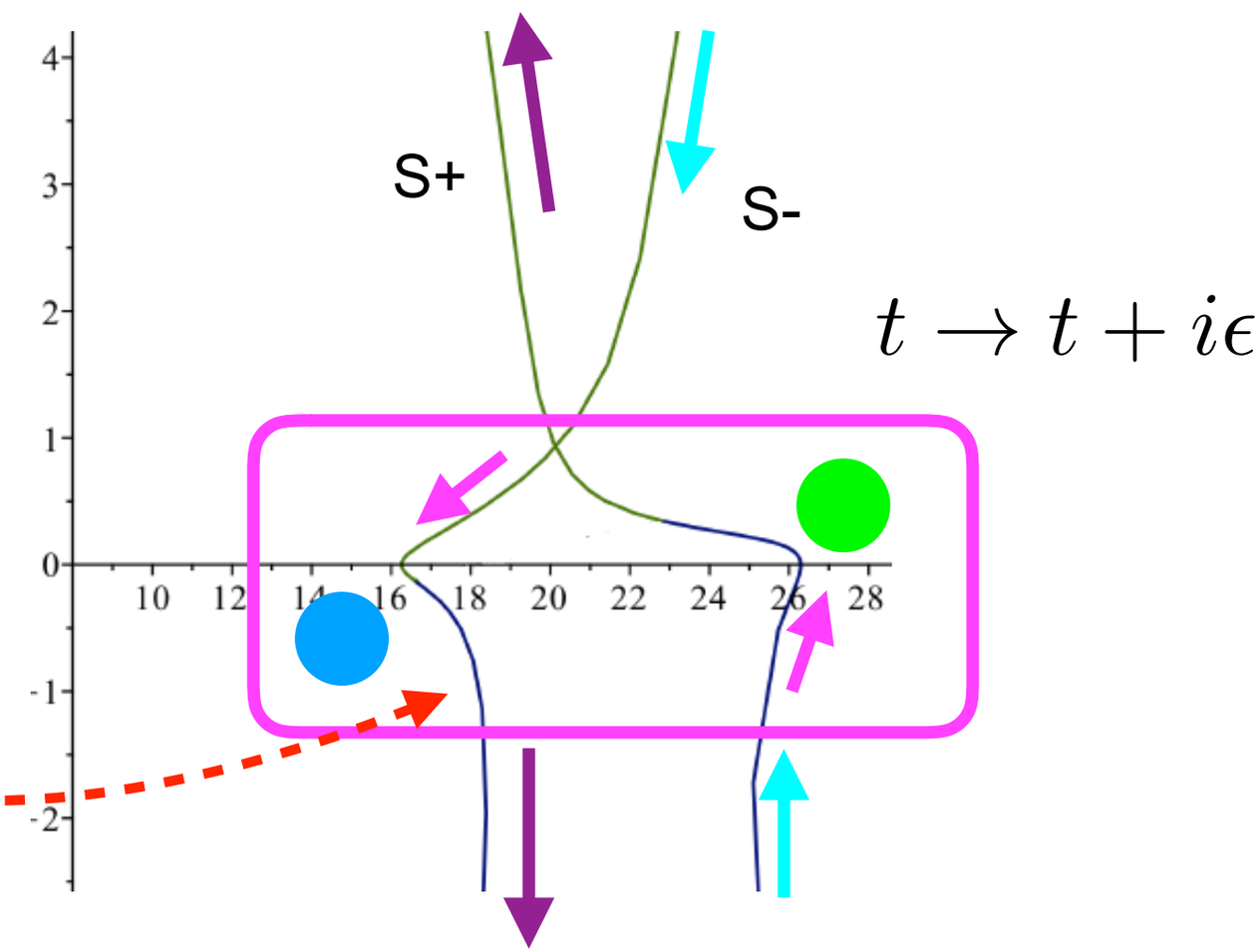
$$A(t) = \frac{1}{m_\Lambda^2 - t} = b(s) + \sum_{l>0}(2l+1)b_l(s)P_l(z_s)$$

$$\frac{1}{\pi}\int_{s_{tr}} ds'\rho(s')\frac{b(s')}{s'-s}$$

- One of the singularities of b(s), (s_-) is close to the physical region and pinches the integration contour



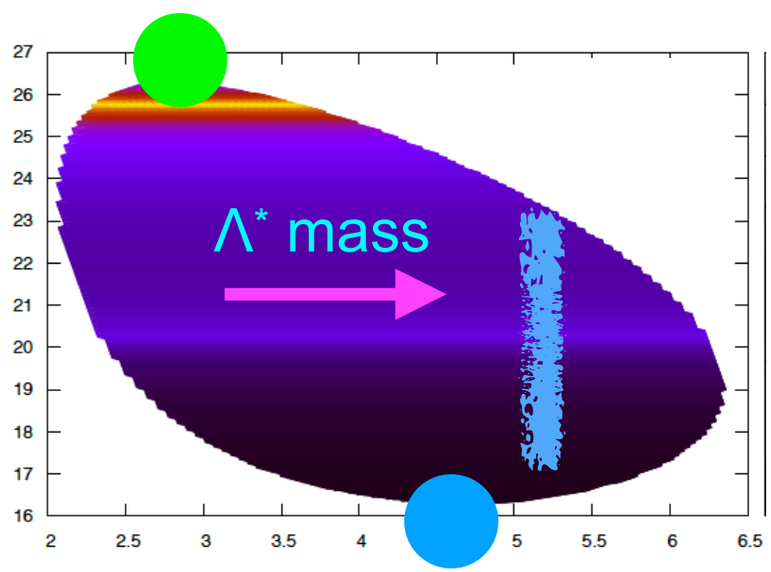


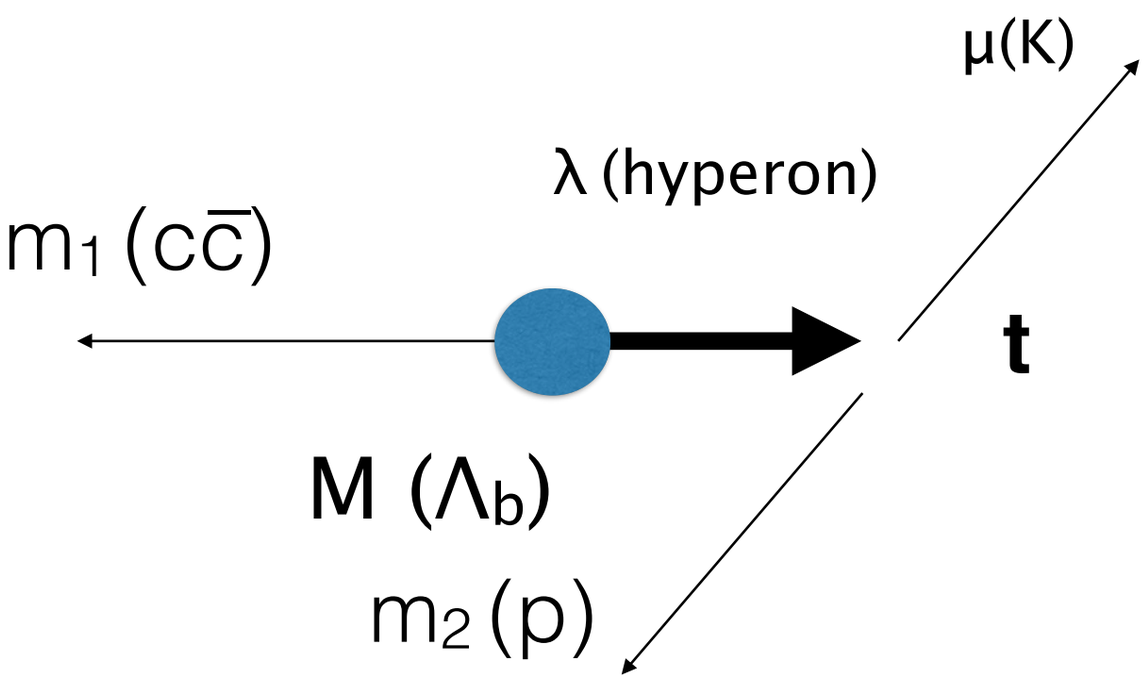


- Λ* Region of interest

Classical picture

Coleman-Norton





t-channel resonance can produce s-channel "band" if:

