Special moments....

Forty Years of Quark-Gluon Plasma

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Jean-Paul Blaizot, IPhT-Saclay

Special moments....

Hiking with Edward

Arguing with Edward...

'Special moments' can also provide simple tools to understand the emergence of hydrodynamical behavior in expanding quark-gluon plasmas

How much entropy is produced in strongly coupled Quark-Gluon Plasma (sQGP) by dissipative effects?

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(Dated: June 14, 2013)

We argue that estimates of dissipative effects based on the first-order hydrodynamics with shear viscosity are potentially misleading because higher order terms in the gradient expansion of the dissipative part of the stress tensor tend to reduce them. Using recently obtained sound dispersion relation in thermal $\mathcal{N}$=4 supersymmetric plasma, we calculate the *resummed* effect of these high order terms for Bjorken expansion appropriate to RHIC/LHC collisions. A reduction of entropy production is found to be substantial, up to an order of magnitude.

The hydrodynamic description of matter produced in heavy ion collisions works amazingly well !...

even in situations where, a priori, it should not ...

(e.g. in presence of strong gradients)

Fluid behavior requires (some degree of) local equilibration (='thermalization'). How is this achieved?

Usual picture:

- **microscopic degrees of freedom relax quickly towards local equilibrium**
- **long wavelength modes, associated to conservation laws, relax on longer time scales**

Thermalization

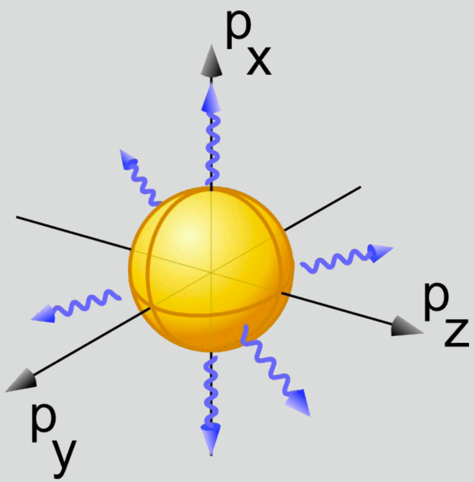
Two main issues

i) relative populations of different momentum modes

ii) isotropy of momentum distribution

Main topic for the rest of this talk

"Isotropization"

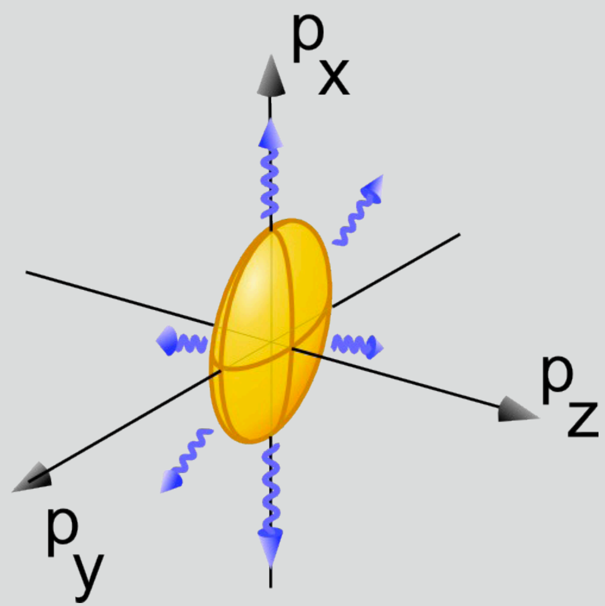


Longitudinal expansion hinders isotropization

The fast expansion of the matter along the collision axis tends to drive the momentum distribution to a very flat distribution along the z direction

Translates into the existence of two different pressures

P_L (longitudinal) P_T (transverse)



Anisotropy relaxes slowly,

like a 'collective' variable associated to a conservation law

Hydrodynamic behavior may emerge

before local isotropization if achieved

"Hydrodynamization"

First hints came from holographic descriptions

Ideal hydrodynamics of boost invariant systems

(Bjorken flow)

energy density

Equation of motion $\partial_\tau(\tau\epsilon) = -P_L$ $\qquad \dfrac{\partial\epsilon}{\partial\tau} = -\dfrac{\epsilon + P_L}{\tau}$

Three independent components $\epsilon, P_\perp, P_L$ **but:** $\epsilon = 2P_\perp + P_L$

conformal symmetry

In local equilibrium $P_\perp = P_L = \epsilon/3$ (equation of state)

Then $\epsilon \sim \tau^{-4/3}$ $\qquad T \sim \tau^{-1/3}$ $\qquad (\epsilon \sim T^4)$

Viscous hydrodynamics $P_\perp - P_L = \dfrac{\eta}{\tau}$ (gradient expansion)

