Dynamical generation of field mixing*

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*M.B., P.Jizba, N.E.Mavromatos and L. Smaldone, *Chiral symmetry-breaking schemes and dynamical generation of masses and field mixing,* arXiv:1807.07616 [hep-th];

Research program

- Quantum field theory of field mixing;

M.B. and G.Vitiello, Annals Phys. (1995)

⋮

- Field mixing for accelerated observer

M. B., G. Lambiase and G. Luciano, Phys. Rev D (2017)

- Neutrino mixing in accelerated proton decay

M. B., G. Lambiase, G. Luciano and L. Petruzziello, Phys. Rev D (2018)

G. Cozzella, S. A. Fulling, et al., Phys. Rev D (2018)

- Flavor-Energy uncertainty relations for neutrino oscillations

M. B., P. Jizba and L. Smaldone, arXiv:1810.01648

- Entanglement in neutrino oscillations

Summary

Motivations

• CKM quark mixing, meson mixing, massive neutrino mixing (and oscillations) play a crucial role in phenomenology;

• Theoretical interest: origin of mixing in the Standard Model;

• Bargmann superselection rule[†]: coherent superposition of states with different masses is not allowed in non-relativistic QM;

• Necessity of a QFT treatment: problems in defining Hilbert space for mixed particles[‡]; oscillation formulas[§];

[†] V.Bargmann, Ann. Math. (1954); D.M.Greenberger, Phys. Rev. Lett. (2001).
[‡] C.W.Kim and A.Pevsner, *Neutrinos in Physics and Astrophysics*, (Harwood, 1993). C.Giunti, J. Phys. G (2007).
[§] M.Beuthe, Phys. Rep. (2003).

Prelude

Neutrino oscillations in QM *

Pontecorvo mixing relations

$$|\nu_e\rangle = \quad \cos\theta\,|\nu_1\rangle \,+\, \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle \,+\, \cos\theta\,|\nu_2\rangle$$

– Time evolution:

$$|\nu_e(t)\rangle = \cos\theta\, e^{-iE_1 t}\,|\nu_1\rangle \,+\, \sin\theta\, e^{-iE_2 t}\,|\nu_2\rangle$$

– Flavor oscillations:

$$P_{\nu_e\to\nu_e}(t) \;=\; |\langle\nu_e|\nu_e(t)\rangle|^2 \,=\, 1-\sin^2 2\theta\,\sin^2\left(\frac{\Delta E}{2}t\right) \,=\, 1-P_{\nu_e\to\nu_\mu}(t)$$

– Flavor conservation:

$$|\langle\nu_e|\nu_e(t)\rangle|^2 \,+\, |\langle\nu_\mu|\nu_e(t)\rangle|^2 \,=\, 1$$

*S.M.Bilenky and B.Pontecorvo, Phys. Rep. (1978)

Mixing of neutrino fields

– Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\, \nu_1(x) + \sin\theta\, \nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\, \nu_1(x) + \cos\theta\, \nu_2(x)$$

ν_1, ν_2 are fields with definite masses.

– Mixing transformations connect the two quadratic forms:

$$\mathcal{L} = \bar{\nu}_1\,(i\,\partial\!\!\!/ - m_1)\,\nu_1 + \bar{\nu}_2\,(i\,\partial\!\!\!/ - m_2)\,\nu_2$$

and

$$\mathcal{L} = \bar{\nu}_e\,(i\,\partial\!\!\!/ - m_e)\,\nu_e + \bar{\nu}_\mu\,(i\,\partial\!\!\!/ - m_\mu)\,\nu_\mu - m_{e\mu}\,(\,\bar{\nu}_e\,\nu_\mu + \bar{\nu}_\mu\,\nu_e\,)$$

with

$$m_e = m_1\cos^2\theta + m_2\sin^2\theta,\; m_\mu = m_1\sin^2\theta + m_2\cos^2\theta,\; m_{e\mu} = (m_2 - m_1)\sin\theta\cos\theta.$$

– ν_i are free Dirac field operators:

$$\nu_i(x) = \sum_{\mathbf{k},r} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} \Big[u^r_{\mathbf{k},i}(t)\, \alpha^r_{\mathbf{k},i} \,+\, v^r_{-\mathbf{k},i}(t)\, \beta^{r\dagger}_{-\mathbf{k},i} \Big], \quad i = 1, 2.$$

– Anticommutation relations:

$$\{\nu_i^\alpha(x), \nu_j^{\beta\dagger}(y)\}_{t=t'} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\delta_{ij}\,; \quad \{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\} = \{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\} = \delta^3(\mathbf{k}-\mathbf{q})\delta_{rs}\delta_{ij}$$

– Orthonormality and completeness relations:

$$u^r_{\mathbf{k},i}(t) \,=\, e^{-i\omega_{k,i}t}\, u^r_{\mathbf{k},i} \quad ; \quad v^r_{\mathbf{k},i}(t) \,=\, e^{i\omega_{k,i}t}\, v^r_{\mathbf{k},i} \quad ; \quad \omega_{k,i} = \sqrt{k^2 + m_i^2}$$

$$u^{r\dagger}_{\mathbf{k},i} u^s_{\mathbf{k},i} = v^{r\dagger}_{\mathbf{k},i} v^s_{\mathbf{k},i} = \delta_{rs} \quad , \quad u^{r\dagger}_{\mathbf{k},i} v^s_{-\mathbf{k},i} = 0 \quad , \quad \sum_r (u^{r\alpha*}_{\mathbf{k},i} u^{r\beta}_{\mathbf{k},i} + v^{r\alpha*}_{-\mathbf{k},i} v^{r\beta}_{-\mathbf{k},i}) = \delta_{\alpha\beta}\ .$$

– Fock space for ν_1, ν_2:

$$\mathcal{H}_{1,2} = \Big\{ \alpha^\dagger_{1,2}\ ,\ \beta^\dagger_{1,2}\ ,\ |0\rangle_{1,2} \Big\}.$$

– Vacuum state $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$.

Rotation

– Pontecorvo mixing can be seen as arising by the application to the vacuum state $|0\rangle_{1,2}$ of the rotated operators:

$$R(\theta)^{-1}\alpha^{r\dagger}_{\mathbf{k},1}R(\theta) = \cos\theta\,\alpha^{r\dagger}_{\mathbf{k},1} + \sin\theta\,\alpha^{r\dagger}_{\mathbf{k},2},$$

$$R(\theta)^{-1}\alpha^{r\dagger}_{\mathbf{k},2}R(\theta) = \cos\theta\,\alpha^{r\dagger}_{\mathbf{k},2} - \sin\theta\,\alpha^{r\dagger}_{\mathbf{k},1},$$

and similar ones for $\beta^{r\dagger}_{\mathbf{k},i}$.

– The generator $R(\theta)$ is:

$$R(\theta) \equiv \exp\left\{\theta\sum_{\mathbf{k},r}\left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{\mathbf{k},1}\beta^{r}_{\mathbf{k},2}\right) - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{\mathbf{k},2}\beta^{r}_{\mathbf{k},1}\right)\right]\right\},$$

• The above unitary operator leaves the vacuum invariant:

$$R(\theta)|0\rangle_{1,2} = |0\rangle_{1,2}$$

Consider the action of the rotation on the field ν_1 for example:

$$R^{-1}(\theta)\,\nu_1(x)\,R(\theta) \;=\; \cos\theta\,\nu_1(x) \;+\; \sin\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\left(\alpha^r_{\mathbf{k},2}\,u^r_{\mathbf{k},1}(t)+\beta^{r\dagger}_{\mathbf{k},2}\,v^r_{-\mathbf{k},1}(t)\right),$$

• Problem in the last term in the r.h.s. which appears as the expansion of the field in the “wrong” basis.

Bogoliubov transformation

– We can recover the wanted expression by means of a Bogoliubov transformation:

$$\tilde{\alpha}^{r\dagger}_{\mathbf{k},i} = \cos\Theta_{\mathbf{k},i}\, \alpha^{r\dagger}_{\mathbf{k},i} - \epsilon^r\, \sin\Theta_{\mathbf{k},i}\, \beta^r_{-\mathbf{k},i},$$

$$\tilde{\beta}^{r\dagger}_{-\mathbf{k},i} = \cos\Theta_{\mathbf{k},i}\, \beta^{r\dagger}_{-\mathbf{k},i} + \epsilon^r\, \sin\Theta_{\mathbf{k},i}\, \alpha^r_{\mathbf{k},i}, \qquad i = 1, 2,$$

with $\tilde{\alpha}^{r\dagger}_{\mathbf{k},i} \equiv B_i^{-1}(\Theta_i)\, \alpha^{r\dagger}_{\mathbf{k},i}\, B_i(\Theta_i)$, etc..

– Generator

$$B_i(\Theta_i) = \exp\left\{ \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \Theta_{\mathbf{k},i}\, \epsilon^r \left[\alpha^r_{\mathbf{k},i} \beta^r_{-\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i} \alpha^{r\dagger}_{\mathbf{k},i} \right] \right\}.$$

Let us see this for the field ν_1.

$$B_2^{-1}(\Theta_2)\, R^{-1}(\theta)\, \nu_1(x)\, R(\theta)\, B_2(\Theta_2) =$$

$$= \cos\theta\, \nu_1(x) + \sin\theta \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\tilde{\alpha}^r_{\mathbf{k},2}\, u^r_{\mathbf{k},1}(t) + \tilde{\beta}^{r\dagger}_{\mathbf{k},2}\, v^r_{-\mathbf{k},1}(t)\right)$$

$$= \cos\theta\, \nu_1(x) + \sin\theta \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\alpha^r_{\mathbf{k},2}\, \tilde{u}^r_{\mathbf{k},1}(t) + \beta^{r\dagger}_{\mathbf{k},2}\, \tilde{v}^r_{-\mathbf{k},1}(t)\right) \,,$$

where

$$\tilde{u}^r_{\mathbf{k},1}(t) = \cos\Theta_{\mathbf{k},2}\, u^r_{\mathbf{k},1}(t) + \epsilon^r \sin\Theta_{\mathbf{k},2}\, v^r_{-\mathbf{k},1}(t)\,,$$

$$\tilde{v}^r_{-\mathbf{k},1}(t) = \cos\Theta_{\mathbf{k},2}\, v^r_{-\mathbf{k},1}(t) - \epsilon^r \sin\Theta_{\mathbf{k},2}\, u^r_{\mathbf{k},1}(t)\,.$$

For

$$\tilde{\Theta}_{\mathbf{k},2} = \cos^{-1}\left(u^{r\dagger}_{\mathbf{k},2}(t) u^{r}_{\mathbf{k},1}(t) \right)$$

the above Bogoliubov transformation implements the mass shift

$$\Delta m = m_2 - m_1$$

such that $\tilde{u}^{r}_{\mathbf{k},1}(t) = u^{r}_{\mathbf{k},2}(t)$ and $\tilde{v}^{r}_{-\mathbf{k},1}(t) = v^{r}_{-\mathbf{k},2}(t)$.