all particles on-shell

m_2 and m_1 collinear

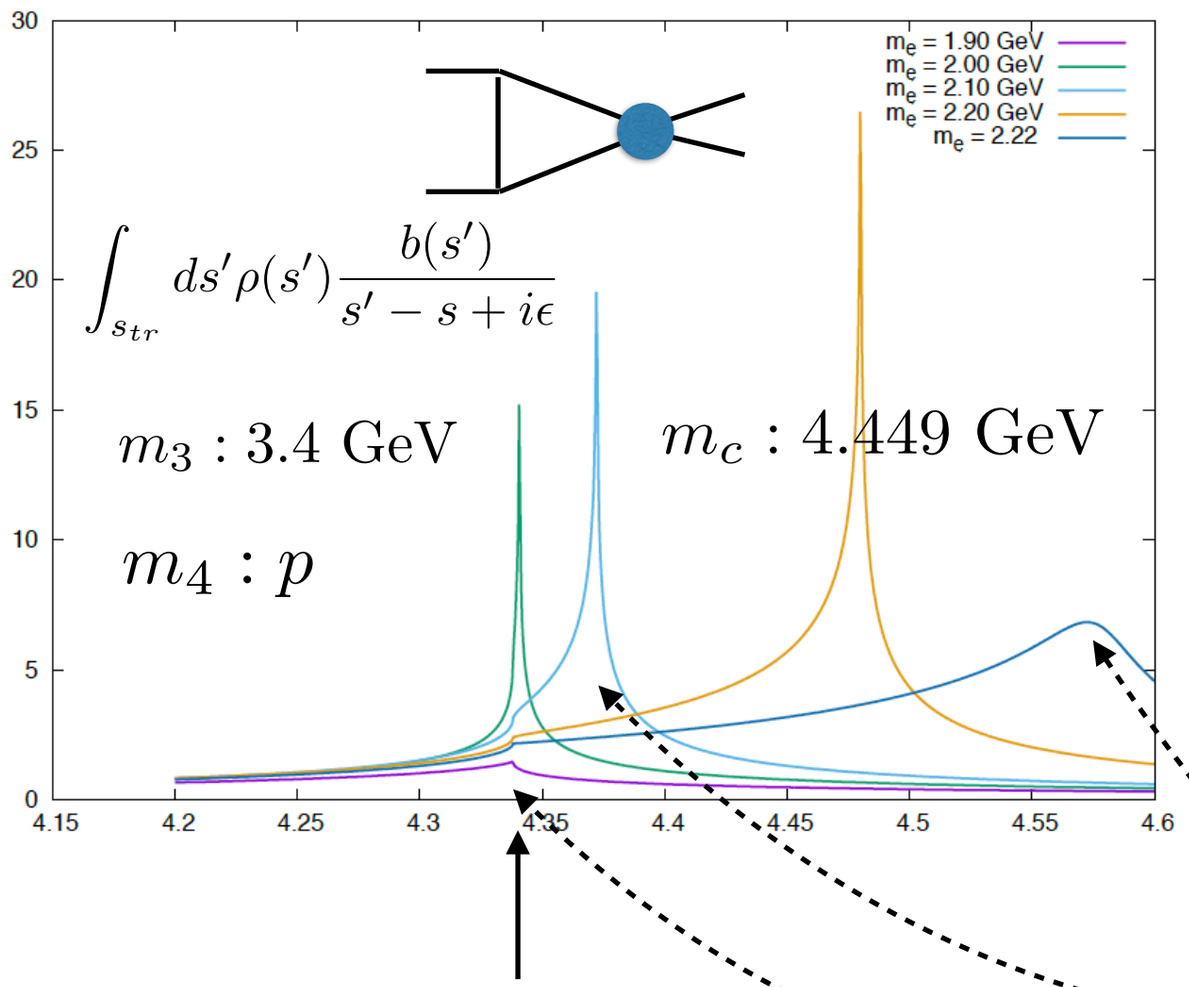
$v(m_2) > v(m_1)$

Example : Pc Kinematics

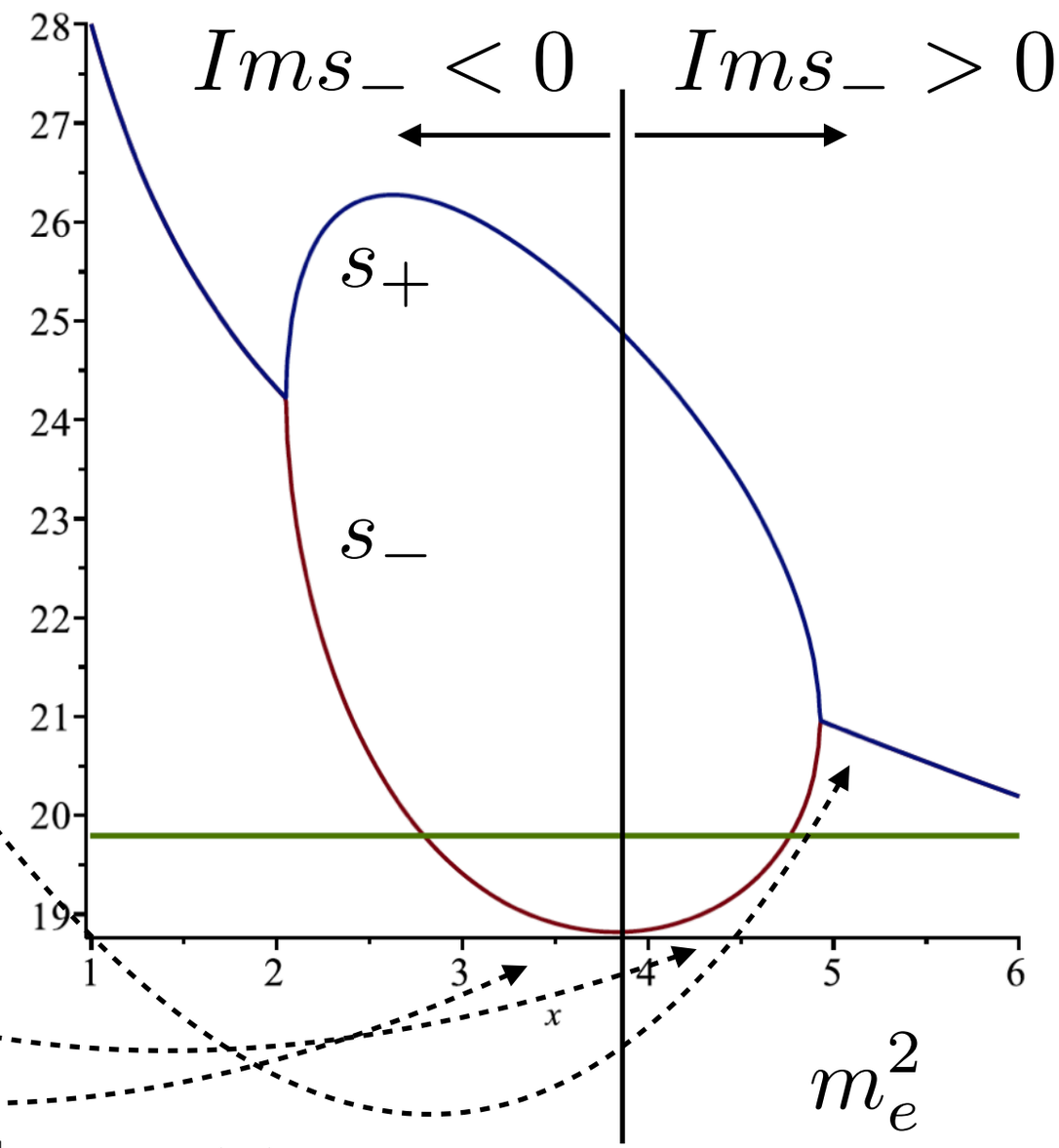
m_1	m_3
m_2	m_4

$$s_\pm = -m_e^2 + p_2^2 + p_3^2 + \frac{(m_e^2 + p_1^2 - p_3^2)(m_e^2 + p_4^2 - p_2^2)}{2m_e^2} \pm \frac{\lambda^{1/2}(m_e^2, p_1^2, p_3^2)\lambda^{1//2}(m_e^2, p_2^2, p_4^2)}{2m_e^2}$$

$$m_1 : \Lambda_b \quad m_2 : K$$



$m_3 + m_1$ threshold

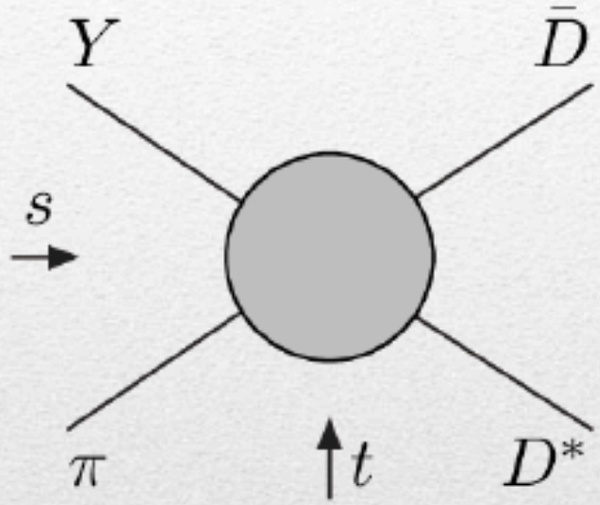


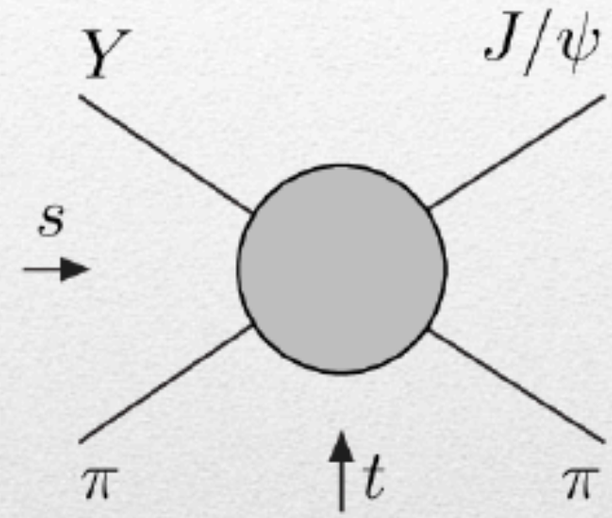
- Singularities of b(s) are at $s=s_\pm$ $\quad b_l(s) = \frac{1}{2}\int_{-1}^{1} dz_s \frac{P_l(z_s)}{m_\Lambda^2 - t(s,z)}$

Amplitude analysis for $Z_c(3900)$

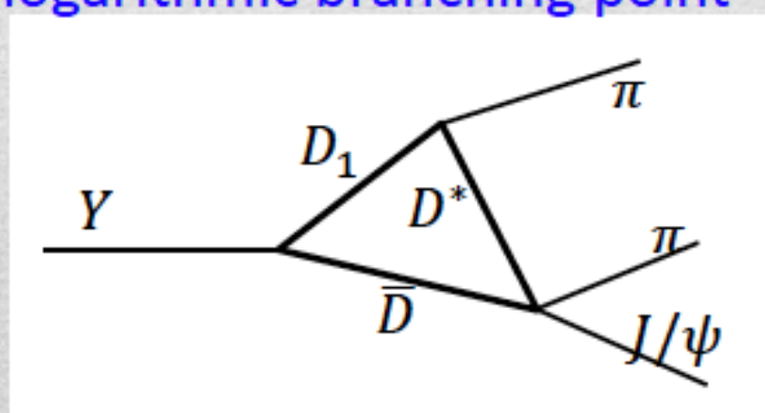
One can test different parametrizations of the amplitude, which correspond to different singularities → different natures

AP *et al.* (JPAC), arXiv:1612.06490





Triangle rescattering,
logarithmic branching point

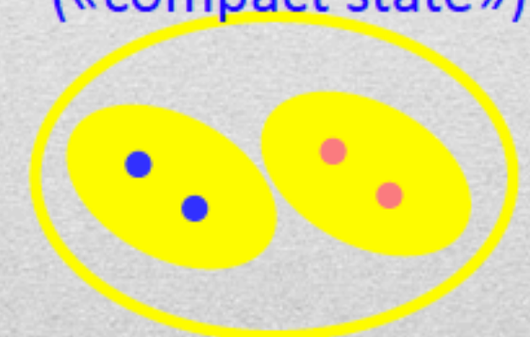


Szczepaniak, PLB747, 410-416
Szczepaniak, PLB757, 61-64
Guo *et al.* PRD92, 071502

(anti)bound state,
II/IV sheet pole
(«molecule»)

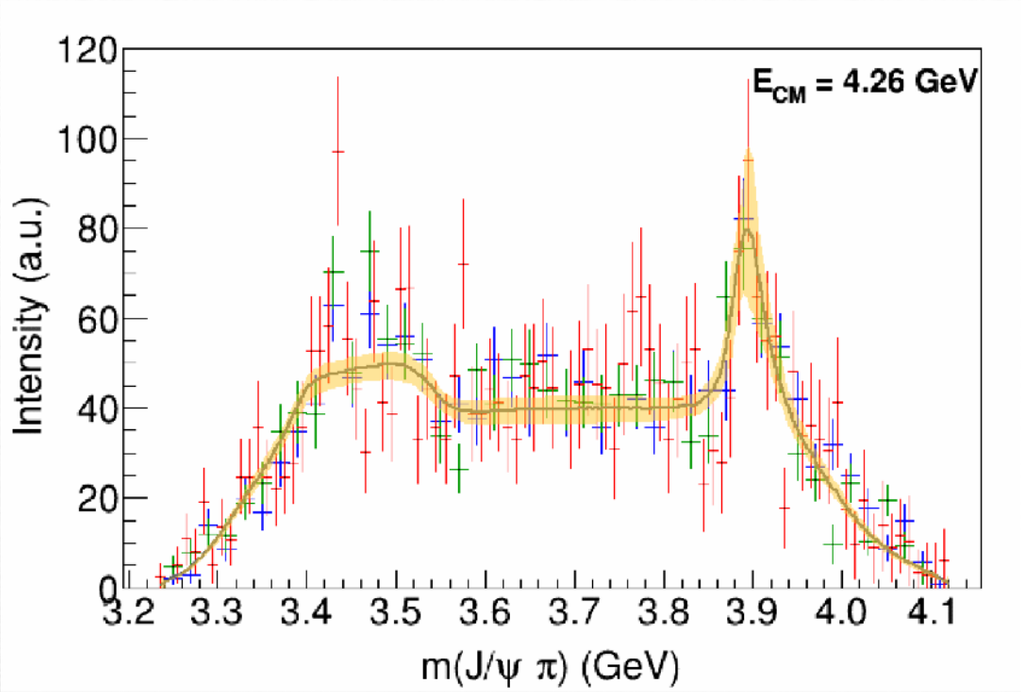
Tornqvist, Z.Phys. C61, 525
Swanson, Phys.Rept. 429
Hanhart *et al.* PRL111, 132003

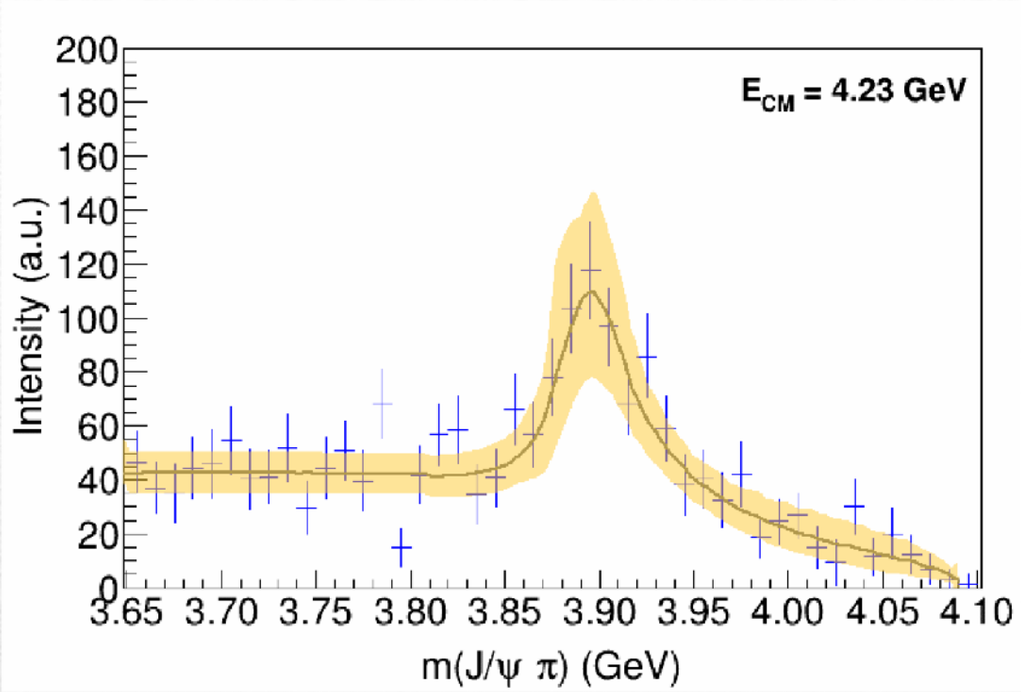
Resonance,
III sheet pole
(«compact state»)