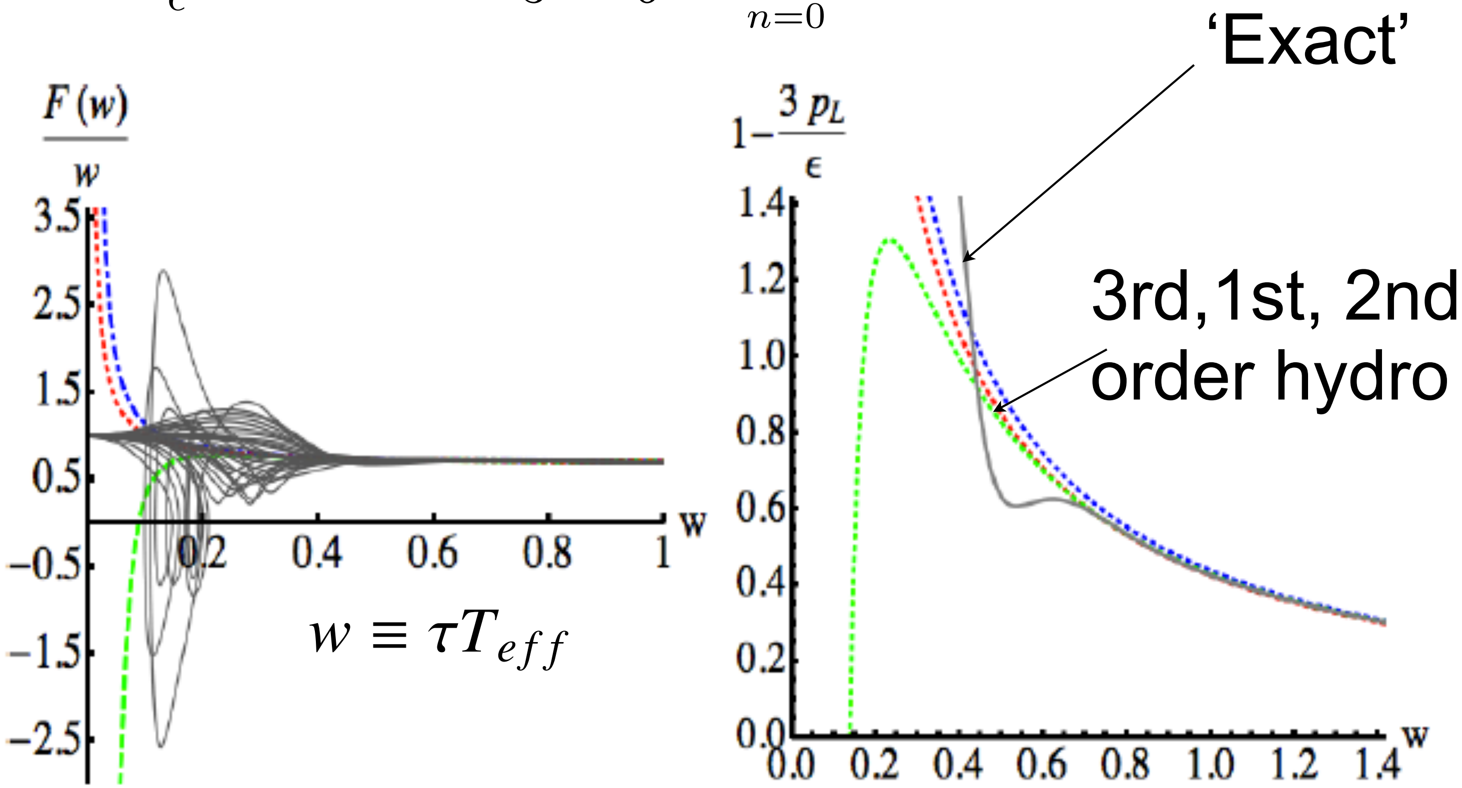
In boost invariant systems, the gradient expansion is an expansion in inverse powers of

$$\frac{1}{w} = \frac{1/T}{\tau} \sim \text{Knudsen number} \sim \frac{\text{micro}}{\text{macro}}$$

Holographic description of a boost invariant plasma

(Heller, Janik, Witaszczyk, [1103.3452])

Define $\mathcal{R} = \dfrac{\mathcal{P}_T - \mathcal{P}_L}{\epsilon}$ $\qquad f \equiv \dfrac{2}{3} + \dfrac{\mathcal{R}}{6} = \displaystyle\sum_{n=0}^{\infty} f_n w^{-n}$



Viscous hydro can cope with partial thermalization, and large differences between longitudinal and transverse pressures

The gradient expansion is divergent

$$f \equiv \frac{2}{3} + \frac{\mathcal{R}}{6} = \sum_{n=0}^{\infty} f_n w^{-n} \qquad\qquad f_n \sim n!$$

f_n **has been calculated up to n=240 (!)** (Heller, Janik, Witaszczyk , 2013)

Sophisticated resummation yields a 'transseries'

(Heller, Spalinski , 2015)

$$f = \sum_{n=0}^{\infty} f_n w^{-n} + c\, e^{-\frac{3}{2C_{\tau\Pi}} w} \left(w^{\frac{C_\eta - 2C_{\lambda_1}}{C_{\tau\Pi}}} \sum_{n=0}^{\infty} f_n^{(1)} w^{-n} \right) + \ldots$$

Similar features are observed in kinetic theory

(Heller, Kurkela, Spalinski , Svensson, 2016)

Simple kinetic equation

- **Relaxation time approximation**

$$\left[\partial_\tau - \frac{p_z}{\tau}\partial_{p_z}\right] f(\boldsymbol{p}/T) = -\frac{f(\boldsymbol{p},\tau) - f_{\rm eq}(p,\tau)}{\tau_R}$$