• The action of $B_2^{-1}(\tilde{\Theta}_2)\, R^{-1}(\theta)$ produces the desired transformation (rotation) of the field ν_1.

– Similar reasoning for ν_2, using $B_1^{-1}(\tilde{\Theta}_1)\, R^{-1}(\theta)$.

Mixing generator

Neutrino mixing in QFT

- Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\, \nu_1(x) + \sin\theta\, \nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\, \nu_1(x) + \cos\theta\, \nu_2(x)$$

can be written as†

$$\nu_e^\alpha(x) = G_\theta^{-1}(t)\, \nu_1^\alpha(x)\, G_\theta(t)$$

$$\nu_\mu^\alpha(x) = G_\theta^{-1}(t)\, \nu_2^\alpha(x)\, G_\theta(t)$$

– Mixing generator:

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x}\, \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

For ν_e, we get $\frac{d^2}{d\theta^2}\, \nu_e^\alpha = -\nu_e^\alpha$ with i.c. $\nu_e^\alpha\big|_{\theta=0} = \nu_1^\alpha$, $\frac{d}{d\theta}\nu_e^\alpha\big|_{\theta=0} = \nu_2^\alpha$.

† M.B. and G.Vitiello, Annals Phys. (1995)

• The vacuum $|0\rangle_{1,2}$ is not invariant under the action of $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\;|0\rangle_{1,2}$$

• Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V\to\infty}\; {}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty} e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\;\ln\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right)^2} = 0$$

with

$$|V_\mathbf{k}|^2 \;\;\equiv\;\; \sum_{r,s} |\; v^{r\dagger}_{-\mathbf{k},1} u^s_{\mathbf{k},2}\;|^2 \neq 0 \quad for \quad m_1 \neq m_2$$

.

Quantum Field Theory vs. Quantum Mechanics

- Quantum Mechanics:

- finite ♯ of degrees of freedom.

- unitary equivalence of the representations of the canonical commutation relations (von Neumann theorem).

- Quantum Field Theory:

- infinite ♯ of degrees of freedom.

- ∞ many unitarily inequivalent representations of the field algebra $\Leftrightarrow$ many vacua .

- The mapping between interacting and free fields is “weak”, i.e. representation dependent (LSZ formalism)*. Example: theories with spontaneous symmetry breaking.

*F.Strocchi, *Elements of Quantum Mechanics of Infinite Systems* (W. Sc., 1985).

- The "flavor vacuum" $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state[†]:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r}\Big[(1-\sin^2\theta\,|V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\,\cos\theta\,|V_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\,\epsilon^r \sin^2\theta\,|V_{\mathbf{k}}||U_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\,|V_{\mathbf{k}}|^2\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\,|0\rangle_{1,2}$$

- Condensation density:

$${}_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; {}_{e,\mu}\langle 0(t)|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; \sin^2\theta\,|V_{\mathbf{k}}|^2$$

vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing).

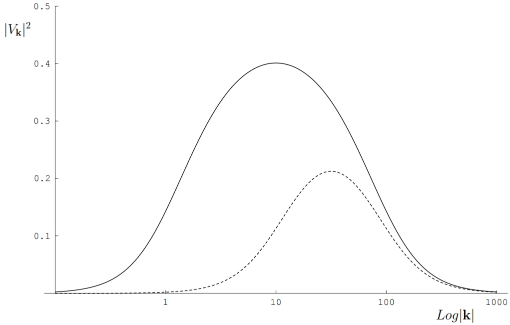
– Condensate structure as in systems with SSB (e.g. superconductors)

– Exotic condensates: mixed pairs

– Note that $|0\rangle_{e\,\mu} \neq |a\rangle_1 \otimes |b\rangle_2 \Rightarrow$ entanglement.

[†] A. Perelomov, *Generalized Coherent States,* (Springer V., 1986)

Condensation density for mixed fermions



Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

- $V_{\mathbf{k}} = 0$ when $m_1 = m_2$ and/or $\theta = 0$.
- Max. at $k = \sqrt{m_1 m_2}$ with $V_{max} \to \frac{1}{2}$ for $\frac{(m_2 - m_1)^2}{m_1 m_2} \to \infty$.
- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.

• Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\,\beta^r_{-\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right)$$

• Mixing transformation = Rotation + Bogoliubov transformation .

– Bogoliubov coefficients:

$$U_{\mathbf{k}}(t) = u^{r\dagger}_{\mathbf{k},2} u^r_{\mathbf{k},1}\, e^{i(\omega_{k,2}-\omega_{k,1})t} \quad ; \quad V_{\mathbf{k}}(t) = \epsilon^r\, u^{r\dagger}_{\mathbf{k},1} v^r_{-\mathbf{k},2}\, e^{i(\omega_{k,2}+\omega_{k,1})t}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$$

Decomposition of mixing generator *

Mixing generator function of m_1, m_2, and θ. Try to disentangle the mass dependence from the one by the mixing angle.

Let us define:

$$R(\theta) \equiv \exp\left\{\theta \sum_{\mathbf{k},r} \left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{\mathbf{k},1}\beta^{r}_{\mathbf{k},2}\right)e^{i\psi_k} - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{\mathbf{k},2}\beta^{r}_{\mathbf{k},1}\right)e^{-i\psi_k}\right]\right\},$$

$$B_i(\Theta_i) \equiv \exp\left\{\sum_{\mathbf{k},r} \Theta_{\mathbf{k},i}\ \epsilon^r \left[\alpha^{r}_{\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}e^{i\phi_{k,i}}\right]\right\}, \qquad i = 1, 2$$

Since $[B_1, B_2] = 0$ we put

$$B(\Theta_1, \Theta_2) \equiv B_1(\Theta_1)\, B_2(\Theta_2)$$

*M.B., M.V.Gargiulo and G.Vitiello, Phys. Lett. B (2017)

• We find:

$$G_\theta \;=\; B(\Theta_1,\Theta_2)\;\; R(\theta)\;\; B^{-1}(\Theta_1,\Theta_2)$$

which is realized when the $\Theta_{\mathbf{k},i}$ are chosen as:

$$U_{\mathbf{k}} = e^{-i\psi_k}\;\cos(\Theta_{\mathbf{k},1}-\Theta_{\mathbf{k},2})\,; \qquad V_{\mathbf{k}} = e^{\frac{(\phi_{k,1}+\phi_{k,2})}{2}}\;\sin(\Theta_{\mathbf{k},1}-\Theta_{\mathbf{k},2})$$

The $B_i(\Theta_{\mathbf{k},i})$, $i=1,2$ are Bogoliubov transformations implementing a mass shift, and $R(\theta)$ is a rotation.

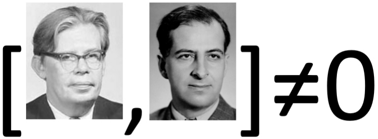
– Their action on the vacuum is given by:

$$|\widetilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1,\Theta_2)|0\rangle_{1,2} \;\;=\;\; \prod_{\mathbf{k},r,i}\Big[\cos\Theta_{\mathbf{k},i} + \epsilon^r \sin\Theta_{\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},i}\Big]|0\rangle_{1,2}$$

$$R^{-1}(\theta)|0\rangle_{1,2} \;\;=\;\; |0\rangle_{1,2}\;.$$

Bogoliubov vs Pontecorvo

Bogoliubov and Pontecorvo do not commute!



As a result, flavor vacuum gets a non-trivial term:

$$|0\rangle_{e,\mu} \equiv G_\theta^{-1}|0\rangle_{1,2} \; = \; |0\rangle_{1,2} \; + \; \left[B(m_1, m_2)\;,\; R^{-1}(\theta)\right] \; |\widetilde{0}\rangle_{1,2}$$

- Non-diagonal Bogoliubov transformation

$$|0\rangle_{e,\mu} \cong \left[\mathbb{1} + \theta\, a \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \; \tilde{V}_{\mathbf{k}} \sum_r \epsilon^r \left(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\right)\right]|0\rangle_{1,2}\,,$$

with $a \equiv \frac{(m_2-m_1)^2}{m_1 m_2}$.

Currents & Charges

Currents and charges for mixed fermions *

– Lagrangian in the mass basis:

$$\mathcal{L} = \bar{\nu}_m \left(i \not\partial - M_d\right) \nu_m$$

where $\nu_m^T = (\nu_1, \nu_2)$ and $M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$.

- $\mathcal{L}$ invariant under global $U(1)$ with conserved charge Q= total charge.

– Consider now the $SU(2)$ transformation:

$$\nu'_m = e^{i\alpha_j \tau_j} \nu_m \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

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*M. B., P. Jizba and G. Vitiello, Phys. Lett. B (2001);
E. Noether, Nachr. Gesellschaft. Wiss. Gottingen (1918).

The associated currents are:

$$\delta\mathcal{L} \;=\; i\alpha_j\,\bar{\nu}_m\,[\tau_j, M_d]\,\nu_m \;=\; -\alpha_j\,\partial_\mu J^\mu_{m,j}$$

$$J^\mu_{m,j} \;=\; \bar{\nu}_m\,\gamma^\mu\,\tau_j\,\nu_m$$

– The charges $Q_{m,j}(t) \equiv \int d^3\mathbf{x}\, J^0_{m,j}(x)$, satisfy the $su(2)$ algebra:

$$[Q_{m,j}(t), Q_{m,k}(t)] \;=\; i\,\epsilon_{jkl}\,Q_{m,l}(t)\,.$$

– Casimir operator proportional to the total charge: $C_m = \frac{1}{2}Q$.

- $Q_{m,3}$ is conserved ⇒ charge conserved separately for ν_1 and ν_2:

$$Q_1 \;=\; \frac{1}{2}Q + Q_{m,3} \;=\; \int d^3\mathbf{x}\,\nu_1^\dagger(x)\,\nu_1(x)$$

$$Q_2 \;=\; \frac{1}{2}Q - Q_{m,3} \;=\; \int d^3\mathbf{x}\,\nu_2^\dagger(x)\,\nu_2(x).$$

These are the flavor (Noether) charges in the absence of mixing.

The currents in the flavor basis

– Lagrangian in the flavor basis:

$$\mathcal{L} = \bar{\nu}_f \left(i \not\partial - M\right) \nu_f$$

where $\nu_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$.

– Consider the $SU(2)$ transformation:

$$\nu_f' = e^{i\alpha_j \tau_j} \, \nu_f \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

– The charges $Q_{f,j} \equiv \int d^3\mathbf{x} \, J^0_{f,j}$ satisfy the $su(2)$ algebra:

$$[Q_{f,j}(t), Q_{f,k}(t)] = i \, \epsilon_{jkl} \, Q_{f,l}(t).$$

– Casimir operator proportional to the total charge $C_f = C_m = \frac{1}{2} Q$.