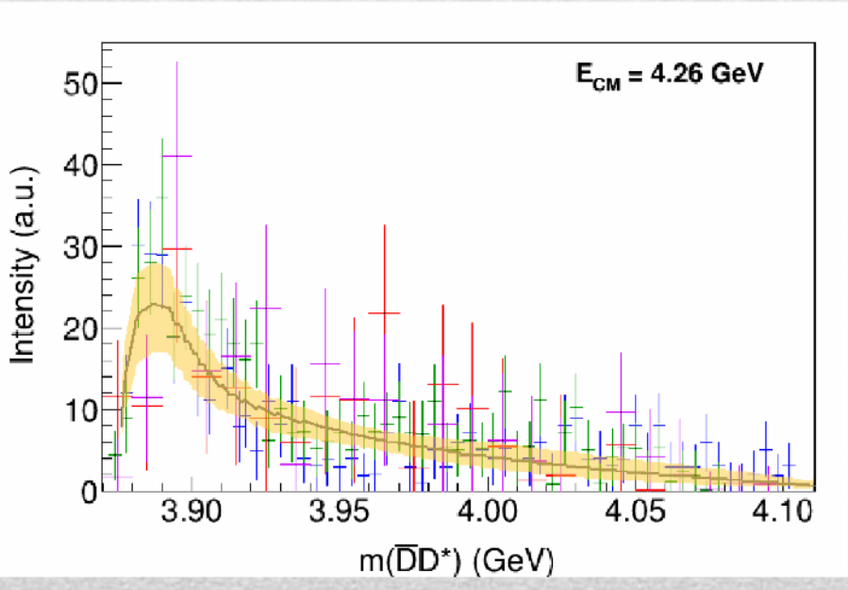


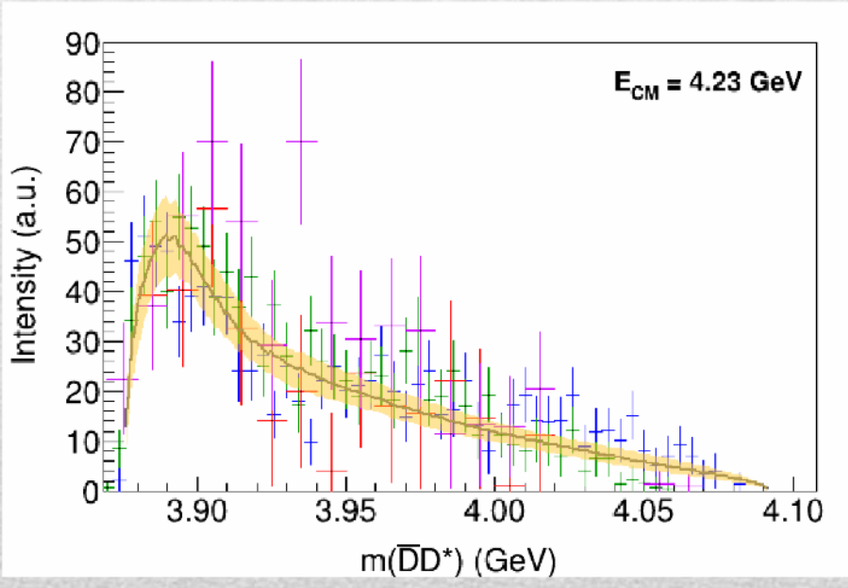
Maiani *et al.*, PRD71, 014028
Faccini *et al.*, PRD87, 111102
Esposito *et al.*, Phys.Rept. 668

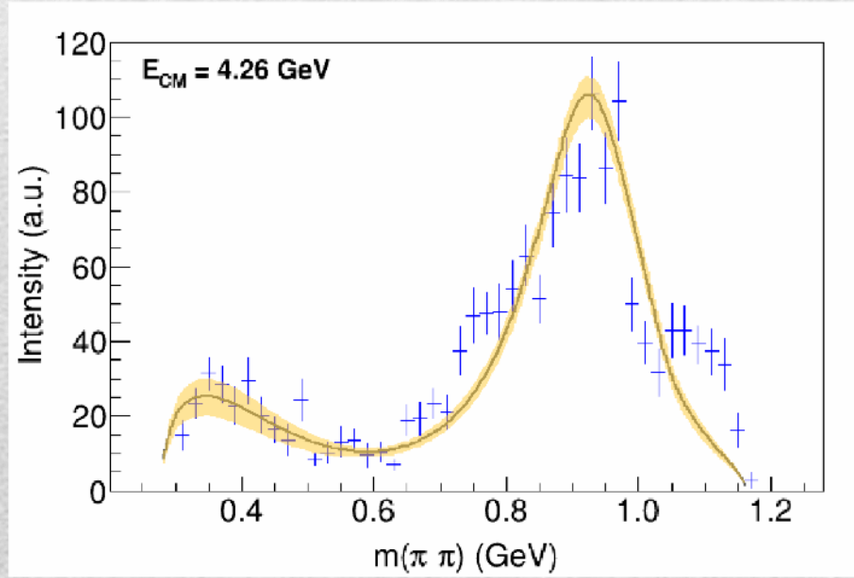
Fit: III


E_{CM} = 4.26 GeV
Intensity (a.u.)
m(J/ψ π) (GeV)

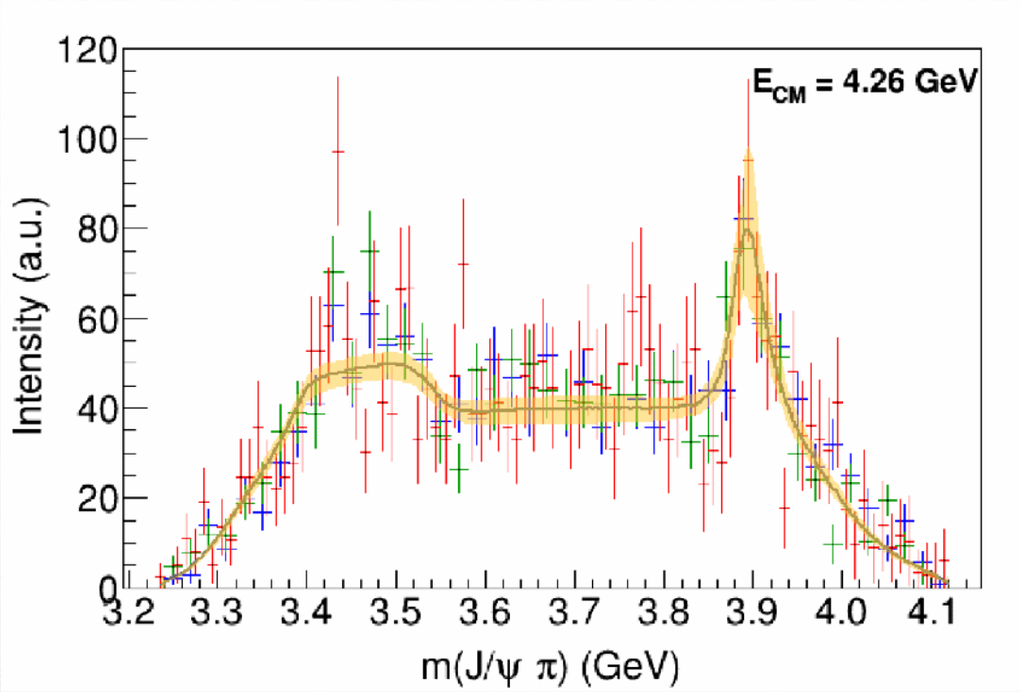

E_{CM} = 4.23 GeV
Intensity (a.u.)
m(J/ψ π) (GeV)

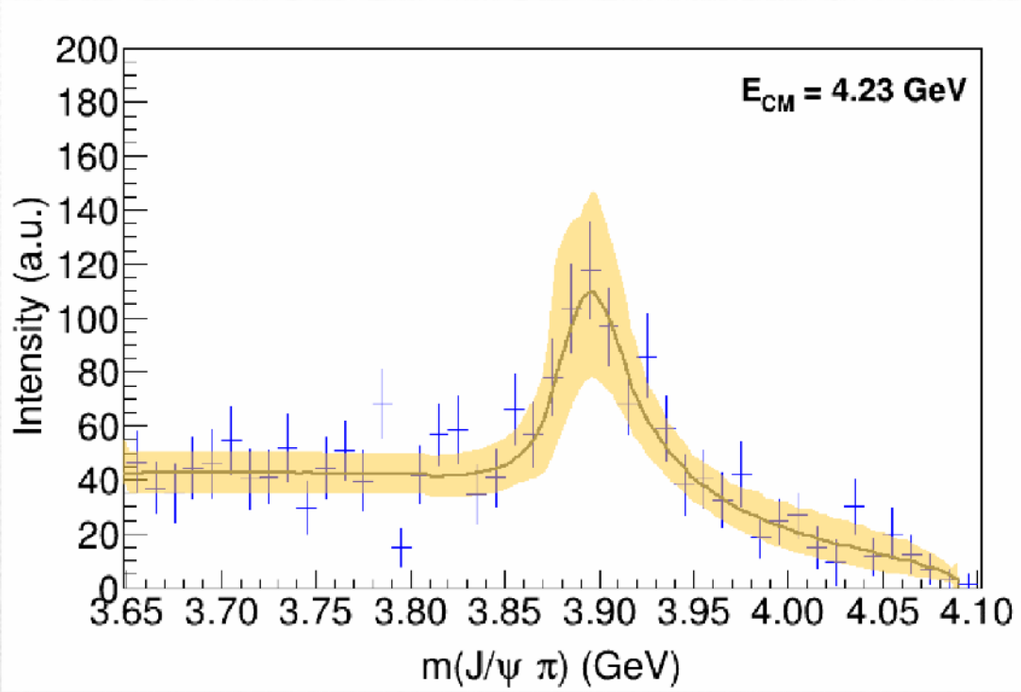

E_{CM} = 4.26 GeV
Intensity (a.u.)
m($\bar{D}D^*$) (GeV)


E_{CM} = 4.23 GeV
Intensity (a.u.)
m($\bar{D}D^*$) (GeV)


E_{CM} = 4.26 GeV
Intensity (a.u.)
m(π π) (GeV)

Fit: III


E_{CM} = 4.26 GeV
Intensity (a.u.)
m(J/ψ π) (GeV)


E_{CM} = 4.23 GeV
Intensity (a.u.)
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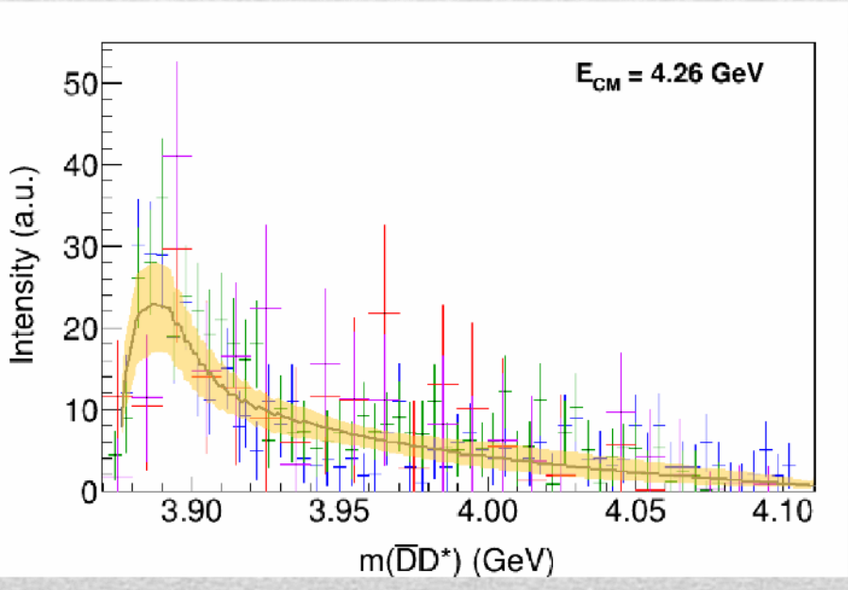
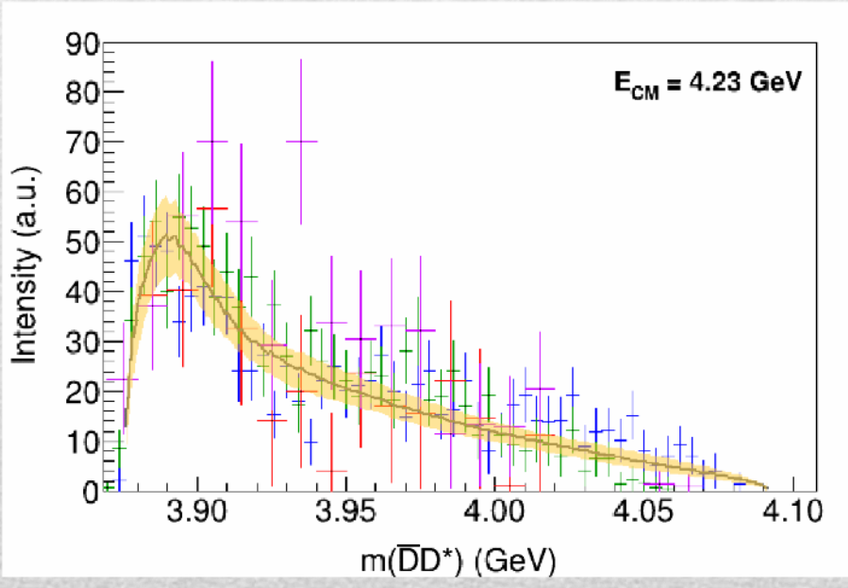