(derivative at constant $p_z\tau$)

- **Free streaming (e.g. in absence of collisions)**

$$f(t,\boldsymbol{p}) = f_0(\boldsymbol{p}_\perp, p_z t/t_0)$$

- **Solved long ago by Baym** [PLB 138 (1984) 18]

$$\epsilon(\tau) = \mathrm{e}^{-(\tau-\tau_0)/\theta}\epsilon^{(0)}(\tau) + \mathrm{e}^{-\tau/\theta}\int_{\tau_0}^{\tau}\frac{\mathrm{d}\tau'}{\tau_R}\mathrm{e}^{\tau'/\tau_R}\frac{\tau'}{\tau}\epsilon(\tau')h(\tau'/\tau)$$

(free streaming)

$$h(x) \equiv \int_0^1 d\mu\sqrt{1-\mu^2+\mu^2x^2}$$

(angular integral)

Here come the special moments

(JPB, Li Yan, 2017, 2018)

Special moments $p_z = p\cos\theta$

$$\mathcal{L}_n \equiv \int_p p^2 P_{2n}(\cos\theta) f(\boldsymbol{p})$$

(Legendre polynomial)

$$P_0(z) = 1 \qquad P_2(z) = \frac{1}{2}(3z^2 - 1)$$

Why moments ?

- **There is too much information in the distribution function**
- **We want to focus on the angular degrees of freedom**

The energy momentum tensor is described by first two moments

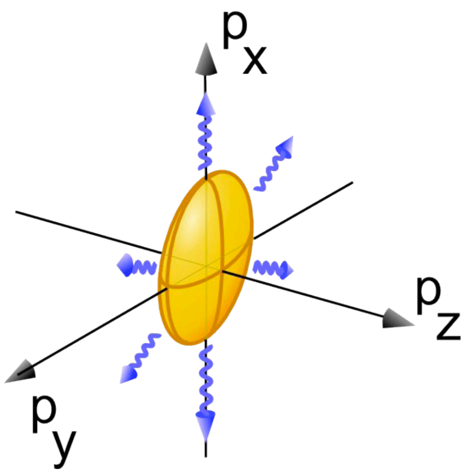
$$T^{\mu\nu} = \int_p f(\boldsymbol{p}) p^\mu p^\nu \qquad \mathcal{L}_0 = \varepsilon \qquad \mathcal{L}_1 = \mathcal{P}_L - \mathcal{P}_T$$

Isotropization

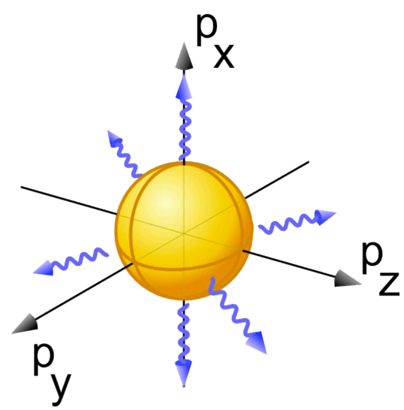
Two antagonistic agents

free streaming and **collisions**

drives the momentum distribution
toward a flat distribution in the longitudinal direction



drive the momentum distribution
toward an isotropic distribution



Coupled equations for the moments

$$\frac{\partial \mathcal{L}_n}{\partial \tau} = -\frac{1}{\tau}\left[a_n \mathcal{L}_n + b_n \mathcal{L}_{n-1} + c_n \mathcal{L}_{n+1}\right] - \frac{\mathcal{L}_n}{\tau_R} \quad (n \geq 1)$$

(Free streaming)

(collisions)

$$\frac{\partial \mathcal{L}_0}{\partial \tau} = -\frac{1}{\tau}\left[a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1\right]$$

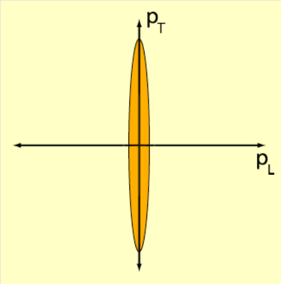
$a_0 = 4/3, \ c_0 = 2/3$

$$\mathcal{L}_0 = \varepsilon \qquad \mathcal{L}_1 = \mathcal{P}_L - \mathcal{P}_T$$

- The coefficients a_n, b_n, c_n are pure numbers
- The competition between expansion and collisions is made obvious
- Interesting system of coupled linear equations
- Emergence of hydrodynamics is transparent: equations for the lowest moments
- Provides much insight on various versions of viscous hydrodynamics