- $Q_{f,3}$ is not conserved $\Rightarrow$ exchange of charge between ν_e and ν_μ.

Define the flavor charges as:

$$Q_e(t) \equiv \frac{1}{2}Q + Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x)$$

$$Q_\mu(t) \equiv \frac{1}{2}Q - Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

where $Q_e(t) + Q_\mu(t) = Q$.

– We have:

$$Q_e(t) = \cos^2\theta\, Q_1 + \sin^2\theta\, Q_2 + \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

$$Q_\mu(t) = \sin^2\theta\, Q_1 + \cos^2\theta\, Q_2 - \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

In conclusion:

– In presence of mixing, neutrino flavor charges are defined as

$$Q_e(t) \equiv \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x) \quad ; \qquad Q_\mu(t) \equiv \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

– They are not conserved charges ⇒ flavor oscillations.

– They are still (at tree level) conserved in the vertex ⇒ define flavor neutrinos as their eigenstates

- Problem: find the eigenstates of the above charges.

- Flavor charge operators are diagonal in the flavor ladder operators:

$$:: Q_\sigma(t) :: \;\equiv\; \int d^3\mathbf{x} \; :: \nu_\sigma^\dagger(x)\, \nu_\sigma(x) ::$$

$$= \; \sum_r \int d^3\mathbf{k} \, \Big(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t) \alpha^r_{\mathbf{k},\sigma}(t) \, - \, \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \beta^r_{-\mathbf{k},\sigma}(t) \Big) \, , \quad \sigma = e, \mu.$$

Here $:: \ldots ::$ denotes normal ordering w.r.t. flavor vacuum:

$$:: A :: \equiv \; A \, - \, {}_{e,\mu}\langle 0 | A | 0 \rangle_{e,\mu}$$

- Define flavor neutrino states with definite momentum and helicity:

$$|\nu^r_{\mathbf{k},\sigma}\rangle \, \equiv \, \alpha^{r\dagger}_{\mathbf{k},\sigma}(0) \, |0\rangle_{e,\mu}$$

– Such states are eigenstates of the flavor charges (at t=0):

$$:: Q_\sigma :: |\nu^r_{\mathbf{k},\sigma}\rangle \, = \, |\nu^r_{\mathbf{k},\sigma}\rangle$$

Neutrino oscillation formula (QFT)

– We have, for an electron neutrino state:

$$\mathcal{Q}_{\mathbf{k},\sigma}(t) \equiv \langle \nu^r_{\mathbf{k},e}| :: Q_\sigma(t) :: |\nu^r_{\mathbf{k},e}\rangle$$

$$= \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2$$

• Neutrino oscillation formula (exact result)*:

$$\mathcal{Q}_{\mathbf{k},e}(t) = 1 - |U_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\, t\right) - |V_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}\, t\right)$$

$$\mathcal{Q}_{\mathbf{k},\mu}(t) = |U_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\, t\right) + |V_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}\, t\right)$$

- For $k \gg \sqrt{m_1 m_2}$, $|U_\mathbf{k}|^2 \to 1$ and $|V_\mathbf{k}|^2 \to 0$.

*M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999).

Lepton charge violation for Pontecorvo states[†]

– Pontecorvo states:

$$
\begin{aligned}
|\nu^r_{\mathbf{k},e}\rangle_P &= \cos\theta\,|\nu^r_{\mathbf{k},1}\rangle + \sin\theta\,|\nu^r_{\mathbf{k},2}\rangle \\
|\nu^r_{\mathbf{k},\mu}\rangle_P &= -\sin\theta\,|\nu^r_{\mathbf{k},1}\rangle + \cos\theta\,|\nu^r_{\mathbf{k},2}\rangle\,,
\end{aligned}
$$

are *not* eigenstates of the flavor charges.

⇒ *violation of lepton charge conservation* in the production/detection vertices, at tree level:

$$
{}_P\langle\nu^r_{\mathbf{k},e}|:Q_e(0):|\nu^r_{\mathbf{k},e}\rangle_P = \cos^4\theta + \sin^4\theta + 2|U_{\mathbf{k}}|\sin^2\theta\cos^2\theta < 1,
$$

for any $\theta \neq 0$, $\mathbf{k} \neq 0$ and for $m_1 \neq m_2$.

[†] M. B., A. Capolupo, F. Terranova and G. Vitiello, Phys. Rev. **D** (2005)
C. C. Nishi, Phys. Rev. **D** (2008).

Other results

- Rigorous mathematical treatment for any number of flavors [*]
- Three flavor fermion mixing: CP violation[†];
- QFT spacetime dependent neutrino oscillation formula[‡];
- Boson mixing[§];
- Majorana neutrinos[¶];

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[*] K. C. Hannabuss and D. C. Latimer, J. Phys. A (2000); J. Phys. A (2003);
[†] M.B., A.Capolupo and G.Vitiello, Phys. Rev. **D** (2002)
[‡] M.B., P. Pires Pachêco and H. Wan Chan Tseung, Phys. Rev. **D**, (2003).
[§] M.B., A.Capolupo, O.Romei and G.Vitiello, Phys. Rev. **D**(2001); M.Binger and C.R.Ji. Phys. Rev. **D**(1999); C.R.Ji and Y.Mishchenko, Phys. Rev. **D**(2001); Phys. Rev. **D**(2002).
[¶] M.B. and J.Palmer, Phys. Rev. **D** (2004)

- Flavor vacuum and cosmological constant[*]
- Flavor vacuum induced by condensation of D-particles.[†]
- Geometric phase for mixed particles[‡].

[*] M.B., A.Capolupo, S.Capozziello, S.Carloni and G.Vitiello Phys. Lett. A (2004);
[†] N.E.Mavromatos and S.Sarkar, New J. Phys. (2008); N.E.Mavromatos, S.Sarkar and W.Tarantino, Phys. Rev. D (2008); Phys. Rev. D (2011).
[‡] M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999)

Dynamical generation of flavor mixing

• The non trivial nature of flavor vacuum suggests a dynamical origin;

• We consider dynamical symmetry breaking in a toy model with two flavors and quartic interaction term, as a generalization of Nambu and Jona-Lasinio model*;

• The approach of Umezawa, Takahashi and Kamefuchi for describing mass generation using inequivalent representations† is suitable for our purposes.

* M.B., P.Jizba, G.Lambiase and N.E.Mavromatos, J. Phys. Conf. Ser. (2014)

† H. Umezawa, Y. Takahashi and S. Kamefuchi, Ann. Phys. (1964)

Dynamical mass generation and inequivalent reps.

Consider a free Dirac field (at finite volume V):

$$\psi = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u_{\mathbf{k}} \; a^r_{\mathbf{k}} e^{-ik\cdot x} \, + \, v_{\mathbf{k}} \; b^{r\dagger}_{\mathbf{k}} e^{ik\cdot x} \right] , \qquad a^r_{\mathbf{k}}|0\rangle = b^r_{\mathbf{k}}|0\rangle = 0$$

The *same* field operator can be expanded as

$$\psi \; = \; \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k}}(\vartheta,\varphi) \alpha^r_{\mathbf{k}} \, e^{i\mathbf{k}\cdot\mathbf{x}} + v^r_{\mathbf{k}}(\vartheta,\varphi) \beta^{r\dagger}_{\mathbf{k}} \, e^{-i\mathbf{k}\cdot\mathbf{x}} \right],$$

with

$$\begin{aligned}
\alpha^r_{\mathbf{k}} \; &= \; \cos\vartheta^r_{\mathbf{k}} \; a^r_{\mathbf{k}} \; + \; \sin\vartheta^r_{\mathbf{k}} \; e^{i\varphi^r_k} \; b^{r\dagger}_{-\mathbf{k}} \\
\beta^r_{-\mathbf{k}} \; &= \; \cos\vartheta^r_{\mathbf{k}} \; b^r_{-\mathbf{k}} \; - \; \sin\vartheta^r_{\mathbf{k}} \; e^{i\varphi^r_k} \; a^{r\dagger}_{\mathbf{k}}
\end{aligned}$$

and

$$\begin{aligned}
u^r_{\mathbf{k}}(\vartheta,\phi) \; &= \; u^r_{\mathbf{k}} \cos\vartheta_k \; + \; v^r_{-\mathbf{k}} \, e^{-i\varphi^r_k} \sin\vartheta_k \, , \\
v^r_{\mathbf{k}}(\vartheta,\phi) \; &= \; v^r_{\mathbf{k}} \cos\vartheta_k \; - \; u^r_{-\mathbf{k}} \, e^{i\varphi^r_k} \sin\vartheta_k \, .
\end{aligned}$$

The above is a Bogoliubov transformation, inducing inequivalent representations for different values of the parameters (ϑ, φ):

$$|0(\vartheta, \varphi)\rangle = \prod_{\mathbf{k},r} \left[\cos\vartheta_k - e^{i\varphi_k^r} \sin\vartheta_k \; a_{\mathbf{k}}^{r\dagger} \, b_{-\mathbf{k}}^{r\dagger} \right] |0\rangle$$

with $\alpha_{\mathbf{k}}^r \, |0(\vartheta, \varphi)\rangle \; = \; \beta_{\mathbf{k}}^r \, |0(\vartheta, \varphi)\rangle \; = \; 0 \, .$

In the infinite volume limit, one has the following relations:

$$V\text{-lim}\left[\int d^3\boldsymbol{x}\,\bar\psi_\alpha(x)\psi_\beta(x)\right] = \int d^3\boldsymbol{x}\,:\bar\psi_\alpha(x)\psi_\beta(x): + \int d^3\boldsymbol{x}\,iS^-_{\alpha\beta}(\vartheta,\varphi)\,,$$

$$V\text{-lim}\left[\int d^3\boldsymbol{x}\,\bar\psi_\alpha(x)\psi_\beta(x)\bar\psi_\gamma(x)\psi_\delta(x)\right] =$$

$$= iS^-_{\alpha\beta}(\vartheta,\varphi)\int d^3\boldsymbol{x}\,:\bar\psi_\gamma(x)\psi_\delta(x): + iS^+_{\gamma\delta}(\vartheta,\varphi)\int d^3\boldsymbol{x}\,:\bar\psi_\alpha(x)\psi_\beta(x):$$

$$+ \; iS^-_{\alpha\delta}(\vartheta,\varphi)\int d^3\boldsymbol{x}\,:\bar\psi_\gamma(x)\psi_\beta(x): + iS^+_{\gamma\beta}(\vartheta,\varphi)\int d^3\boldsymbol{x}\,:\bar\psi_\alpha(x)\psi_\delta(x):$$

$$+\int d^3\boldsymbol{x}\sum_{\text{contractions}} S^+(\vartheta,\varphi)S^+(\vartheta,\varphi)\,.$$