E_{CM} = 4.26 GeV
Intensity (a.u.)
m($\bar{D}D^*$) (GeV)


E_{CM} = 4.23 GeV
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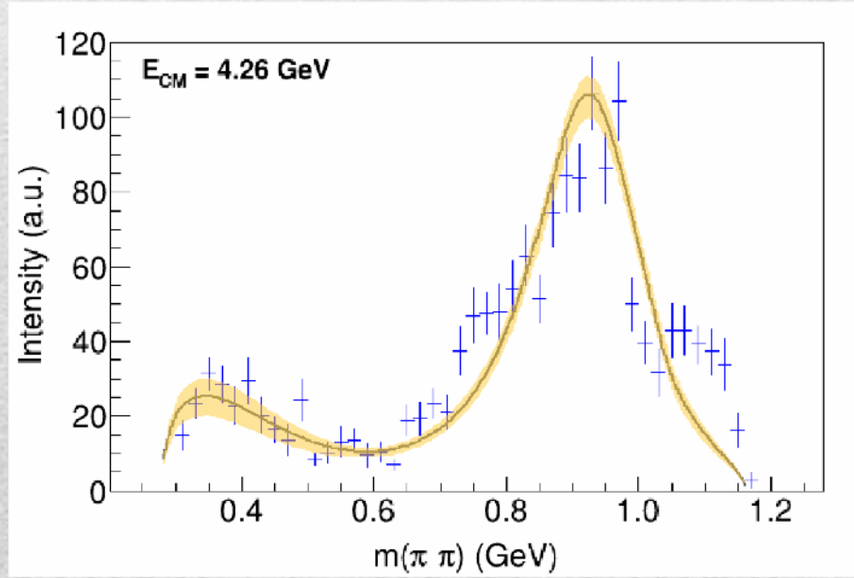

E_{CM} = 4.26 GeV
Intensity (a.u.)
m(π π) (GeV)

Possible tests in photo production

Potential for cusps in

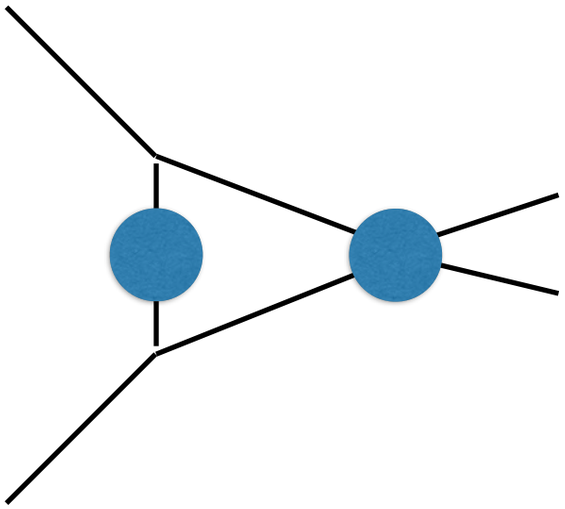
$$\gamma p \to X p \to \omega \pi \pi p$$

$$\gamma p \to X p \to \phi \pi \pi p$$

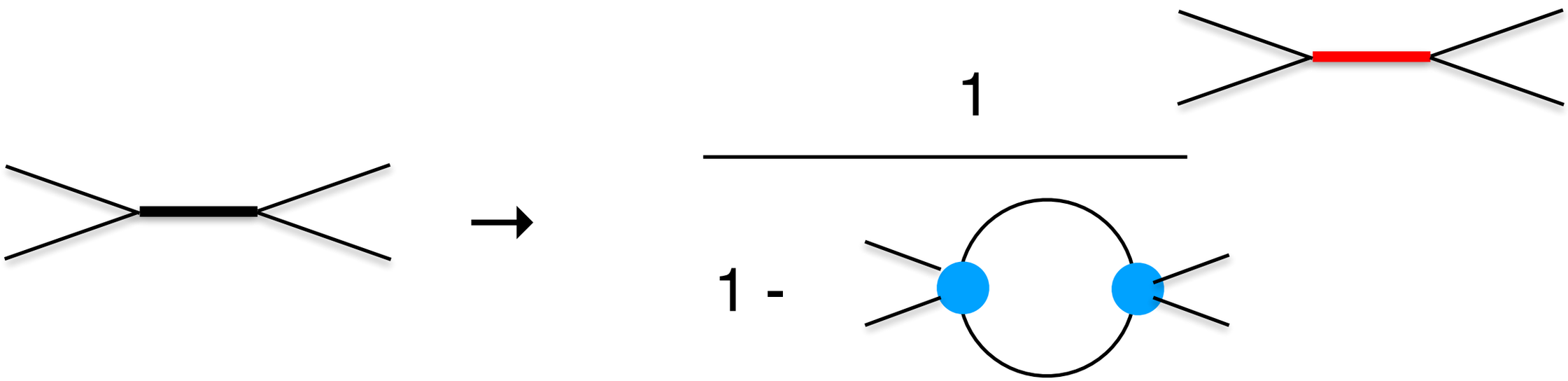
$$\gamma p \to X p \to \phi K \bar{K} p$$

Etc.

If not triangle (exchange) singularity then what?



- Thresholds are not responsible for cusps. They are windows to 2nd sheet singularities
- Peaks come from either cross (cusps), or direct channel singularity (resonances)



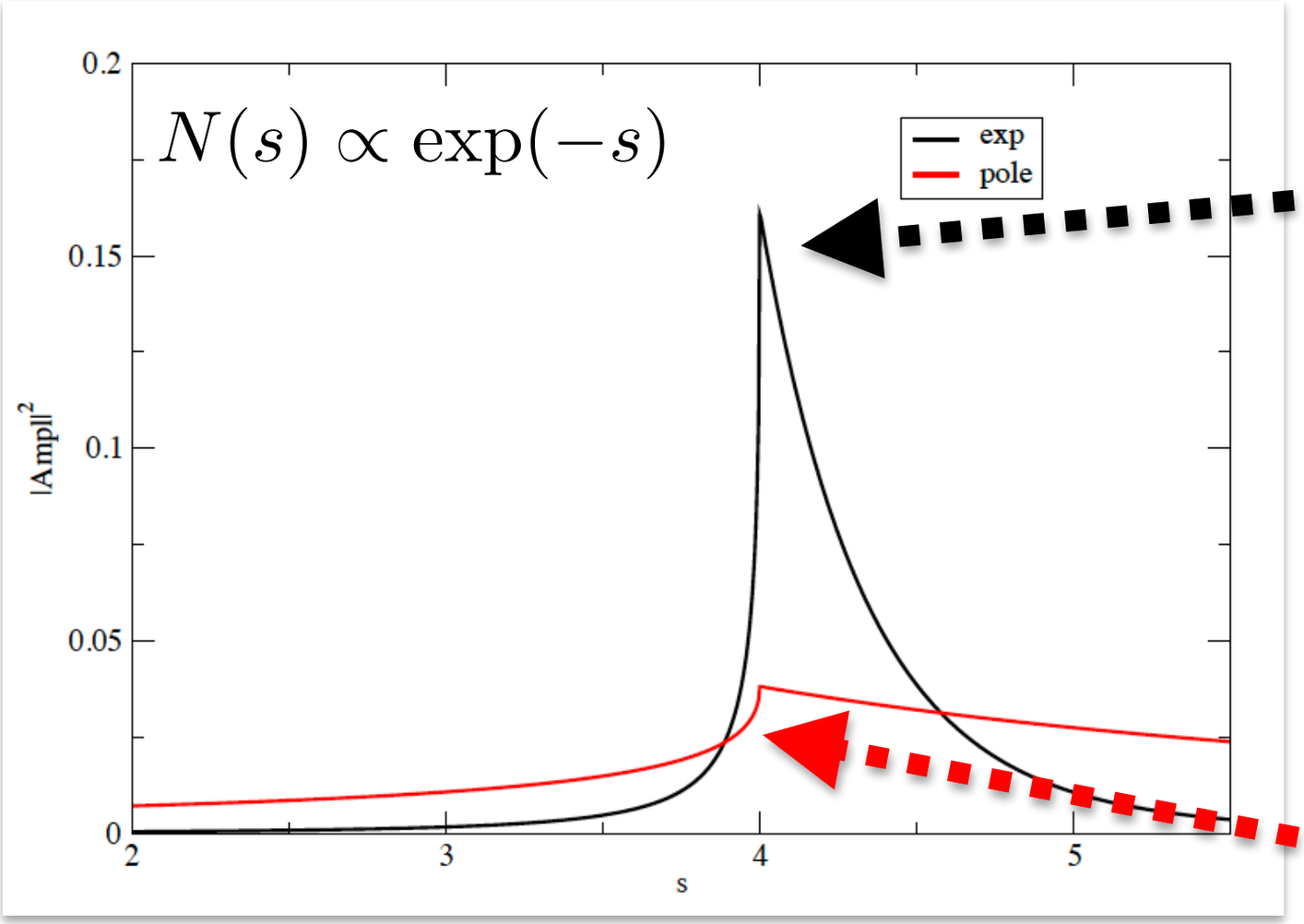
$$\frac{1}{1 - \bigcirc}$$

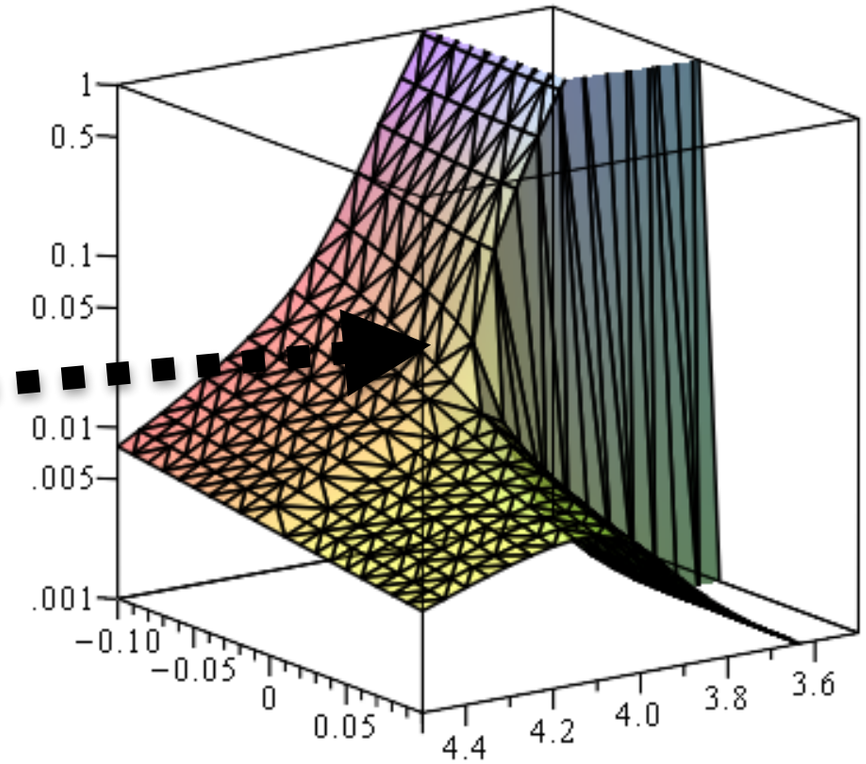
- The form factor model is neither one!