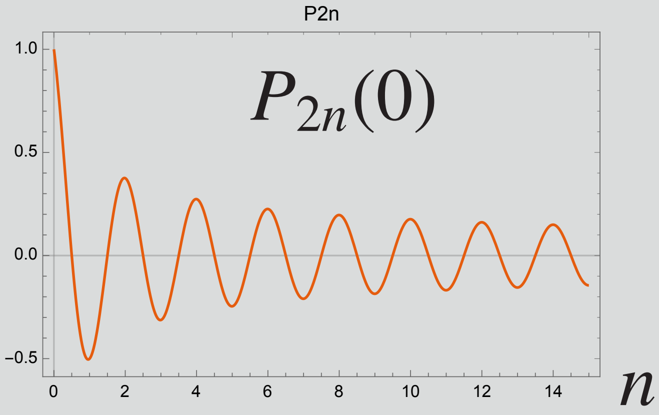
Free streaming solution

$$\mathcal{L}_n^{(0)}(t) = \frac{t_0}{t}\epsilon_0\,\mathcal{F}_n\left(\frac{t_0}{t}\right)$$



Poor convergence: all moments are important at late time

$$\mathcal{F}_n(x \to 0) \to \frac{\pi}{4}P_{2n}(0)$$

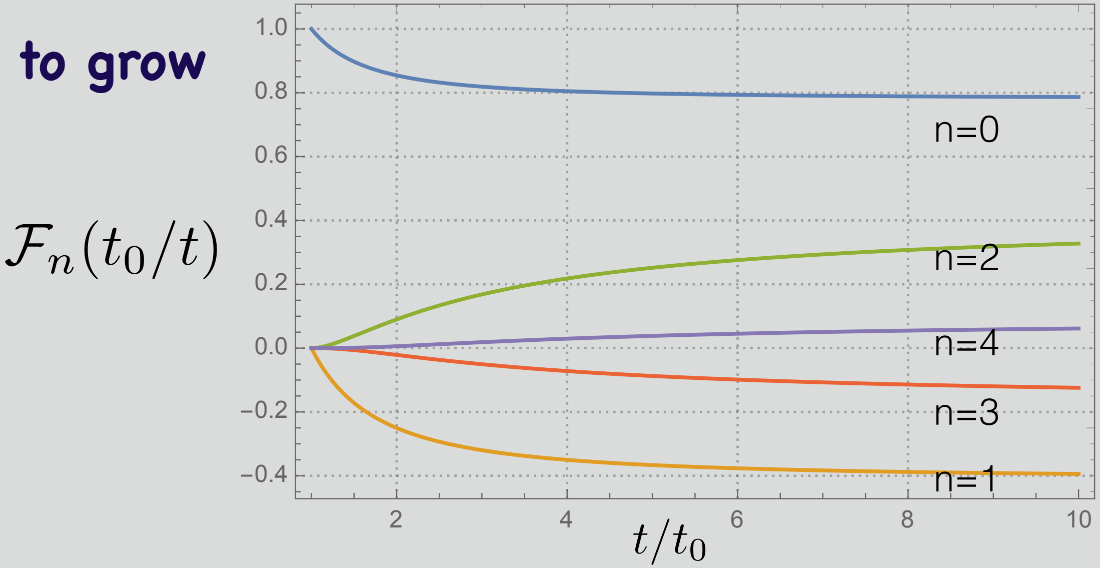


NB

$$g_n(\tau) \equiv \tau\partial_\tau \ln \mathcal{L}_n$$

$$g_n(\tau \to \infty) \to -1$$

But higher moments take time to grow



Free streaming
Simple truncations work

$$\frac{\partial \mathcal{L}_n}{\partial \tau} = -\frac{1}{\tau}\left[a_n \mathcal{L}_n + b_n \mathcal{L}_{n-1} + c_n \mathcal{L}_{n+1}\right]$$