$S^\pm_{\alpha\beta}(\theta,\varphi)$ are free two-point Wightman functions evaluated in $|0(\theta,\varphi)\rangle$:

$$iS^+_{\alpha\beta}(\vartheta,\varphi) = \langle 0(\vartheta,\varphi)|\bar\psi_\alpha(x)\psi_\beta(x)|0(\vartheta,\varphi)\rangle,$$

$$iS^-_{\alpha\beta}(\vartheta,\varphi) = \langle 0(\theta,\varphi)|\bar\psi_\alpha(x)\psi_\beta(x)|0(\vartheta,\varphi)\rangle$$

We consider the following hamiltonian:

$$H = H_0 + H_{\rm int} ,$$

$$H_0 = \int d^3\mathbf{x}\, \bar{\psi}\, (-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m)\, \psi ,$$

$$H_{\rm int} = \lambda \int d^3\mathbf{x} \left[\left(\bar{\psi}\psi\right)^2 - \left(\bar{\psi}\gamma^5\psi\right)^2 \right] .$$

In the lowest order in the Yang-Feldman eq. the V-limit of H gives:

$$V\text{-lim}\,[H] = \bar{H}_0 + c - number.$$

with

$$\bar{H}_0 = H_0 + \delta H_0$$

$$\delta H_0 = \int d^3x \left(f\bar{\psi}\psi + ig\bar{\psi}\gamma_5\psi \right)$$

where f, g depend on the set of parameters (ϑ, φ):

$$f = \lambda\, C_s\,, \qquad g = \lambda\, C_p\,.$$

$$C_p \equiv i \lim_{V\to\infty} \langle 0(\vartheta,\varphi)|\bar{\psi}(x)\gamma_5\psi(x)|0(\vartheta,\varphi)\rangle = \frac{2}{(2\pi)^3}\int d^3\mathbf{k}\, \sin 2\vartheta_k \;\sin\varphi_k$$

$$C_s \equiv \lim_{V\to\infty} \langle 0(\vartheta,\varphi)|\bar{\psi}(x)\psi(x)|0(\vartheta,\varphi)\rangle = -\frac{2}{(2\pi)^3}\int d^3\mathbf{k}\, \left[\frac{m}{\omega_k}\cos 2\vartheta_k - \frac{k}{\omega_k}\sin 2\vartheta_k \cos\varphi_k\right].$$

We then require that $\bar{H}_0$ has the form of a free Hamiltonian:

$$\bar{H}_0 \;=\; \sum_r \int d^3\mathbf{k}\; E_{\mathbf{k}} \left(\alpha_{\mathbf{k}}^{r\dagger}\, \alpha_{\mathbf{k}}^{r} + \beta_{\mathbf{k}}^{r\dagger}\, \beta_{\mathbf{k}}^{r} \right) + W_0 .$$

with

$$E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + M^2} \quad ; \qquad W_0 = -2 \int d^3\mathbf{k}\; E_{\mathbf{k}} .$$

by fixing the Bogoliubov transformation parameters. One obtains:

$$\cos 2\vartheta_{\mathbf{k}}^{r} \;=\; \frac{1}{E_k}\left[\omega_k + f\,\frac{m}{\omega_k}\right]$$

$$\cos\varphi(\mathbf{k},\mathbf{r}) \;=\; -f\,\frac{k}{\omega_k}\,\frac{1}{\sqrt{g^2 + f^2 (k^2/\omega_k^2)}}$$

$$M^2 \;=\; (m+f)^2 + g^2 \, .$$

Two possibilities:

$$C_p = 0, \qquad M = m - \frac{2\lambda}{(2\pi)^3} M \int \frac{d^3\mathbf{k}}{E_k},$$

$$m = 0, \qquad 1 + \frac{2\lambda}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{E_k} = 0.$$

The second case is only allowed for $\lambda < 0$.

Dynamical generation of flavor mixing

– We consider the following hamiltonian:

$$H = H_0 + H_{int}$$

$$H_0 = \int d^3x \bar{\Psi} \left(-i\gamma^i \partial_i + M_0\right) \Psi$$

with $\Psi^T = (\psi_{\rm I}, \psi_{\rm II})$ and $M_0 = diag(m_{\rm I}, m_{\rm II})$.

– The interaction Hamiltonian $H_{\rm int}$ can be assumed in the generic form

$$\mathcal{H}_{\rm int} = \left(\bar{\boldsymbol{\psi}}\, \Gamma\, \boldsymbol{\psi}\right) \left(\bar{\boldsymbol{\psi}}\, \Gamma'\, \boldsymbol{\psi}\right),$$

where Γ and Γ' are some doublet spinor matrices.

– The V-limit renormalization term $\delta\mathcal{H}_0$ has the following structure

$$\delta\mathcal{H}_0 = \delta\mathcal{H}_0^{\rm I} + \delta\mathcal{H}_0^{\rm II} + \delta\mathcal{H}_{\rm mix}$$

$$= f_{\rm I}\, \bar{\psi}_{\rm I}\psi_{\rm I} + f_{\rm II}\, \bar{\psi}_{\rm II}\psi_{\rm II} + h\left(\bar{\psi}_{\rm I}\psi_{\rm II} + \bar{\psi}_{\rm II}\psi_{\rm I}\right).$$

Generalized Bogoliubov transformation

– We consider the 4×4 canonical transformation

$$\begin{pmatrix} \alpha_A \\ \alpha_B \\ \beta_A^\dagger \\ \beta_B^\dagger \end{pmatrix} = \begin{pmatrix} c_\theta\, \rho_{A\,\mathrm{I}} & s_\theta\, \rho_{A\,\mathrm{II}} & c_\theta\, \lambda_{A\,\mathrm{I}} & s_\theta\, \lambda_{A\,\mathrm{II}} \\ -s_\theta\, \rho_{B\,\mathrm{I}} & c_\theta\, \rho_{B\,\mathrm{II}} & -s_\theta\, \lambda_{B\,\mathrm{I}} & c_\theta\, \lambda_{B\,\mathrm{II}} \\ c_\theta\, \lambda_{A\,\mathrm{I}} & s_\theta\, \lambda_{A\,\mathrm{II}} & c_\theta\, \rho_{A\,\mathrm{I}} & s_\theta\, \rho_{A\,\mathrm{II}} \\ -s_\theta\, \lambda_{B\,\mathrm{I}} & c_\theta\, \lambda_{B\,\mathrm{II}} & -s_\theta\, \rho_{B\,\mathrm{I}} & c_\theta\, \rho_{B\,\mathrm{I}} \end{pmatrix} \begin{pmatrix} a_{\mathrm{I}} \\ a_{\mathrm{II}} \\ b_{\mathrm{I}}^\dagger \\ b_{\mathrm{II}}^\dagger \end{pmatrix}$$

where $c_\theta \equiv \cos\theta$, $s_\theta \equiv \sin\theta$ and

$$\rho_{ab} \equiv \cos\frac{\chi_a - \chi_b}{2}, \quad \lambda_{ab} \equiv \sin\frac{\chi_a - \chi_b}{2}\,, \quad \chi_a \equiv \cot^{-1}\left[\frac{k}{m_a}\right],\ a,b = I, II, A, B.$$

Thus we have three parameters (θ, m_A, m_B) to fix in terms of $(f_{\mathrm{I}}, f_{\mathrm{II}}, h)$ in order to diagonalize the Hamiltonian.

Partial diagonalization

A possible representation is obtained by a partial diagonalization of $\bar{H}_0$, leaving untouched $\delta\mathcal{H}_{mix}$:

$$\bar{\mathcal{H}}_0 \;=\; \sum_{\sigma=e,\mu} \bar{\psi}_\sigma \left(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} \;+\; m_\sigma\right)\psi_\sigma \;+\; h\left(\bar{\psi}_e\psi_\mu \;+\; \bar{\psi}_\mu\psi_e\right).$$

Such a representation is obtained by setting

$$\begin{aligned} \theta &\;\to\; 0\,, \\ m_A &\;\to\; m_e \;\equiv\; m_{\rm I} + f_{\rm I}\,, \\ m_B &\;\to\; m_\mu \;\equiv\; m_{\rm II} + f_{\rm II}\,. \end{aligned}$$

The vacuum is denoted as

$$|0(\theta = 0, m_e, m_\mu)\rangle \;\equiv\; |0\rangle_{e\mu}\,,$$

In this representation we have

$${}_{e,\mu}\langle 0|\bar{H}_0|0\rangle_{e,\mu} \;=\; -2\int d^3\mathbf{k}\;\left(\sqrt{k^2 \;+\; m_e^2} \;+\; \sqrt{k^2 \;+\; m_\mu^2}\right),$$

since ${}_{e,\mu}\langle 0|\delta\mathcal{H}_{\rm mix}|0\rangle_{e,\mu} \;=\; 0.$

Another possibility is to require that $\bar{\mathcal{H}}_0$ becomes fully diagonal in two fermion fields, ψ_1 and ψ_2, with masses m_1 and m_2:

$$\bar{\mathcal{H}}_0 \ = \ \sum_{j=1,2} \bar{\psi}_j \left(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} \ + \ m_j\right)\psi_j \,.$$

The condition for the complete diagonalization is found to be:

$$\theta \quad \rightarrow \quad \bar{\theta} \ \equiv \ \frac{1}{2}\,\tan^{-1}\left[\frac{2h}{m_\mu - m_e}\right],$$

$$m_{_{A,B}} \quad \rightarrow \quad m_{1,2} \ \equiv \ \frac{1}{2}\left(m_e + m_\mu \mp \sqrt{(m_\mu - m_e)^2 + 4h^2}\right),$$

where we introduced the notation $m_e = m_{_{\rm I}} + f_{_{\rm I}}$, $m_\mu = m_{_{\rm II}} + f_{_{\rm II}}$.

We set

$$|0(\bar{\theta}, m_1, m_2)\rangle \ \equiv \ |0\rangle_{1,2}\,,$$

The vev of the Hamiltonian in this representation has the form:

$${}_{1,2}\langle 0|\bar{H}_0|0\rangle_{1,2} \ = \ -2\int d^3\mathbf{k} \ \left(\sqrt{k^2 \ + \ m_1^2} \ + \ \sqrt{k^2 \ + \ m_2^2}\right).$$

Patterns of Dynamical Symmetry Breaking

Chiral symmetry

Let us consider a Lagrangian $\mathcal{L}$, invariant under the *chiral-flavor* group $SU(2)_A \times SU(2)_V \times U(1)_V$. Consider a flavor fermion-doublet:

$$\boldsymbol{\psi} \;=\; \begin{bmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{bmatrix} .$$

The chiral transformations act on it as

$$\mathbf{g}\boldsymbol{\psi} \;=\; \exp\left[i\left(\phi + \boldsymbol{\omega}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\omega}_5\cdot\frac{\boldsymbol{\sigma}}{2}\gamma_5\right)\right]\boldsymbol{\psi}\,,$$

where $\sigma_i\,, i = 1, 2, 3$ are the Pauli matrices, ϕ is a real number and $\boldsymbol{\omega}$, $\boldsymbol{\omega}_5$ are vectors with real components.