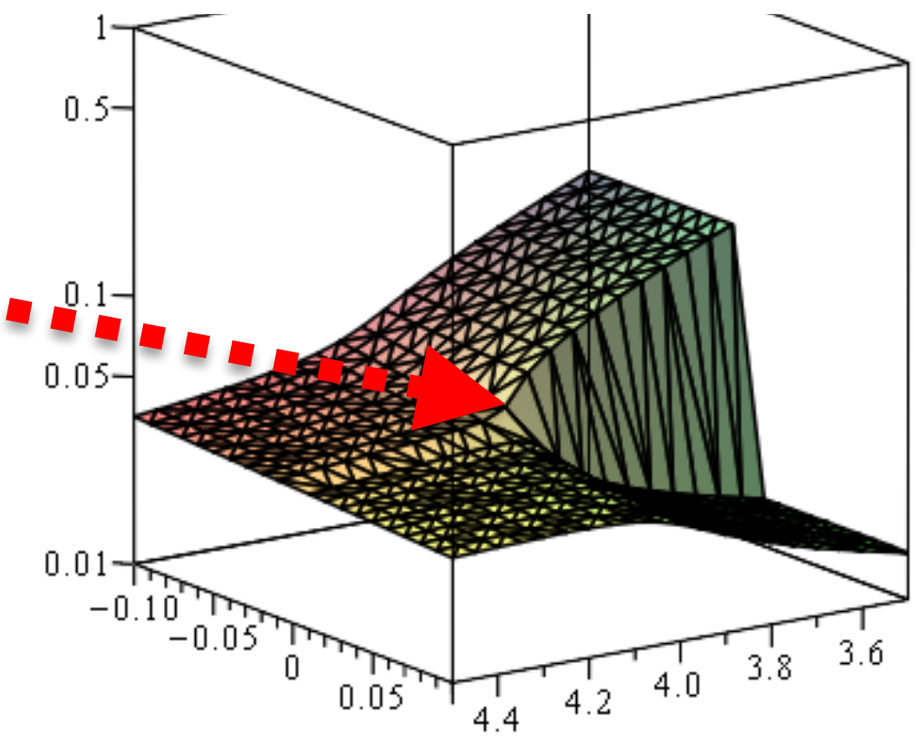
$$\bullet = N(s) \sim exp(-s/\Lambda^2)$$

E.Swanson, Phys.Rev.D91, 034009 (2015)

$$\text{Ampl} \propto \int_4 ds' \sqrt{1 - \frac{4}{s'}} \frac{N(s')}{s' - s}$$

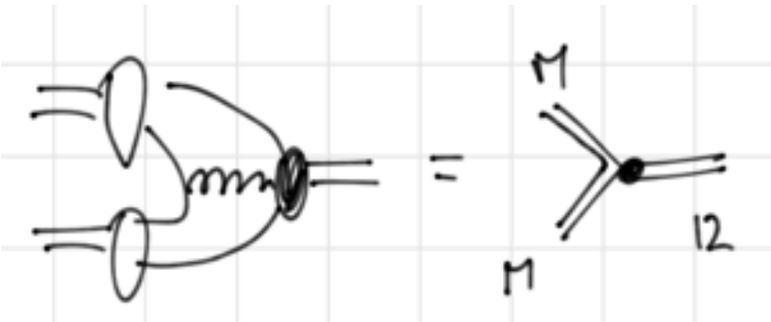


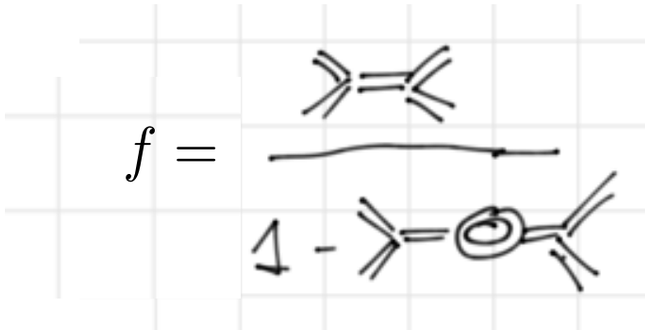


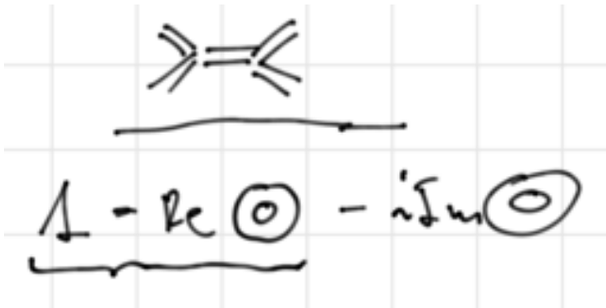


- The cusp is generated by singularity at infinity
- Are such singularities physical ?

Scattering through resonances







$$f(s) = \frac{1}{K^{-1}(s) - i\Gamma(s)}$$

∞ number of poles : confinement

$K^{-1}(s)$ needs to have ∞ number of poles (K(s) needs zeros)

Quadratically spaced radial trajectories

$$K(s) = \sum_{r=1}^{\infty} \frac{g_r^2}{m_r^2 - s} \rightarrow \sum_r \frac{1}{r^2 - s} \sim \frac{\cos(\pi\sqrt{s})}{\sin(\pi\sqrt{s})}$$

Linearly spaced radial trajectories (Veneziano)

$$K(s) \sim \frac{\Gamma(a-s)}{\Gamma(b-s)}$$

Exponential form factors related to infinite number of particles (confinement) !

Veneziano model for Dalitz plot analysis

$$M = \epsilon_{\mu\nu\alpha\beta} p_1^\mu p_2^\nu p_3^\alpha \epsilon^\beta A(s,t,u) \qquad A = \sum_{n,m} c_{n,m} \left[\frac{\Gamma(n-\alpha(s))\Gamma(n-\alpha(t))}{\Gamma(n+m-\alpha(s)-\alpha(t))} + (s,u) + (t,u) \right]$$

- V → PPP (ω, J/ψ → 3π)

- In the past, fit c(n,m)'s to the data. Need various conspiracy relations. (e.g. Lovelace, Phys. Lett. B28, 265 (1968), Altarelli, Rubinstein, Phys. Rev. 183, 1469 (1969))

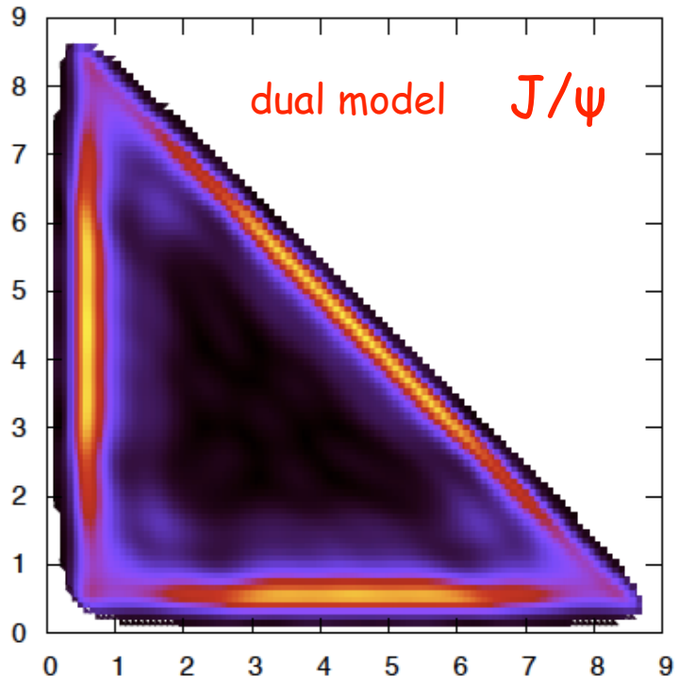
$$\mathcal{A}_n(s,t;N) = \frac{2n-\alpha_s-\alpha_t}{(n-\alpha_s)(n-\alpha_t)} \sum_{i=1}^{n} a_{n,i}(-\alpha_s-\alpha_t)^{i-1} \times \frac{\Gamma(N+1-\alpha_s)\Gamma(N+1-\alpha_t)}{\Gamma(N+1-n)\Gamma(N+n+1-\alpha_s-\alpha_t)}.$$

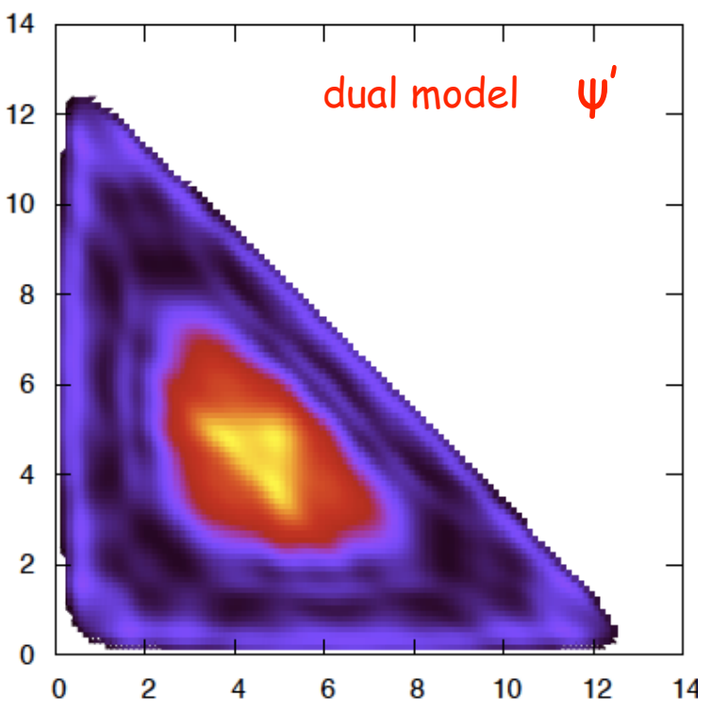
- Allows for imaginary non-linear (and complex) trajectories without introducing "ancestors"

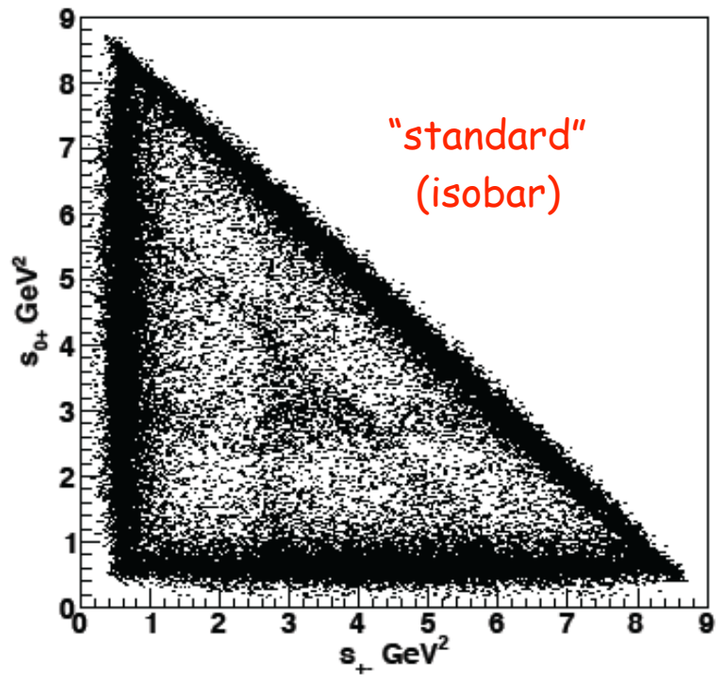
A.S., M.R. Pennington, Phys .Lett. B737, 283 (2014).