$$\partial_t \vec{\mathcal{L}} = -M\vec{\mathcal{L}}$$

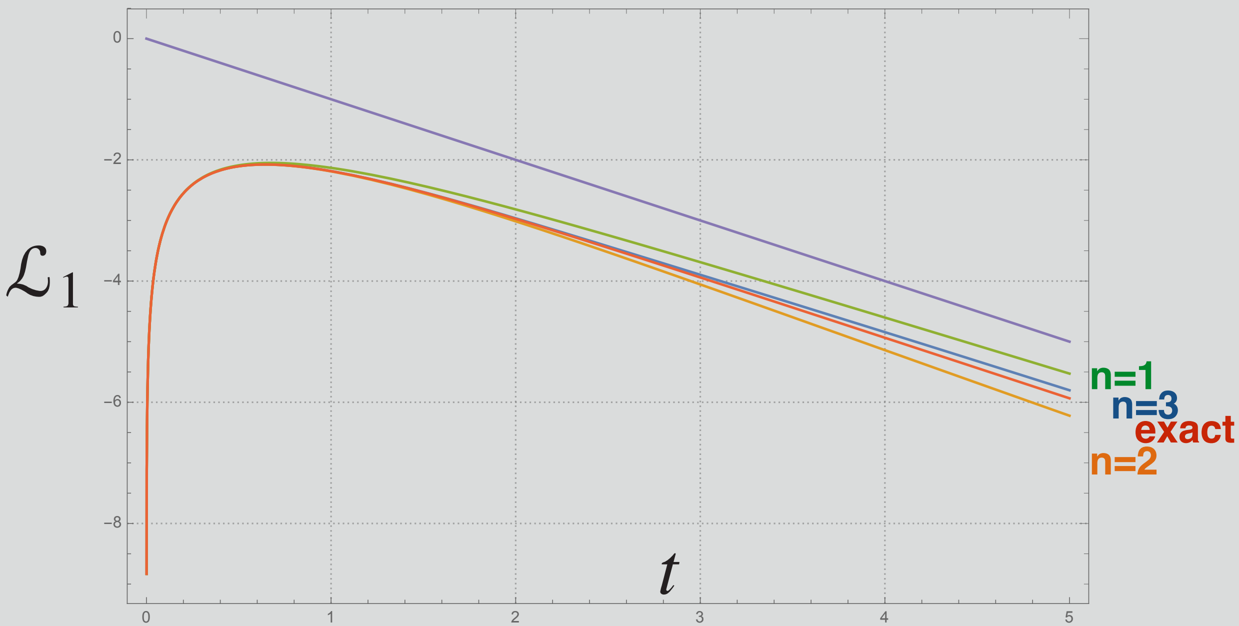
$$(t = \ln(\tau/\tau_0))$$

Keeping only the first two moments one gets

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathcal{L}_0\\ \mathcal{L}_1\end{pmatrix} = -\begin{pmatrix}\frac{4}{3} & \frac{2}{3}\\ \frac{8}{15} & \frac{38}{21}\end{pmatrix}\begin{pmatrix}\mathcal{L}_0\\ \mathcal{L}_1\end{pmatrix}$$

Two eigenmodes $\lambda_0 = 0.929366$ $\lambda_1 = 2.21349$

Truncations are reasonably accurate



Free streaming fixed point

One can transform the coupled linear equations into a single non linear differential equation

$$\tau\frac{\mathrm{d}g_0}{\mathrm{d}\tau} + g_0^2 + (a_0 + a_1)g_0 + a_0 a_1 - c_0 b_1 = 0.$$

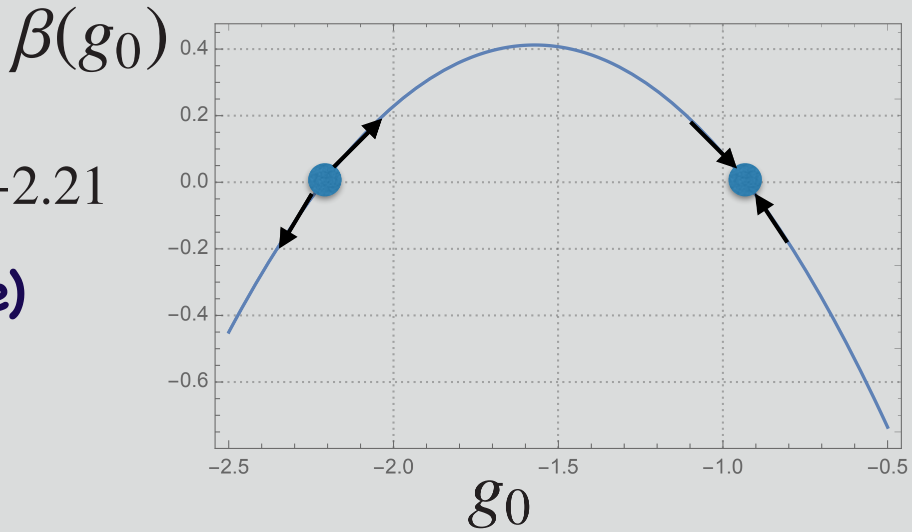
$$g_n(\tau) \equiv \tau\partial_\tau \ln \mathcal{L}_n$$

Write this as

$$\tau\frac{dg_0}{d\tau} = \beta(g_0) \qquad \beta(g_0) = -g_0^2 - (a_0 + a_1)g_0 - a_0 a_1 + c_0 b_1$$

$g_0^* = -\lambda_1 = -2.21$

(unstable)