Charges and Currents

- The vector and axial Noether currents are:

$$J^{\mu} = \overline{\psi}\gamma^{\mu}\psi; \qquad \boldsymbol{J}^{\mu} = \overline{\boldsymbol{\psi}}\gamma^{\mu}\frac{\boldsymbol{\sigma}}{2}\boldsymbol{\psi}; \qquad \boldsymbol{J}_5^{\mu} = \overline{\boldsymbol{\psi}}\gamma^{\mu}\gamma_5\frac{\boldsymbol{\sigma}}{2}\boldsymbol{\psi}$$

and the conserved Noether charges:

$$Q = \int d^3\mathbf{x}\, \psi^{\dagger}\psi; \qquad \boldsymbol{Q} = \int d^3\mathbf{x}\, \boldsymbol{\psi}^{\dagger}\frac{\boldsymbol{\sigma}}{2}\boldsymbol{\psi}; \qquad \boldsymbol{Q}_5 = \int d^3\mathbf{x}\, \boldsymbol{\psi}^{\dagger}\frac{\boldsymbol{\sigma}}{2}\gamma_5\boldsymbol{\psi}$$

Explicit Symmetry Breaking

- Symmetry is explicitly broken by a mass term:

$$\mathcal{L}_M \;=\; -\overline{\boldsymbol{\psi}}\, M\, \boldsymbol{\psi}$$

In fact

$$\begin{aligned} \partial_\mu J^\mu &= 0\,, \\ \partial_\mu \boldsymbol{J}^\mu &= \frac{i}{2}\overline{\boldsymbol{\psi}}\,[M, \boldsymbol{\sigma}]\,\boldsymbol{\psi}\,, \\ \partial_\mu \boldsymbol{J}_5^\mu &= \frac{i}{2}\overline{\boldsymbol{\psi}}\gamma_5\,\{M, \boldsymbol{\sigma}\}\,\boldsymbol{\psi} \end{aligned}$$

- If the mass matrix is proportional to the identity

$$M \;=\; m_0 \mathbb{1}$$

the axial symmetry is explicitly broken:

$$\begin{aligned} \partial_\mu J^\mu &= \partial_\mu \boldsymbol{J}^\mu \;=\; 0 \\ \partial_\mu \boldsymbol{J}_5^\mu &= i\, m_0\, \overline{\boldsymbol{\psi}}\, \gamma_5\, \boldsymbol{\psi} \end{aligned}$$

The residual symmetry is $H = U(2)_V$.

- Adding a mass-shift:

$$M \;=\; m_0\,\mathbb{1} \;+\; m_3\,\sigma_3$$

the isospin symmetry is broken to $H = U(1)_V \times U(1)^3_V$

$$\begin{aligned}
\partial_\mu J^\mu \;&=\; \partial_\mu J^\mu_3 \;=\; 0; \quad \partial_\mu \boldsymbol{J}^\mu_5 \;\neq\; 0\,,\\
\partial_\mu J^\mu_1 \;&=\; \frac{m_3}{2}\,\overline{\boldsymbol{\psi}}\,\sigma_2\,\boldsymbol{\psi}\,,\\
\partial_\mu J^\mu_2 \;&=\; -\frac{m_3}{2}\,\overline{\boldsymbol{\psi}}\,\sigma_1\,\boldsymbol{\psi}
\end{aligned}$$

- Adding an off-diagonal component:

$$M \;=\; m_0\,\mathbb{1} \;+\; m_3\,\sigma_3 \;+\; m_1\,\sigma_1 \;+\; m_2\,\sigma_2$$

A residual phase symmetry $H = U(1)_V$ survives:

$$\begin{aligned}
\partial_\mu J^\mu &= 0\,, \qquad \partial_\mu \boldsymbol{J}_5^\mu \neq 0\,,\\
\partial_\mu J_1^\mu &= \overline{\boldsymbol{\psi}}\,(m_2\,\sigma_3 - m_3\,\sigma_2)\,\boldsymbol{\psi}\\
\partial_\mu J_2^\mu &= -\overline{\boldsymbol{\psi}}\,(m_1\sigma_3 \,-\, m_3\sigma_1)\,\boldsymbol{\psi}\\
\partial_\mu J_3^\mu &= \overline{\boldsymbol{\psi}}\,(m_1\sigma_2 - m_2\sigma_1)\,\boldsymbol{\psi}
\end{aligned}$$

Conservation of the total flavor charge Q.

Dynamical Generation of Fermion Mixing

- Dynamical generation of mixing occurs if*

$$SU(2)_A \times SU(2)_V \times U(1)_V \longrightarrow U(1)_V$$

at the ground state level. SSB is characterized by the existence of some local operator(s) ϕ so that on the vacuum $|\Omega\rangle$

$$\langle [N_i, \phi(0)] \rangle \ = \ \langle \varphi_i(0) \rangle \ \ \equiv \ v_i \ \neq \ 0 \, ,$$

where $\langle \ldots \rangle \equiv \langle \Omega | \ldots | \Omega \rangle$. Here v_i are the *order parameters* and N_i represent group generators from G/H.

In our case N_i will be given by $\boldsymbol{Q}$ and $\boldsymbol{Q}_5$ according to the chosen SSB scheme.

*M.B., P.Jizba, N.E.Mavromatos and L. Smaldone, arXiv:1807.07616 [hep-th].

We introduce the fermion bilinears

$$\Phi_k \;=\; \overline{\boldsymbol{\psi}}\, \sigma_k\, \boldsymbol{\psi}\,, \quad \Phi_k^5 \;=\; \overline{\boldsymbol{\psi}}\, \sigma_k\, \gamma_5 \boldsymbol{\psi}\,, \quad \sigma_0 \;\equiv\; \mathbb{1}\,,\,.$$

with $k = 0, 1, 2, 3$. For simplicity we now assume $\langle \mathbf{\Phi}^5 \rangle = 0$.

– Order parameters:

$$v_\alpha \;=\; \langle \Phi_\alpha(0) \rangle\,, \qquad \alpha = 0, 1, 2, 3$$

Dynamical Mass Generation

- Dynamical mass generation:

$$SU(2)_A \times SU(2)_V \times U(1)_V \longrightarrow SU(2)_V \times U(1)_V$$

Order parameters

$$\langle \Phi_0 \rangle \;=\; v_0 \;\neq 0\,, \qquad \langle \Phi_k \rangle \;=\; 0\,, \quad k = 1, 2, 3\,.$$

By means of ε-term $\mathcal{L}_\varepsilon \;=\; \varepsilon_0 \Phi_0$, Ward-Takahashi identities are derived:

$$i\, v_0 \;=\; \lim_{\varepsilon \to 0} \varepsilon \int \mathrm{d}^4 y \; \left\langle T\left[\Phi^5_k(y)\, \Phi^5_k(0)\right]\right\rangle\,, \quad k = 1, 2, 3.$$

The r.h.s. contains a pole at zero mass: Goldstone theorem

Proof of the above relation

We consider the identity:

$$i\langle \delta\phi(0)\rangle \; = \; \int d^4y \, \langle T[\delta\mathcal{L}(y)\, \phi(0)]\rangle \, ,$$

with $\delta_k\phi(x) \; \equiv \; i\,[N_k, \phi(x)]$.

In particular we will choose $\phi = \delta\Phi$, so that:

$$i\langle \delta^2\Phi(0)\rangle \; = \; \lim_{\varepsilon\to 0} \varepsilon \, \int \mathrm{d}^4y \, \langle T\,[\delta\Phi(y)\delta\Phi(0)]\rangle \, .$$

If we take $N_k = Q_{5,k}$, $\Phi = \Phi_0$ and $\phi = \delta_{5,k}\Phi_0$, we get

$$i\langle \delta^2_{5,k}\Phi_0\rangle \; = \; \lim_{\varepsilon\to 0} \varepsilon \int \mathrm{d}^4y \;\; \langle T\,[\delta_{5,k}\Phi_0(y)\, \delta_{5,k}\Phi_0(0)]\rangle \, , \quad k = 1, 2, 3.$$

From $\delta_{5,k}\Phi_0 = -i\Phi^5_k$ and $\delta^2_{5,k}\Phi_0 = -\Phi_0$, we obtain the above eq.

Dynamical Generation of Different Masses

- Dynamical generation of different masses:

$$SU(2)_A \times SU(2)_V \times U(1)_V \;\longrightarrow\; U(1)^3_V \times U(1)_V$$

Order parameters

$$\langle \Phi_0 \rangle = v_0 \neq 0\,, \qquad \langle \Phi_3 \rangle = v_3 \neq 0\,.$$

With $\mathcal{L}_\varepsilon = \varepsilon(\Phi_0 + \Phi_3)$, W-T identities are derived:

$$i\,v_3 \;=\; -\lim_{\varepsilon\to 0} \varepsilon \int \mathrm{d}^4y\, \langle T\left[\Phi_k(y)\,\Phi_k(0)\right]\rangle\,, \quad k=1,2$$

$$i(v_3+v_0) \;=\; \lim_{\varepsilon\to 0} \varepsilon \int \mathrm{d}^4y\, \langle T\left[(\Phi^5_3(y)+\Phi^5_0(y))(\Phi^5_3(0)+\Phi^5_0(0))\right]\rangle.$$

Dynamical Generation of Mixing

- Dynamical generation of mixing:

$$SU(2)_A \times SU(2)_V \times U(1)_V \longrightarrow U(1)_V$$

Order parameters

$$\langle \Phi_k \rangle = v_k \neq 0\,, \qquad k \;=\; 0,1,2,3\,.$$

Necessary condition for dynamical generation of mixing is the presence of exotic pairs in the vacuum

$$\langle \overline{\psi}_i(x)\,\psi_j(x) \rangle \neq \; 0\,, \qquad i \;\neq\; j\,.$$

- Mixing of fields requires mixing in the vacuum.