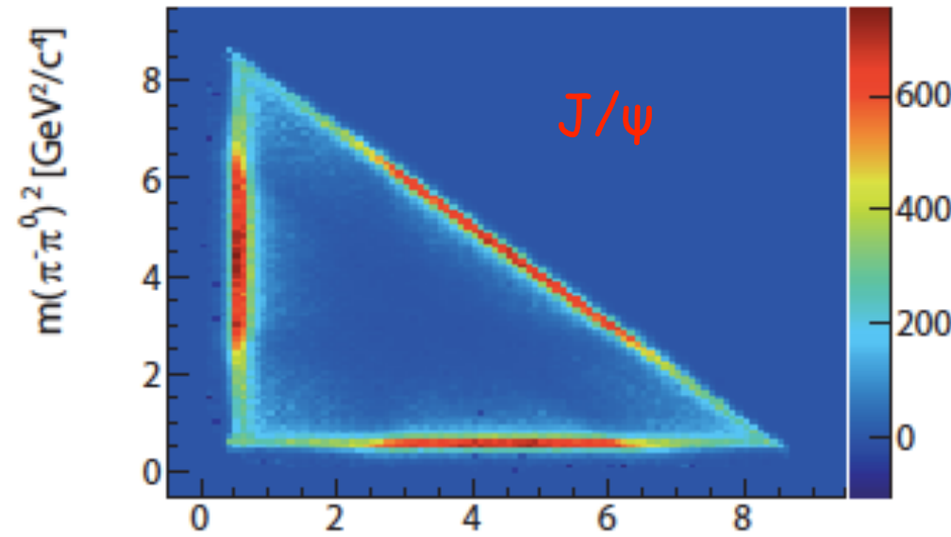
n: number of Regge trajectories
a(n,i): determine resonance couplings
N: determines the onset of Regge behavior
α(s), α(t) = Re α + i Im α: with Im α related to resonance widths

Speaking of the Veneziano amplitude

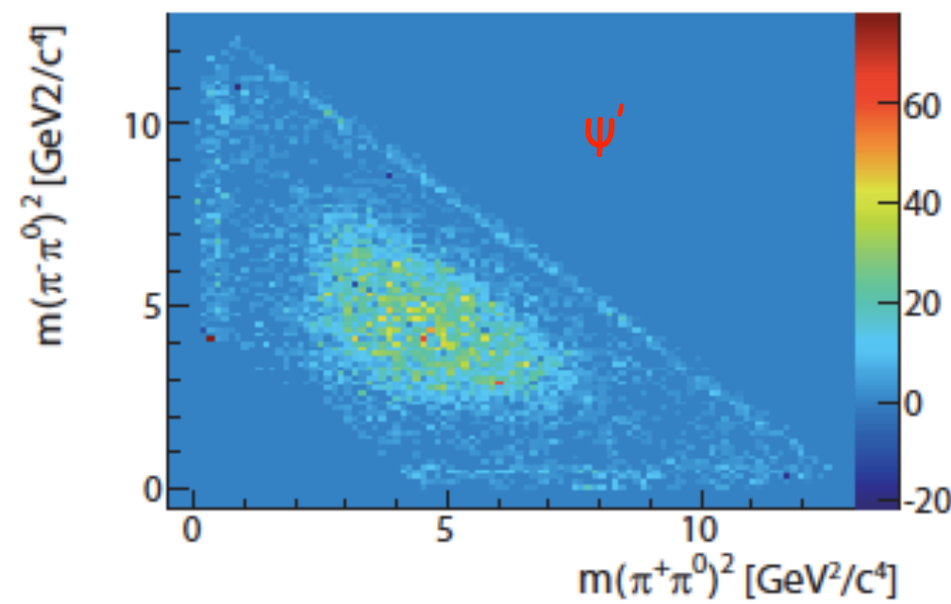








BESIII, Phys.Lett.
B710 (2012) 594-599



Summary

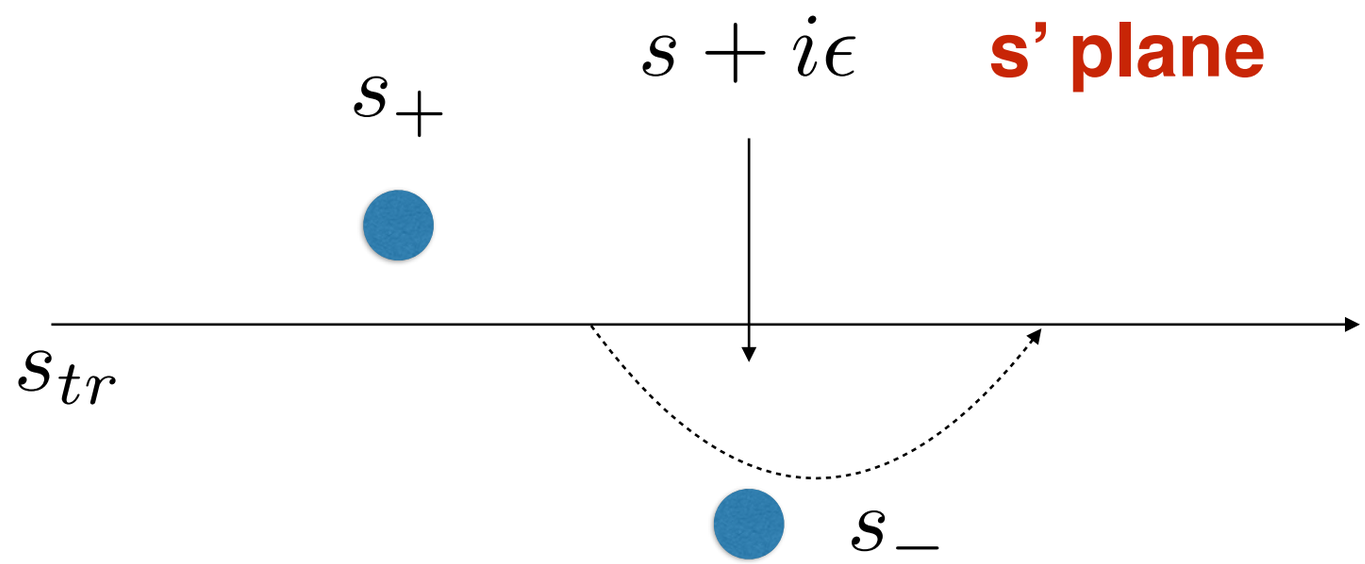
1. Cusps are possible, if kinematics cooperates. In principle systematic approach possible, but …

— f.s.i amplitudes ? (lattice?)

— multiple coupled channels ?

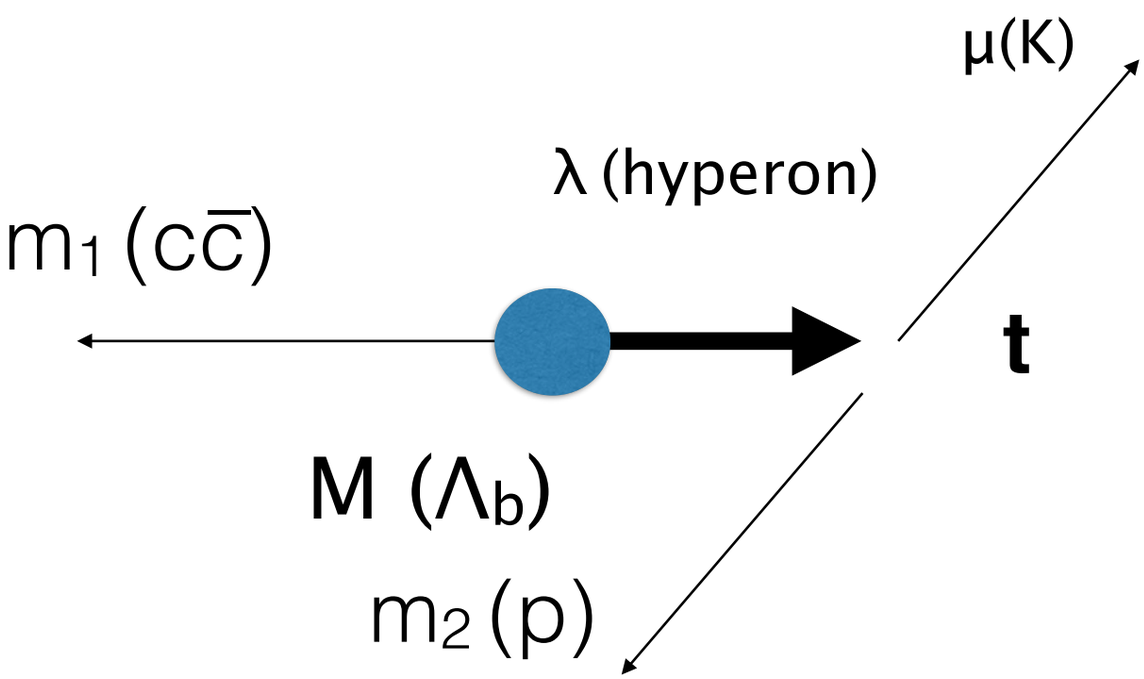
— include in data fits.

$$\frac{1}{\pi}\int_{s_{tr}} ds'\rho(s')\frac{b(s')}{s'-s}$$



One of the singularities of b(s), (s_-) is close to the physical region and pinches the integration contour

Coleman-Norton theorem



t-channel resonance can produce s-channel "band" if:

all particles on-shell

m_2 and m_1 collinear

$v(m_2) > v(m_1)$

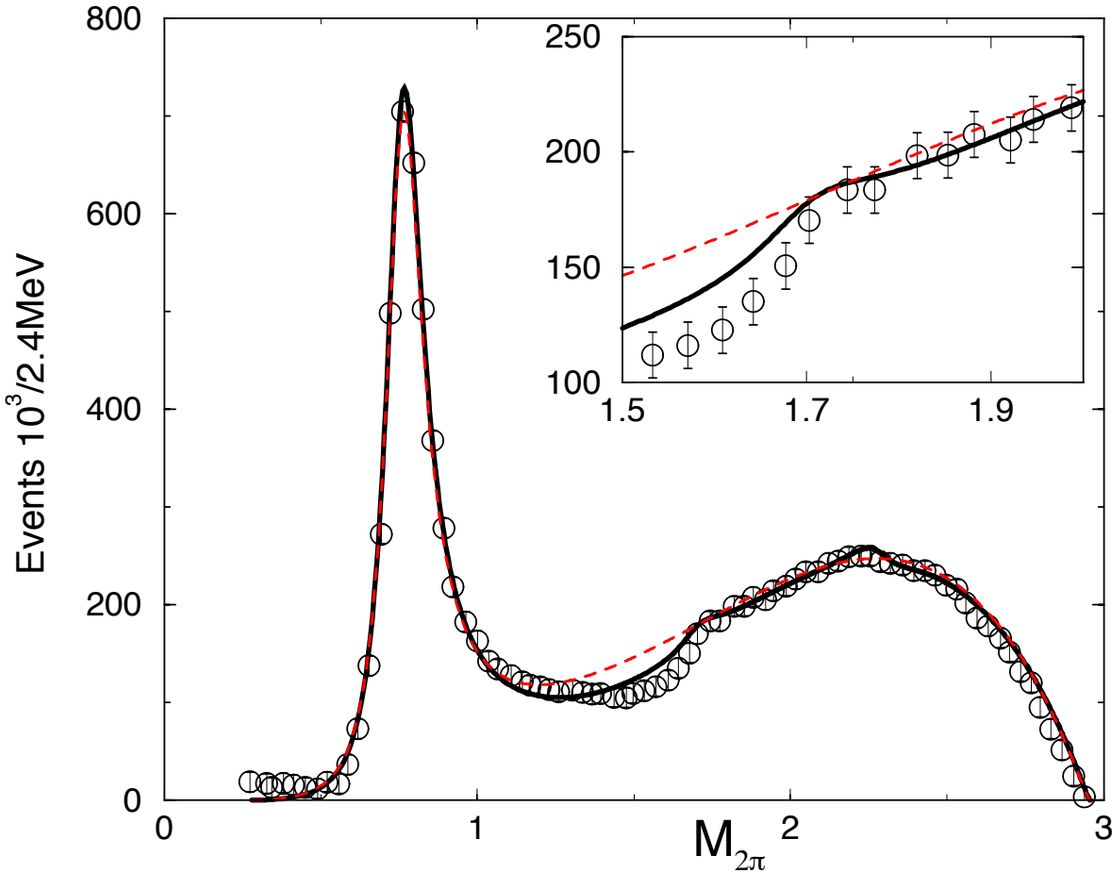


FIG. 2: Dalitz plot projection of the di-pion mass distribution from J/ψ decay. The solid is the result of the fit with three amplitudes and the dashed line with the amplitude $\mathcal{A}_1$ alone. The insert shows the mass region of the ρ_3 and its contribution from the fit with the full set of amplitudes (solid line) as compared. Absence of the structure at 1.7GeV from the fit with the $\mathcal{A}_1$ amplitude is indicated by the dashed line.