$g_0^* = -\lambda_0 = -0.929$

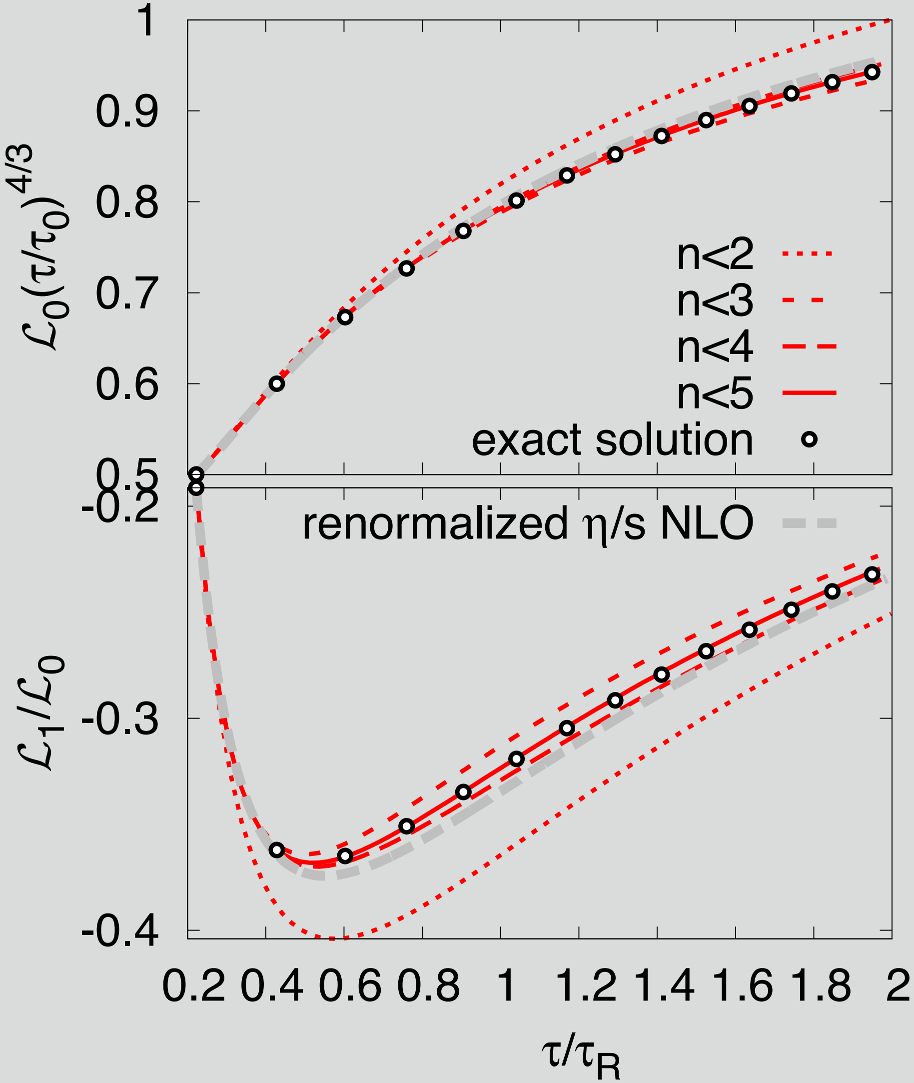
(stable)

NB **exact fixed point**

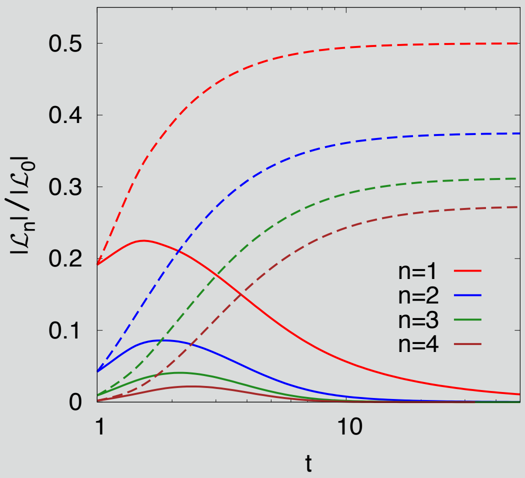
$g_n(\tau \to \infty) \to -1$

Including collisions
Simple truncations work well

(drop all moments beyond a certain n)



First few moments are relevant



moments of free
streaming solution

damping of higher
moments by collisions

A simple model can be constructed

$$\partial_\tau \mathcal{L}_0 = -\frac{1}{\tau}\left(a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1\right)$$

$$\partial_\tau \mathcal{L}_1 = -\frac{1}{\tau}\left(b_1 \mathcal{L}_0 + a_1 \mathcal{L}_1\right) - \frac{1}{\tau_R}\mathcal{L}_1$$

("Almost" viscous hydrodynamics...
in fact, better!)

This simple equations capture much of the physics and illuminates the analytic features of the transition to viscous hydrodynamics

The hydrodynamic fixed point

$$\partial_\tau \mathcal{L}_0 = -\frac{1}{\tau}\left(a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1\right) \qquad \partial_\tau \mathcal{L}_1 = -\frac{1}{\tau}\left(b_1 \mathcal{L}_0 + a_1 \mathcal{L}_1\right) - \frac{1}{\tau_R}\mathcal{L}_1$$