WT identities and mixing generation

With $\mathcal{L}_\varepsilon = \varepsilon \sum_{k=0}^{3} \Phi_k$ we obtain ($k = 1, 2, 3$ and $j = 1, 3$):

$$
\begin{aligned}
i(v_0 + v_k) &= \lim_{\varepsilon \to 0} \varepsilon \int d^4y \, \langle T[(\Phi_k^5(y) + \Phi_0^5(y))(\Phi_k^5(0) + \Phi_0^5(0))] \rangle, \\
-i(v_2 + v_j) &= \lim_{\varepsilon \to 0} \varepsilon \int d^4y \, \langle T[(\Phi_2(y) - \Phi_j(y))(\Phi_2(0) - \Phi_j(0))] \rangle, \\
-i(v_1 + v_3) &= \lim_{\varepsilon \to 0} \varepsilon \int d^4y \, \langle T[(\Phi_1(y) - \Phi_3(y))(\Phi_1(0) - \Phi_3(0))] \rangle,
\end{aligned}
$$

NG modes associated with fields:

$$
\Phi_2 - \Phi_1, \quad \Phi_2 - \Phi_3, \quad \Phi_1 - \Phi_3, \quad \Phi_2^5 + \Phi_0^5, \quad \Phi_1^5 + \Phi_0^5, \quad \Phi_3^5 + \Phi_0^5
$$

The number of NG modes is thus 6 which coincides with $\dim(G/H)$.

Vacuum Structure in Mean-Field Approximation

Mass vacuum: Equal Masses

In mean field approximation, the mass vacuum, in terms of bare vacuum $|0\rangle$, is

$$|0\rangle_m = \prod_{j=1,2}\prod_{\mathbf{k},r}(\cos\Theta_{\mathbf{k}} - \epsilon^r \sin\Theta_{\mathbf{k}} \tilde{\alpha}^{r\dagger}_{\mathbf{k},j}\tilde{\beta}^{r\dagger}_{-\mathbf{k},j})|0\rangle$$

$$= B(m)|0\rangle\,, \qquad \Theta_{\mathbf{k}} = \frac{1}{2}\cot^{-1}\left(\frac{|\mathbf{k}|}{m}\right)$$

with $\epsilon^r = (-1)^r$. Here m is the physical mass and

$$B(m) = B_1(m)\,B_2(m)$$

where:

$$B_j(m) \;=\; \exp\left[\sum_r \int d^3\mathbf{k}\,\Theta_{\mathbf{k}}\,\epsilon^r\left(\tilde{\alpha}^r_{\mathbf{k},j}\tilde{\beta}^r_{-\mathbf{k},j} - \tilde{\beta}^{r\dagger}_{-\mathbf{k},j}\tilde{\alpha}^{r\dagger}_{\mathbf{k},j}\right)\right]$$

is the generator of a Bogoliubov transformation.

The order parameter is:

$$v_0 \;=\; 2\int d^3\mathbf{k}\,\sin 2\Theta_{\mathbf{k}}$$

Mass vacuum: Different Masses

The mass vacuum is:

$$
\begin{aligned}
|0\rangle_{1,2} &= \prod_{j=1,2}\prod_{\mathbf{k},r}(\cos\Theta_{\mathbf{k},j} - \epsilon^r \sin\Theta_{\mathbf{k},j}\,\tilde{\alpha}^{r\dagger}_{\mathbf{k},j}\tilde{\beta}^{r\dagger}_{-\mathbf{k},j})|0\rangle \\
&= B(m_1,m_2)|0\rangle\,, \qquad \Theta_{\mathbf{k},j} \;=\; \frac{1}{2}\cot^{-1}\left(\frac{|\mathbf{k}|}{m_j}\right), j=1,2
\end{aligned}
$$

$B(m_1, m_2)$ factorizes as the product of Bogoliubov transformations:

$$
B(m_1,m_2) \;=\; B_1(m_1)\,B_2(m_2)
$$

The ladder operators of massive fields are:

$$
\begin{aligned}
\alpha^r_{\mathbf{k},j} &= B(m)\,\tilde{\alpha}^r_{\mathbf{k},j}\,B^{-1}(m) \;=\; \cos\Theta_{\mathbf{k},j}\tilde{\alpha}^r_{\mathbf{k},j} + \epsilon^r\sin\Theta_{\mathbf{k},j}\tilde{\beta}^{r\dagger}_{-\mathbf{k},j} \\
\beta^r_{-\mathbf{k},j} &= B(m)\,\tilde{\beta}^{r\dagger}_{-\mathbf{k},j}\,B^{-1}(m) \;=\; \cos\Theta_{\mathbf{k},j}\tilde{\beta}^r_{-\mathbf{k},j} - \epsilon^r\sin\Theta_{\mathbf{k},j}\tilde{\alpha}^{r\dagger}_{\mathbf{k},j}
\end{aligned}
$$

Flavor Vacuum

Consider the charge

$$
\begin{aligned}
Q_2 &= -\frac{i}{2}\int d^3\mathbf{x}\,\left(\tilde{\psi}_1^\dagger(x)\,\tilde{\psi}_2(x) - \tilde{\psi}_2^\dagger(x)\,\tilde{\psi}_1(x)\right) \\
&= -\frac{i}{2}\int d^3\mathbf{k}\,\left(\tilde{\alpha}^{r\dagger}_{\mathbf{k},1}\tilde{\alpha}^{r}_{\mathbf{k},2} + \tilde{\beta}^{r\dagger}_{\mathbf{k},1}\tilde{\beta}^{r}_{\mathbf{k},2} - \tilde{\alpha}^{r\dagger}_{\mathbf{k},2}\tilde{\alpha}^{r}_{\mathbf{k},1} - \tilde{\beta}^{r\dagger}_{\mathbf{k},2}\tilde{\beta}^{r}_{\mathbf{k},1}\right)
\end{aligned}
$$

Exponentiating it, we obtain the generator of a rotation:

$$
\tilde{R}(\theta) \;=\; \exp\left(2i\theta Q_2\right)
$$

In terms of mass fields:

$$
\tilde{R}(\theta) = \exp\left[\theta\int d^3\mathbf{x}\,\left(\psi_1^\dagger(x)\,\psi_2(x) - \psi_2^\dagger(x)\,\psi_1(x)\right)\right]\Bigg|_{t=0} \;\equiv\; G_\theta(0)
$$

- This is the mixing generator at $t = 0$.

Consider the charge

$$Q_3 = \frac{1}{2}\int d^3\mathbf{x}\,\left(\tilde{\psi}_1^\dagger(x)\,\tilde{\psi}_1(x) - \tilde{\psi}_2^\dagger(x)\,\tilde{\psi}_2(x)\right)$$
$$= \int d^3\mathbf{k}\,\left(\tilde{\alpha}^{r\dagger}_{\mathbf{k},1}\tilde{\alpha}^{r}_{\mathbf{k},1} - \tilde{\beta}^{r\dagger}_{\mathbf{k},1}\tilde{\beta}^{r}_{\mathbf{k},1} - \tilde{\alpha}^{r\dagger}_{\mathbf{k},2}\tilde{\alpha}^{r}_{\mathbf{k},2} + \tilde{\beta}^{r\dagger}_{\mathbf{k},2}\tilde{\beta}^{r}_{\mathbf{k},2}\right)$$

Exponentiating it, we obtain the generator of a phase transformation:

$$P(\theta) = \exp\left(i\phi Q_3\right)$$

The ground state takes the form

$$|0\rangle_{e,\mu} = P^{-1}(\phi)\tilde{R}^{-1}(\theta)B(m_1,m_2)|0\rangle$$

Flavor vacuum: order parameters

The order parameters are now:

$$
\begin{aligned}
v_0 &= \sum_{j=1,2} \int d^3\mathbf{k}\ \sin 2\Theta_{\mathbf{k},j} \\
v_3 &= \cos 2\theta \left(\int d^3\mathbf{k}\ \sin 2\Theta_{\mathbf{k},1} - \int d^3\mathbf{k}\ \sin 2\Theta_{\mathbf{k},2} \right) \\
v_1 &= \cos 2\phi\ \sin 2\theta \left(\int d^3\mathbf{k}\ \sin 2\Theta_{\mathbf{k},1} - \int d^3\mathbf{k}\ \sin 2\Theta_{\mathbf{k},2} \right) \\
v_2 &= i\ \sin 2\phi\ \sin 2\theta \left(\int d^3\mathbf{k}\ \sin 2\Theta_{\mathbf{k},1} - \int d^3\mathbf{k}\ \sin 2\Theta_{\mathbf{k},2} \right)
\end{aligned}
$$

Conclusions

Conclusions and Perspectives

- Mixing transformations are not trivial in Q.F.T. (not just a rotation!) $\Rightarrow$ inequivalent representations.

- The vacuum for mixed fields has the structure of a $SU(N)$ generalized coherent state (condensate of particle-antiparticle pairs).

- Condensate structure of the flavor vacuum $\Rightarrow$ dynamical origin of mixing.

- Lorentz invariance violation (?)

Three flavor mixing; CP violation

Three-flavor fermion mixing in QFT[†]

Mixing relations:

$$\Psi_f(x) \; = \; \mathbf{M}\, \Psi_m(x)$$

where $\Psi_f^T = (\nu_e, \nu_\mu, \nu_\tau)$, $\Psi_m^T = (\nu_1, \nu_2, \nu_3)$ and

$$\mathbf{M} \, = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

with $c_{ij} = \cos\theta_{ij}, s_{ij} = \sin\theta_{ij}$

[†] M.B., A.Capolupo and G.Vitiello, Phys. Rev. **D** (2002)

We have:

$$\nu_\sigma^\alpha(x) = G_\theta^{-1}(t)\, \nu_i^\alpha(x)\, G_\theta(t),$$

where $(\sigma, i) = (e,1), (\mu,2), (\tau,3)$, and

$$\begin{aligned}
G_\theta(t) &= G_{2_3}(t) G_{1_3}(t) G_{12}(t) \\
G_{12}(t) &= exp\left[\theta_{12} \int d^3\mathbf{x} (\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x))\right], \\
G_{13}(t) &= exp\left[\theta_{13} \int d^3\mathbf{x} (\nu_1^\dagger(x)\nu_3(x) e^{-i\delta} - \nu_3^\dagger(x)\nu_1(x) e^{i\delta})\right], \\
G_{23}(t) &= exp\left[\theta_{23} \int d^3\mathbf{x} (\nu_2^\dagger(x)\nu_3(x) - \nu_3^\dagger(x)\nu_2(x))\right],
\end{aligned}$$

Flavor vacuum:

$$|0\rangle_f = G_\theta^{-1}(t)\, |0\rangle_m$$

Flavor annihilation operators:

$$\alpha^r_{\mathbf{k},e} = c_{12}c_{13}\ \alpha^r_{\mathbf{k},1} + s_{12}c_{13}\left(U^{\mathbf{k}*}_{12}\ \alpha^r_{\mathbf{k},2} + \epsilon^r V^{\mathbf{k}}_{12}\ \beta^{r\dagger}_{-\mathbf{k},2}\right) + e^{-i\delta}\ s_{13}\left(U^{\mathbf{k}*}_{13}\ \alpha^r_{\mathbf{k},3} + \epsilon^r V^{\mathbf{k}}_{13}\ \beta^{r\dagger}_{-\mathbf{k},3}\right)\ ,$$

$$\alpha^r_{\mathbf{k},\mu} = \left(c_{12}c_{23} - e^{i\delta}\ s_{12}s_{23}s_{13}\right)\ \alpha^r_{\mathbf{k},2} - \left(s_{12}c_{23} + e^{i\delta}\ c_{12}s_{23}s_{13}\right)\left(U^{\mathbf{k}}_{12}\ \alpha^r_{\mathbf{k},1} - \epsilon^r V^{\mathbf{k}}_{12}\ \beta^{r\dagger}_{-\mathbf{k},1}\right)$$