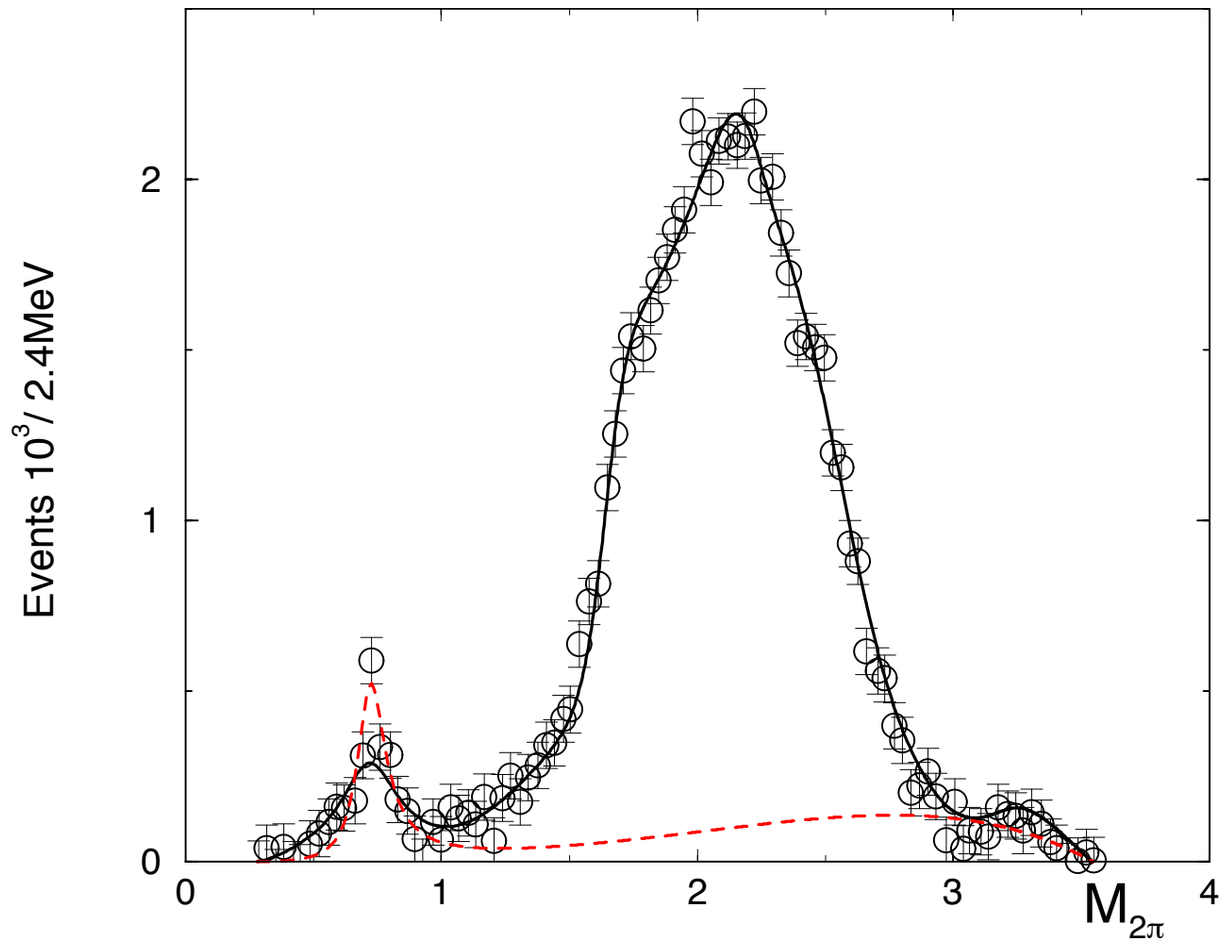
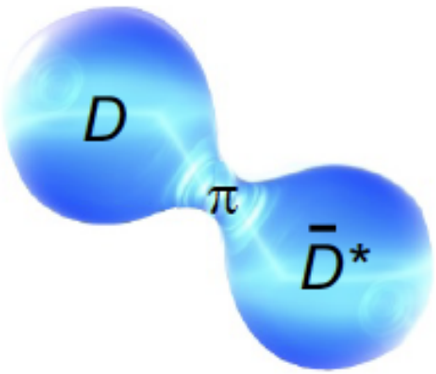
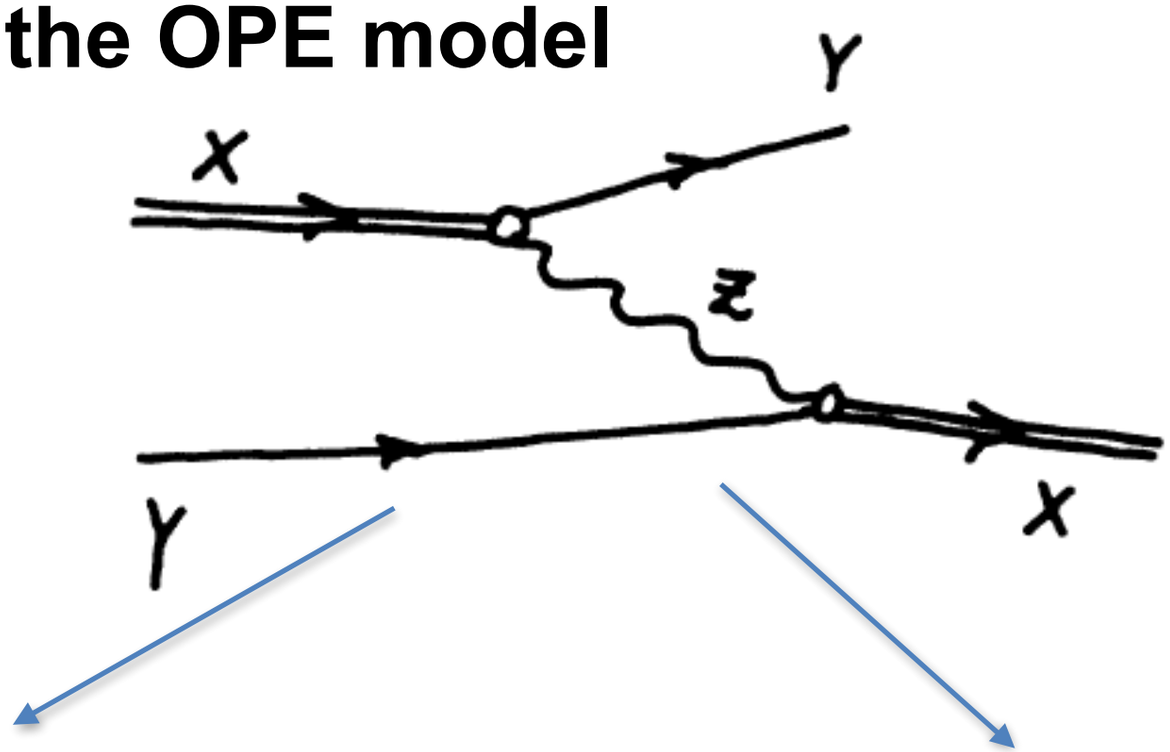


FIG. 3: Dalitz plot projection of the di-pion mass distribution from ψ' decay. The solid is the result of the fit with three amplitudes and the dashed line is the fit with $\mathcal{A}_1$ alone.

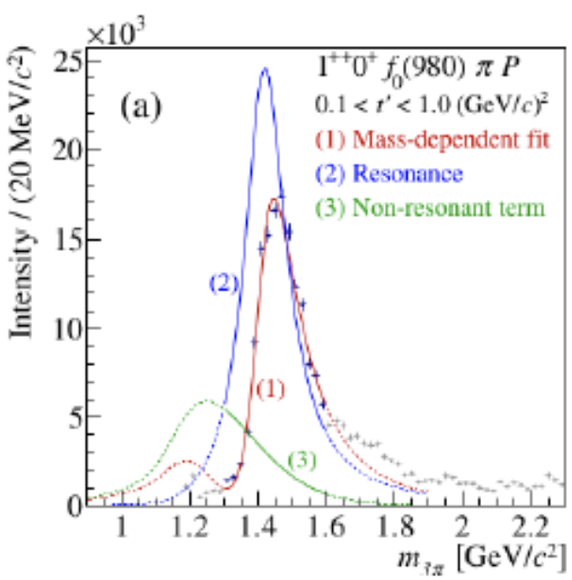
Difficulties with the OPE model

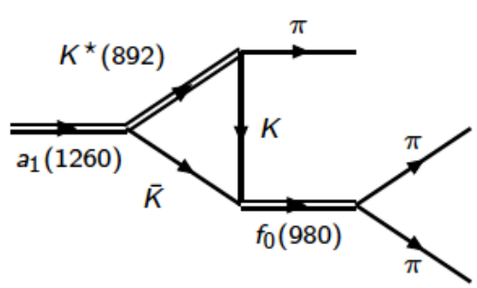




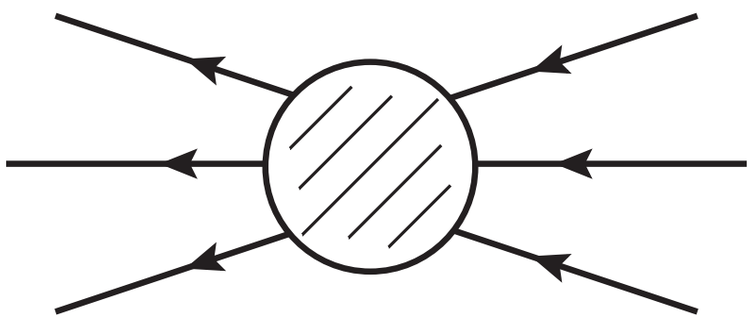
Virtual One Particle Exchange (potential)

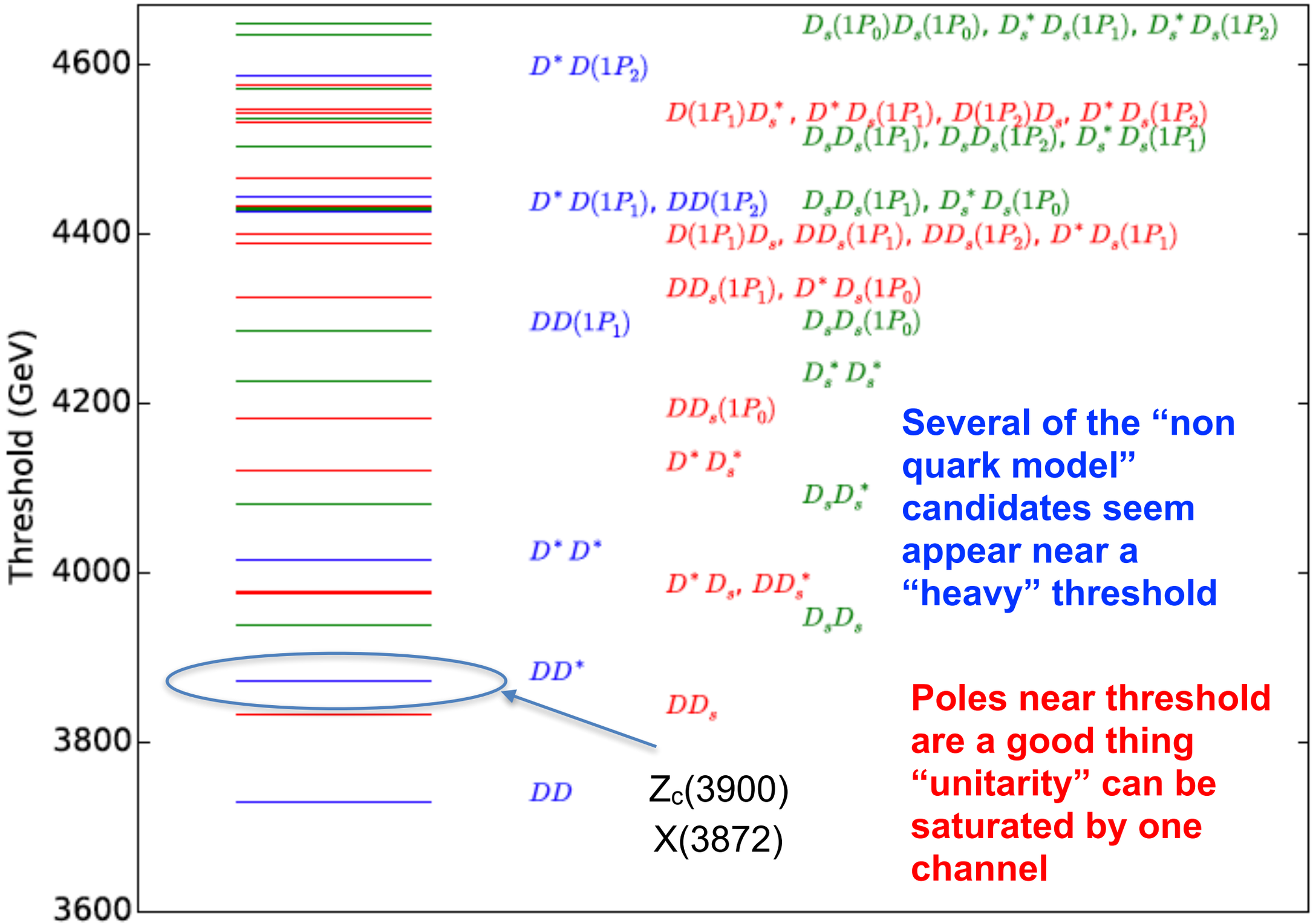
Real One Partice Exchange (3 particle intermediate state)