Complete damping of first moment

$$\tau \partial_\tau \mathcal{L}_0 = -a_0 \mathcal{L}_0 = -(4/3)\mathcal{L}_0$$

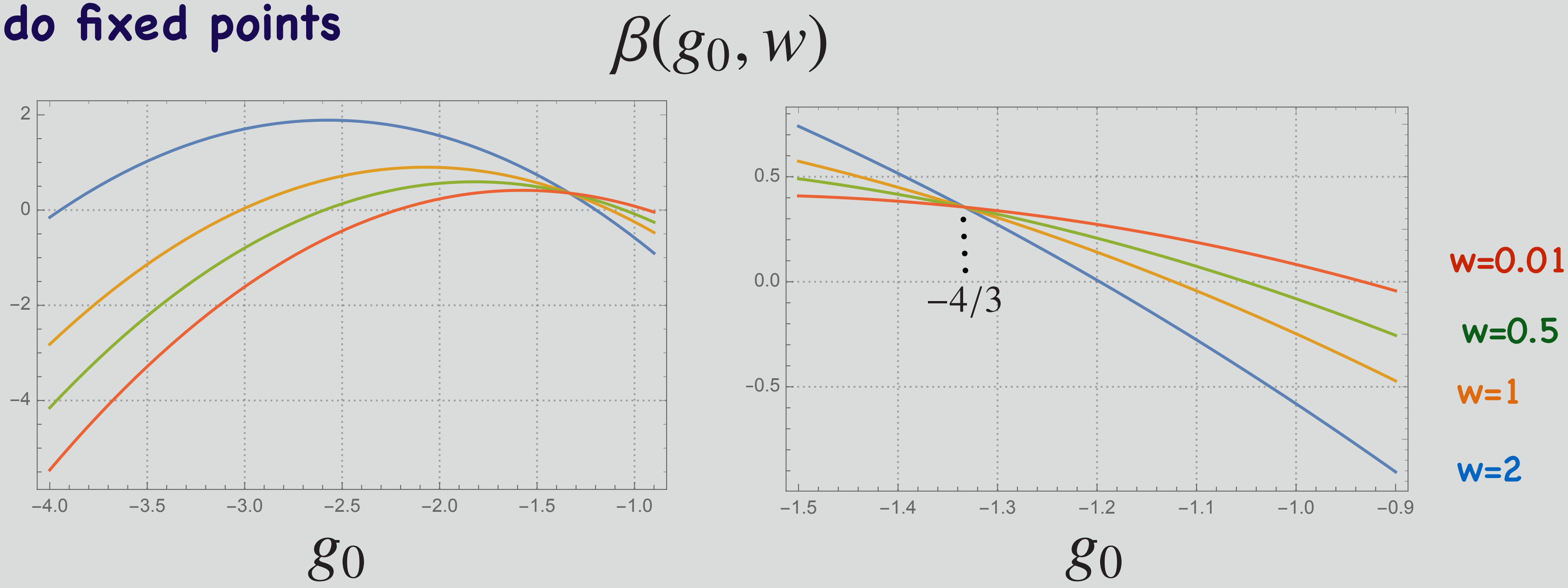
(hydro fixed point)

Modified equation of motion

$$w g_0' + g_0^2 + (a_0 + a_1 + w) g_0 + w a_0 + a_0 a_1 - b_1 c_0 = 0, \qquad \tau_R = \mathrm{Cste}$$

$$\frac{d g_0}{d \ln w} = \beta(g_0, w)$$

Pseudo fixed points



The transition from free streaming to hydrodynamics

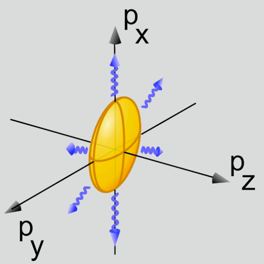
(Attractor solution)

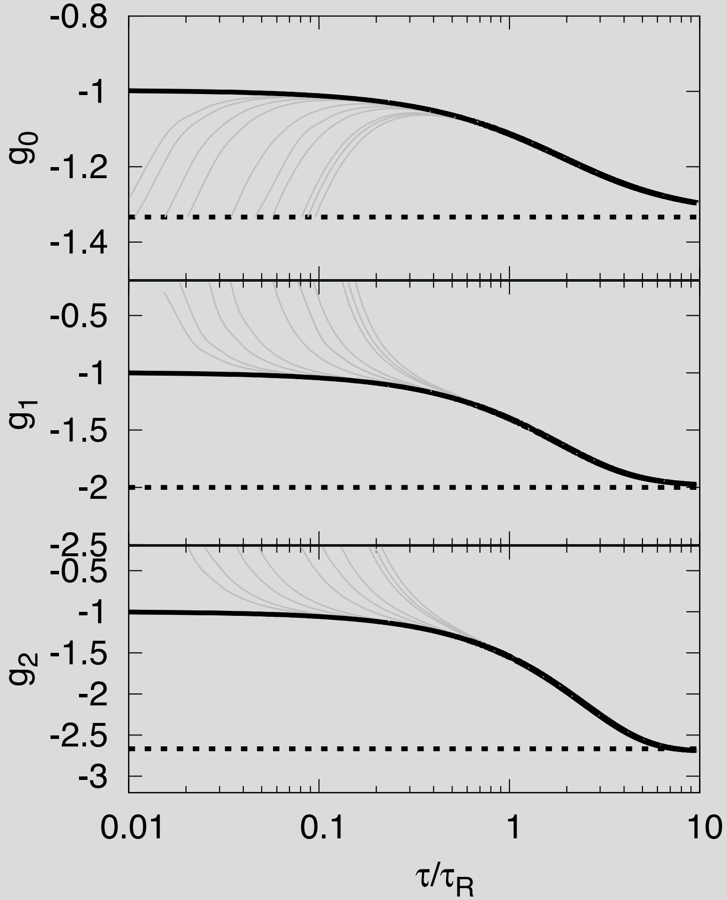
Early and late times are controlled by the free streaming and the hydrodynamic fixed points, respectively

$$g_n(\tau) = \tau\partial_\tau \ln \mathcal{L}_n$$

Free streaming fixed point

$$g_n = -1$$

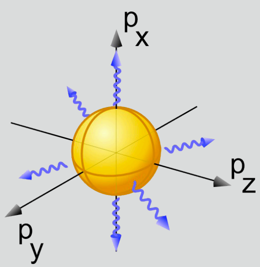




Hydro fixed point

$$g_n = -\frac{4+2n}{3}$$

(Universal!)