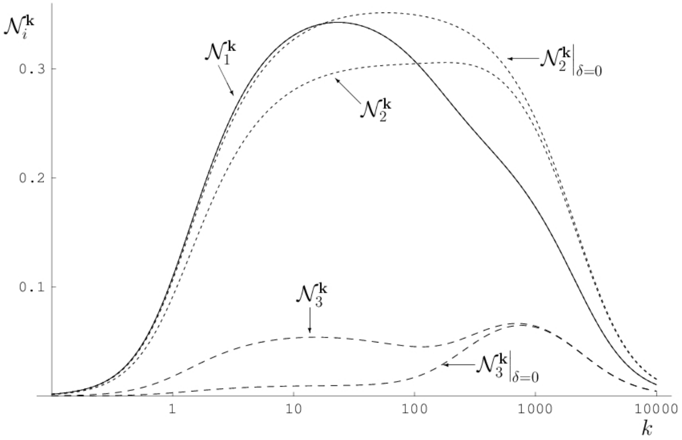
$$+ s_{23}c_{13}\left(U^{\mathbf{k}*}_{23}\ \alpha^r_{\mathbf{k},3} + \epsilon^r V^{\mathbf{k}}_{23}\ \beta^{r\dagger}_{-\mathbf{k},3}\right)\ ,$$

$$\alpha^r_{\mathbf{k},\tau} = c_{23}c_{13}\ \alpha^r_{\mathbf{k},3} - \left(c_{12}s_{23} + e^{i\delta}\ s_{12}c_{23}s_{13}\right)\left(U^{\mathbf{k}}_{23}\ \alpha^r_{\mathbf{k},2} - \epsilon^r V^{\mathbf{k}}_{23}\ \beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$+ \left(s_{12}s_{23} - e^{i\delta}\ c_{12}c_{23}s_{13}\right)\left(U^{\mathbf{k}}_{13}\ \alpha^r_{\mathbf{k},1} - \epsilon^r V^{\mathbf{k}}_{13}\ \beta^{r\dagger}_{-\mathbf{k},1}\right)$$

and similar ones for antiparticles ($\delta \to -\delta$).

Condensation densities



Condensation densities $\mathcal{N}_i^{\mathbf{k}}$ for sample values of masses and mixings

Parameterizations of mixing matrix

$$\nu_\sigma^\alpha(x) = G_\theta^{-1}(t)\,\nu_i^\alpha(x)\,G_\theta(t),$$

Define the more general generators:

$$G_{12} \equiv \exp\left[\theta_{12}\int d^3x\left(\nu_1^\dagger\nu_2 e^{-i\delta_2} - \nu_2^\dagger\nu_1 e^{i\delta_2}\right)\right]$$

$$G_{1_3} \equiv \exp\left[\theta_{1_3}\int d^3x\left(\nu_1^\dagger\nu_3 e^{-i\delta_5} - \nu_3^\dagger\nu_1 e^{i\delta_5}\right)\right]$$

$$G_{2_3} \equiv \exp\left[\theta_{2_3}\int d^3x\left(\nu_2^\dagger\nu_3 e^{-i\delta_7} - \nu_3^\dagger\nu_2 e^{i\delta_7}\right)\right]$$

There are six different matrices obtained by permutations of the above generators.

We can obtain all possible parameterizations of the matrix by setting to zero two of the phases and permuting rows/columns.

Currents and charges for 3-flavor fermion mixing

Lagrangian for three free Dirac fields with different masses

$$\mathcal{L}(x) \,=\, \bar{\Psi}_m(x) \left(i \not\partial - M_d \right) \Psi_m(x)$$

where $\Psi_m^T = (\nu_1, \nu_2, \nu_3)$ and $M_d = diag(m_1, m_2, m_3)$.

The $SU(3)$ transformations:

$$\Psi'_m(x) \,=\, e^{i\alpha_j \lambda_j/2}\, \Psi_m(x) \quad ; \qquad j = 1,..,8$$

with α_j real constants, and λ_j the Gell-Mann matrices, give the currents:

$$J^\mu_{m,j}(x) \,=\, \frac{1}{2} \bar{\Psi}_m(x)\, \gamma^\mu\, \lambda_j\, \Psi_m(x)$$

The combinations:

$$Q_1 \equiv \frac{1}{3}Q + Q_{m,3} + \frac{1}{\sqrt{3}}Q_{m,8},$$

$$Q_2 \equiv \frac{1}{3}Q - Q_{m,3} + \frac{1}{\sqrt{3}}Q_{m,8}$$

$$Q_3 \equiv \frac{1}{3}Q - \frac{2}{\sqrt{3}}Q_{m,8}$$

$$Q_i = \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i}\beta^r_{-\mathbf{k},i} \right), \ \ i = 1, 2, 3.$$

are the Noether charges for the fields ν_i with $\sum_i Q_i = Q$.

Flavor charges:

$$Q_\sigma(t) ::= G_\theta^{-1}(t) : Q_i : G_\theta(t) = \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^r_{\mathbf{k},\sigma}(t) - \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\beta^r_{-\mathbf{k},\sigma}(t) \right)$$

CP violation and $SU(3)$

Modified Gell-Mann matrices:

$$\tilde{\lambda}_1 = \begin{pmatrix} 0 & e^{i\delta_2} & 0 \\ e^{-i\delta_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tilde{\lambda}_2 = \begin{pmatrix} 0 & -ie^{i\delta_2} & 0 \\ ie^{-i\delta_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tilde{\lambda}_4 = \begin{pmatrix} 0 & 0 & e^{-i\delta_5} \\ 0 & 0 & 0 \\ e^{i\delta_5} & 0 & 0 \end{pmatrix}$$

$$\tilde{\lambda}_5 = \begin{pmatrix} 0 & 0 & -ie^{-i\delta_5} \\ 0 & 0 & 0 \\ ie^{i\delta_5} & 0 & 0 \end{pmatrix}, \tilde{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i\delta_7} \\ 0 & e^{-i\delta_7} & 0 \end{pmatrix}, \tilde{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -ie^{i\delta_7} \\ 0 & ie^{-i\delta_7} & 0 \end{pmatrix}$$

$$\tilde{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \tilde{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Dynamical generation of mixing – three flavors

Global *chiral-flavor* group $G = SU(3)_A \times SU(3)_V \times U(1)_V$. Let the fermion field be a flavor triplet

$$\boldsymbol{\psi} \;=\; \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} \,.$$

Under a chiral-group transformation $\mathbf{g}$, we get

$$\boldsymbol{\psi} \;\rightarrow\; \boldsymbol{\psi}' \;=\; \mathbf{g}\boldsymbol{\psi} \;\;=\;\; \exp\left[i\left(\phi + \boldsymbol{\omega}\cdot\frac{\boldsymbol{\lambda}}{2} + \boldsymbol{\omega}_5\cdot\frac{\boldsymbol{\lambda}}{2}\gamma_5\right)\right]\boldsymbol{\psi}\,,$$

where $\lambda_j\,, j = 1, \ldots, 8$ are the Gell-Mann matrices

Noether's currents and (conserved) charges

$$J^\mu \ = \ \overline{\boldsymbol{\psi}}\gamma^\mu \boldsymbol{\psi}\,, \quad \boldsymbol{J}^\mu \ = \ \overline{\boldsymbol{\psi}}\gamma^\mu \frac{\boldsymbol{\lambda}}{2}\boldsymbol{\psi}\,, \quad \boldsymbol{J}_5^\mu \ = \ \overline{\boldsymbol{\psi}}\gamma^\mu\gamma_5 \frac{\boldsymbol{\lambda}}{2}\boldsymbol{\psi}\,,$$

$$Q \ = \ \int\! \mathrm{d}^3\mathbf{x}\, \boldsymbol{\psi}^\dagger \boldsymbol{\psi}\,, \quad \boldsymbol{Q} \ = \ \int\! \mathrm{d}^3\mathbf{x}\, \boldsymbol{\psi}^\dagger \frac{\boldsymbol{\lambda}}{2}\boldsymbol{\psi}\,, \quad \boldsymbol{Q}_5 \ = \ \int\! \mathrm{d}^3\mathbf{x}\, \boldsymbol{\psi}^\dagger \frac{\boldsymbol{\lambda}}{2}\gamma_5\boldsymbol{\psi}\,.$$

From these we recover the Lie algebra of the chiral-flavor group G, i.e.

$$[Q_i, Q_j] \ = \ i\, f_{ijk} Q_k\,, \quad [Q_i, Q_{5,j}] \ = \ i\, f_{ijk} Q_{5,k}\,,$$

$$[Q_{5,i}, Q_{5,j}] \ = \ i\, f_{ijk} Q_k\,, \ [Q, Q_{5,j}] \ = \ [Q, Q_j] \ = \ 0\,,$$

where $i, j, k = 1, \ldots, 8$ and f_{ijk} are the structure constants of $su(3)$.