M. Mikhasenko,B. Ketzer,A.Sarantsev





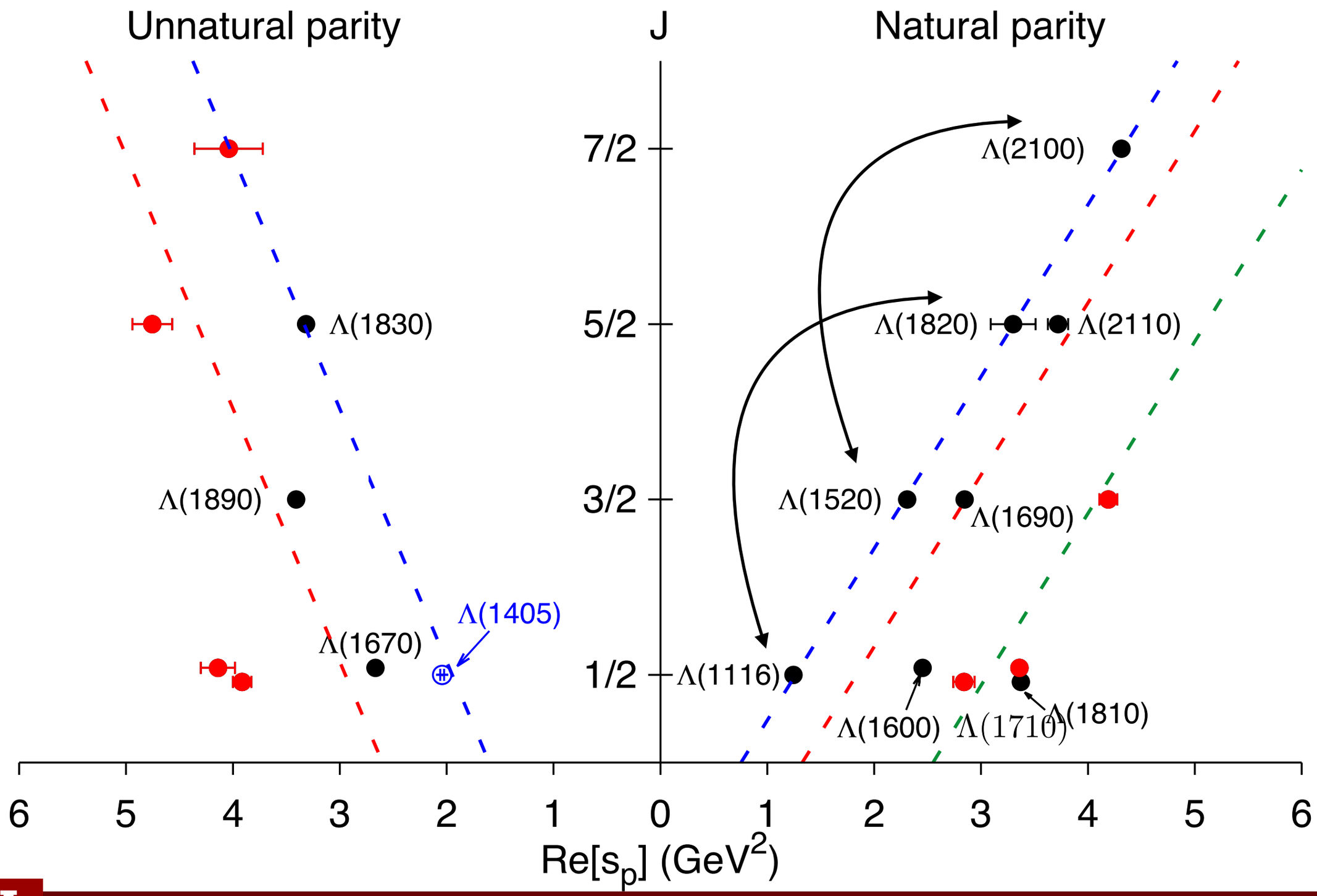
Several of the "non quark model" candidates seem appear near a "heavy" threshold

Poles near threshold are a good thing "unitarity" can be saturated by one channel

E.Eichten

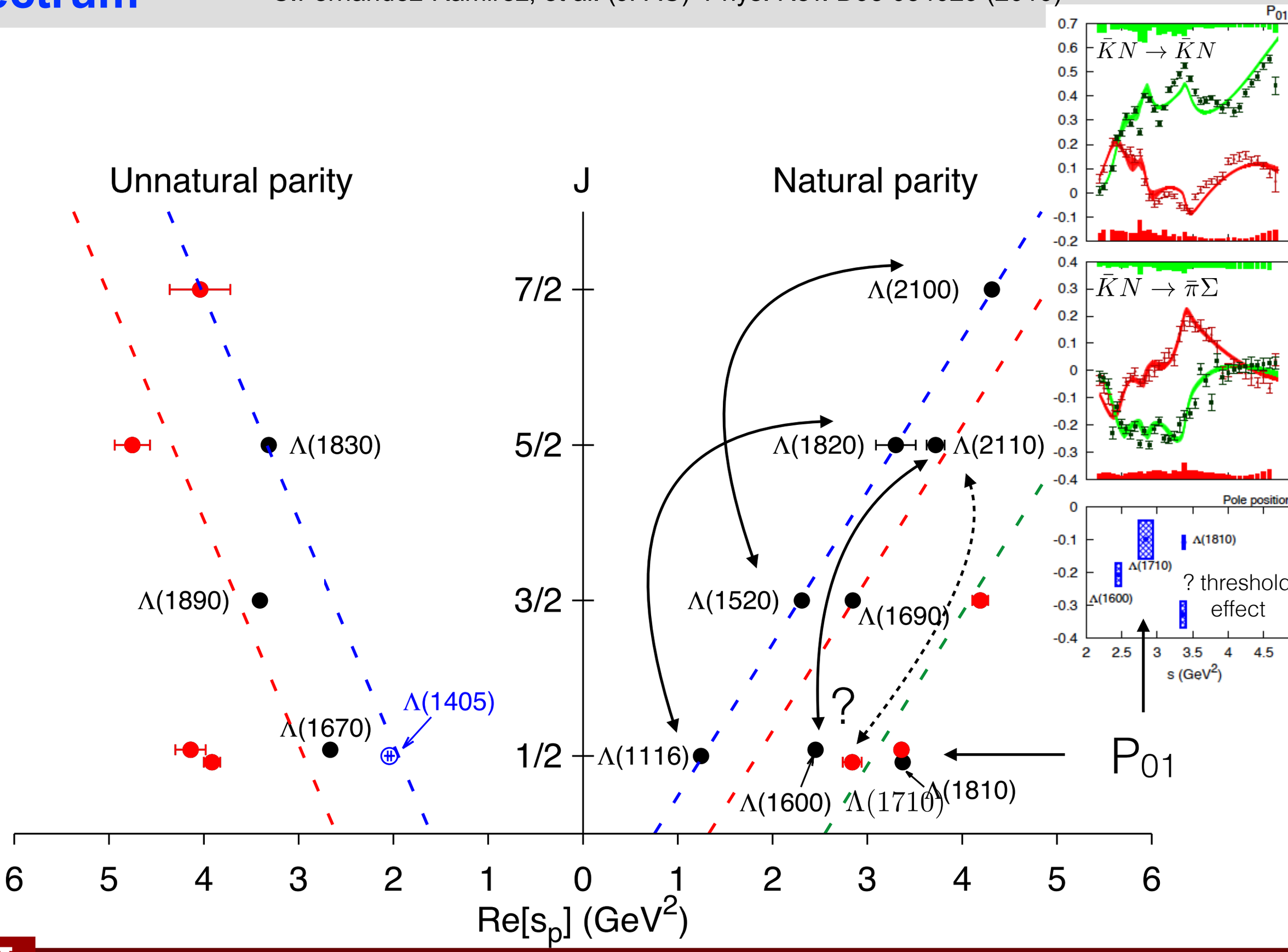
Λ spectrum

C.Fernandez-Ramirez, et al. (JPAC) Phys. Rev. D93 034029 (2016)



Λ spectrum

C.Fernandez-Ramirez, et al. (JPAC) Phys. Rev. D93 034029 (2016)



Λ spectrum

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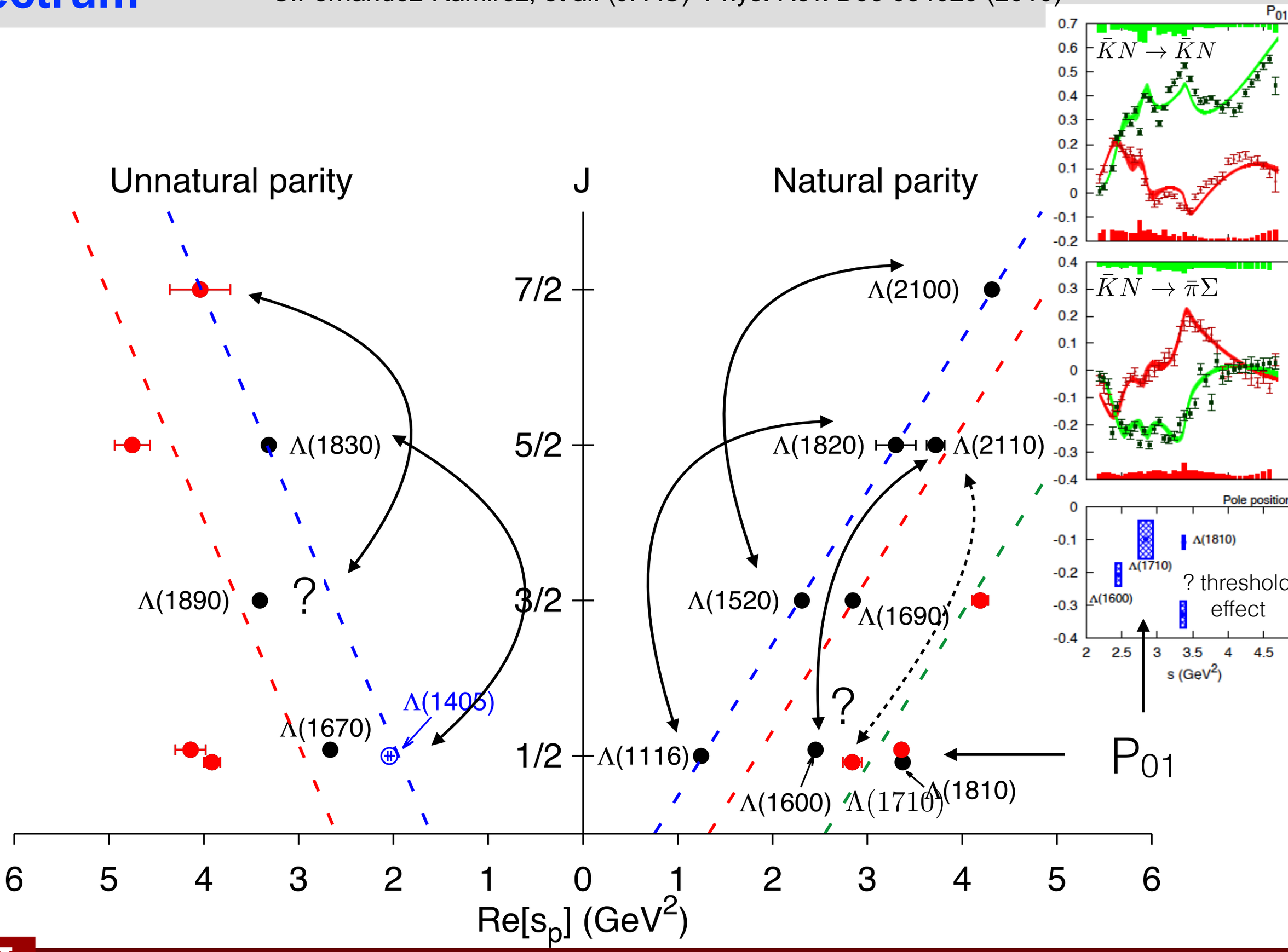

Λ spectrum

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