Renormalization of the viscosity

The effect of higher moments can be viewed as a renormalisation of the equation for the lowest moments, i.e. of the viscous hydrodynamical equations

$$\partial_\tau \mathcal{L}_1 = -\frac{1}{\tau}\left(a_1 \mathcal{L}_1 + b_1 \mathcal{L}_0\right) - \left[1 + \frac{c_1 \tau_R}{\tau}\frac{\mathcal{L}_2}{\mathcal{L}_1}\right]\frac{\mathcal{L}_1}{\tau_R}$$

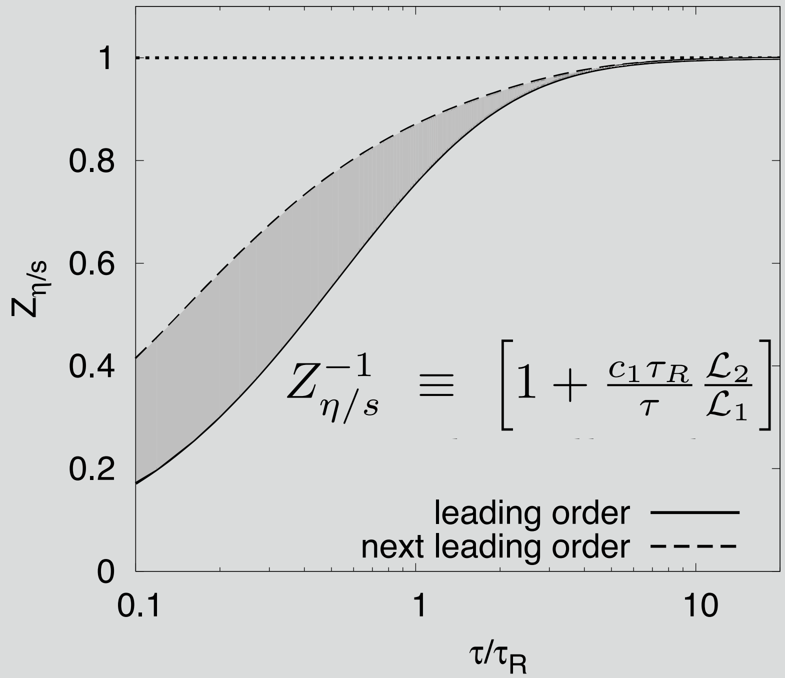
renormalisation of relaxation time,

$$\tau_R \;\rightarrow\; Z_{\eta/s}\tau_R$$

or, equivalently, of the viscosity

$$\eta = \frac{4}{15}\tau_R \epsilon$$

Sizeable reduction of the effective viscosity at early time



[For an early suggestion of such an effect: Lublinsky-Shuryak (2007)]

Variants of viscous hydrodynamics

Different approximate ways to solve the equations for the first two moments

Navier Stokes

$$\partial_\tau \epsilon = -\frac{4}{3}\frac{\epsilon}{\tau} + \frac{\Pi}{\tau} \qquad \Pi = \frac{4\eta}{3\tau} \qquad \partial_\tau \epsilon = -\frac{4}{3}\frac{\epsilon}{\tau} + \frac{4\eta}{3\tau^2} \qquad \Pi = -c_0 \mathcal{L}_1$$

Mueller-Israel-Steward

$$\partial_\tau \Pi = -\frac{\Pi}{\tau_\pi} + \frac{4\eta}{3\tau\tau_\pi} = -\frac{1}{\tau_\pi}\left(\Pi - \frac{4\eta}{3\tau}\right)$$

Second order hydro (DNMR)

$$\partial_\tau \Pi = \frac{4}{3}\frac{\eta}{\tau\tau_\pi} - \beta_{\pi\pi}\frac{\Pi}{\tau} - \frac{\Pi}{\tau_\pi} \qquad \beta_{\pi\pi} = \frac{38}{21} = a_1 \qquad \tau_\pi = \tau_R$$

same as $\frac{\partial \mathcal{L}_1}{\partial \tau} = -b_1 \frac{\epsilon}{\tau} - a_1 \frac{\mathcal{L}_1}{\tau} - \frac{\mathcal{L}_1}{\tau_R}$ **provided** $\frac{4}{3}\frac{\eta}{\tau\tau_\pi} = c_0 b_1 \frac{\varepsilon}{\tau}$

which holds in leading order if $\tau_\pi = \tau_R$

Similar analysis can be made for BRSSS hydro (full second order, conformal), or third order (Jaiswal).

[DNMR= Denicol, Niemi, Molnar, Rischke (2012)]

[BRSSS= Baier, Romatschke, Son, Starinets, Stephanov (2008)]

Conclusions

In high energy collisions, the longitudinal expansion prevents the system to reach full isotropy in a short time (expansion plays a role somewhat similar to a conservation law...)

However strong anisotropy does not hinder the emergence of (viscous) hydrodynamic behavior

A simple picture based on special set of moments of the distribution functions provides much insight into the mathematical structure of viscous hydrodynamics of expanding (boost invariant) systems

Strong reduction of the viscosity at early times due to out of equilibrium effects (coupling to higher moments)

Coupled equations for the first few (two) moments could be a convenient alternative to viscous hydrodynamics