– Explicit breakdown of symmetry: add $\mathcal{L}_M = -\overline{\boldsymbol{\psi}}\, M\, \boldsymbol{\psi}$ to $\mathcal{L}$.

$$\partial_\mu J^\mu = 0\,, \quad \partial_\mu \boldsymbol{J}^\mu = \frac{i}{2}\overline{\boldsymbol{\psi}}\, [M, \boldsymbol{\lambda}]\, \boldsymbol{\psi}\,, \quad \partial_\mu \boldsymbol{J}^\mu_5 = \frac{i}{2}\overline{\boldsymbol{\psi}}\gamma_5\, \{M, \boldsymbol{\lambda}\}\, \boldsymbol{\psi}\,.$$

Introduce the following notation:

$$\Phi_k \;=\; \overline{\boldsymbol{\psi}}\, \lambda_k\, \boldsymbol{\psi}\,, \quad \Phi^5_k \;=\; \overline{\boldsymbol{\psi}}\, \lambda_k\, \gamma_5 \boldsymbol{\psi}\,, \qquad \lambda_0 \;\equiv\; 1\!\!1\,, \quad k = 0, \ldots, 8\,.$$

i) Let $M = m_0\lambda_0$, then (1) reduces to

$$\partial_\mu J^\mu \;=\; \partial_\mu \boldsymbol{J}^\mu \;=\; 0\,, \quad \partial_\mu \boldsymbol{J}^\mu_5 \;=\; i\, m_0\, \boldsymbol{\Phi}^5\,,$$

i.e., the scalar and vector currents remain conserved and the broken-phase symmetry is $H = U(3)_V$.

ii) If $M = m_0\,\lambda_0 \;+\; m_3\lambda_3 + m_8\,\lambda_8$, then the mass matrix assumes the form

$$M \;=\; \begin{bmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{bmatrix},$$

where

$$m_e \;=\; m_0 + m_3 + \frac{1}{\sqrt{3}} m_8\,, \quad m_\mu \;=\; m_0 - m_3 + \frac{1}{\sqrt{3}} m_8\,, \quad m_\tau \;=\; m_0 - \frac{2}{\sqrt{3}} m_8\,.$$

The current divergences read

$$\partial_\mu J^\mu \;=\; \partial_\mu J^\mu_3 \;=\; \partial_\mu J^\mu_8 \;=\; 0\,, \quad \partial_\mu J^\mu_1 \;=\; -m_3 \Phi_2\,, \quad \partial_\mu J^\mu_2 \;=\; m_3 \Phi_1\,,$$

$$\partial_\mu J^\mu_4 \;=\; -\frac{m_3 + \sqrt{3} m_8}{2}\,\Phi_5\,, \quad \partial_\mu J^\mu_5 \;=\; \frac{m_3 + \sqrt{3} m_8}{2}\,\Phi_4\,,$$

$$\partial_\mu J^\mu_6 \;=\; \frac{m_3 - \sqrt{3} m_8}{2}\,\Phi_7\,, \qquad \partial_\mu J^\mu_7 \;=\; -\frac{m_3 - \sqrt{3} m_8}{2}\,\Phi_6\,,$$

$$\partial_\mu \boldsymbol{J}^\mu_5 \;\neq 0$$

The residual symmetry is thus $U(1)_V \times U(1)^3_V \times U(1)^8_V$.

iii) If $M = \sum_{k=0}^{8} m_k$, we can write

$$M = \begin{bmatrix} m_e & m_{e\mu}e^{i\delta_{e\mu}} & m_{e\tau}e^{i\delta_{e\tau}} \\ m_{e\mu}e^{-i\delta_{e\mu}} & m_\mu & m_{\mu\tau}e^{i\delta_{\mu\tau}} \\ m_{e\tau}e^{-i\delta_{e\tau}} & m_{\mu\tau}e^{-i\delta_{\mu\tau}} & m_\tau \end{bmatrix},$$

$$m_{e\mu}e^{i\delta_{e\mu}} = m_1 - i\,m_2\,, \quad m_{e\tau}e^{i\delta_{e\tau}} = m_4 - i\,m_5\,, \quad m_{\mu\tau}e^{i\delta_{\mu\tau}} = m_6 - i\,m_7\,.$$

The current divergences now read:

$$\partial_\mu J_1^\mu = g_{23} + \frac{1}{2}(g_{47} + g_{65})\,, \quad \partial_\mu J_2^\mu = g_{31} + \frac{1}{2}(g_{46} + g_{57})\,,$$

$$\partial_\mu J_3^\mu = g_{12} + \frac{1}{2}(g_{45} + g_{76})\,, \quad \partial_\mu J_4^\mu = \frac{1}{2}\left(g_{71} + g_{62} + g_{53} + \sqrt{3}g_{58}\right),$$

$$\partial_\mu J_5^\mu = \frac{1}{2}\left(g_{16} + g_{72} + g_{34} + \sqrt{3}g_{84}\right), \quad \partial_\mu J_6^\mu = \frac{1}{2}\left(g_{51} + g_{24} + \sqrt{3}g_{78} + g_{37}\right),$$

$$\partial_\mu J_7^\mu = \frac{1}{2}\left(g_{14} + g_{25} + \sqrt{3}g_{86} + g_{63}\right), \quad \partial_\mu J_8^\mu = \frac{\sqrt{3}}{2}(g_{45} + g_{67})\,,$$

$$\partial_\mu \boldsymbol{J}_5^\mu \neq 0, \quad \partial_\mu J^\mu = 0$$

where $g_{ij} \equiv m_i\,\Phi_j - m_j\,\Phi_i\,, \qquad h_{ij} \equiv m_i\,\Phi_j^5 + m_j\,\Phi_i^5\,.$

The residual symmetry is then $H = U(1)_V$.

Feld mixing is dynamically generated via the SSB scheme

$$SU(3)_A \times SU(3)_V \times U(1)_V \longrightarrow U(1)_V \,.$$

i) SSB sequence corresponding to a single mass generation is

$$SU(3)_A \times SU(3)_V \times U(1)_V \longrightarrow SU(3)_V \times U(1)_V \;\sim\; U(3)_V \,.$$

The order parameters are

$$\langle \Phi_0 \rangle \;=\; v_0 \;\neq 0\,, \qquad \langle \Phi_k \rangle \;=\; 0\,, \quad k = 1, \ldots, 8\,.$$

Taking the ε-term as $\mathcal{L}_\varepsilon = \varepsilon \Phi_0$, we obtain

$$i\frac{2}{3} v_0 \;=\; \lim_{\varepsilon \to 0} \varepsilon \int \mathrm{d}^4 y \, \langle T \left[\Phi_k^5(y)\, \Phi_k^5(0) \right] \rangle \,, \quad k = 1, \ldots, 8\,.$$

Note that we have eight NG fields which equals to $\dim(G/H)$.

Dynamical maps:

$$\Phi_k^5(x) \;=\; \sqrt{Z_{\varphi_k^5}}\, \varphi_k^5(x) + \ldots\,, \qquad k \;=\; 1, \ldots, 8\,.$$

where $\varphi_{5,k}$ are the NG fields, $Z_{\varphi_k^5}$ are the field renormalization constants and the dots denote higher order terms in the Haag expansions.

ii) SSB sequence corresponding to dynamical generation of different masses is

$$SU(3)_A \times SU(3)_V \times U(1)_V \longrightarrow \; U(1)_V \times U(1)^3_V \times U(1)^8_V \, .$$

The order parameters are

$$\langle \Phi_0 \rangle \; = \; v_0 \; \neq 0 \, , \quad \langle \Phi_3 \rangle \; = \; v_3 \; \neq 0 \quad \langle \Phi_8 \rangle \; = \; v_8 \; \neq 0 \, .$$

However, we can also restrict to the case $v_8 = 0$, as it clear from Eqs.(**??**)-(2). This assumption simplifies the formulas without affecting the main reasoning.

The ε-term is now

$$\mathcal{L}_\varepsilon = \varepsilon \left(\Phi_0 + \Phi_3 \right) \, .$$

WT identities:

$$
\begin{aligned}
iv_3 &= -\lim_{\varepsilon\to 0}\varepsilon\int \mathrm{d}^4y\,\langle T\,[\Phi_k(y)\,\Phi_k(0)]\rangle\,, \quad k \;=\; 1,2,4,5,6,7\,,\\
iv_0 &= \lim_{\varepsilon\to 0}\varepsilon\int \mathrm{d}^4y\,\langle T\left[\Phi^5_k(y)\,\Phi^5_k(0)\right]\rangle\,, \quad k=1,2\,,\\
i\left(\frac{2}{3}v_0+v_3\right) &= \lim_{\varepsilon\to 0}\varepsilon\int \mathrm{d}^4y\,\langle T\left[\left(\Phi^5_3(y)+\Phi^{5,12}_0(y)\right)\left(\Phi^5_3(0)+\Phi^{5,12}_0(0)\right)\right]\rangle\,,\\
i\left(v_0+\frac{3}{4}v_3\right) &= \lim_{\varepsilon\to 0}\varepsilon\int \mathrm{d}^4y\,\langle T\left[\Phi^5_k(y)\,\Phi^5_k(0)\right]\rangle\,, \quad k=4,5\,,\\
i\left(\frac{4}{3}v_0-v_3\right) &= \lim_{\varepsilon\to 0}\varepsilon\int \mathrm{d}^4y\,\langle T\left[\Phi^5_k(y)\,\Phi^5_k(0)\right]\rangle\,, \quad k=6,7\,,\\
i\frac{1}{3}\left(2v_0+v_3\right) &= \lim_{\varepsilon\to 0}\varepsilon\int \mathrm{d}^4y\,\langle T\left[\left(\Phi^5_8(y)+\frac{1}{\sqrt{3}}\Phi^5_3(y)\right)\left(\Phi^5_8(0)+\frac{1}{\sqrt{3}}\Phi^5_3(0)\right)\right]\rangle\,,
\end{aligned}
$$

In the above relations we have put

$$\Phi_0^{5,12} \equiv \sum_{k=1}^{2} \overline{\psi}_k \, \gamma_5 \, \psi_k = \frac{2}{3}\Phi_0 + \frac{1}{\sqrt{3}}\Phi_8 \, .$$

We have 14 NG modes, which equals $\dim(G/H)$.

NG dynamical maps

$$\begin{aligned}
\Phi_k(x) &= \sqrt{Z_{\varphi_k}}\, \varphi_k(x) + \ldots \,, \\
\Phi_k^5(x) &= \sqrt{Z_{\varphi_k^5}}\, \varphi_k^5(x) + \ldots \,, \qquad k = 1,2,4,5,6,7 \,, \\
\Phi_8^5(x) + \frac{1}{\sqrt{3}}\Phi_3^5(x) &= \sqrt{Z_{\varphi_{38}^5}}\, \varphi_{38}^5(x) + \ldots \,, \\
\Phi_0^{5,12}(x) + \Phi_3^5(x) &= \sqrt{Z_{\varphi_{03}^5}}\, \varphi_{03}^5(x) + \ldots \,.
\end{aligned}$$

iii) SSB sequence corresponding to dynamical mixing generation is

$$SU(3)_A \times SU(3)_V \times U(1)_V \longrightarrow U(1)_V \,.$$

The order parameters are

$$\langle \Phi_k \rangle \; = \; v_k \; \neq 0 \,, \qquad k = 0, \ldots, 8 \,.$$

implying that

$$\langle \overline{\psi}_i \psi_j \rangle \; \neq \; \langle \overline{\psi}_j \psi_i \rangle \; \neq \; 0 \,, \qquad i \neq j \,.$$

take a ε-term of the form:

$$\mathcal{L}_\varepsilon \; = \; \sum_{k=0}^{8} \varepsilon \, \Phi_k \,.$$

The corresponding WT identities now assume a very long and not illuminating form. Dynamical maps which contain NG fields:

$$\Phi_2 - \Phi_3 + \frac{1}{2} \left(\Phi_4 - \Phi_7 + \Phi_6 - \Phi_5 \right) \; = \; V_1 + \sqrt{Z_1} \varphi_1 \; + \; \ldots \,,$$

etc (15 more) ..

We have 16 independent NG fields, in agreement with $\dim(G/H)$.