On the CP-odd basis invariants of the 2HDM

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based on:
181x.xxxx, AT to appear

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Motivation

Open physical question:

Sufficient conditions for CP conservation in NHDM?

note, solved in special cases [Nishi '07]

More general motivation:

Physical observables must not depend on basis and notation.
⇒ Can only depend on basis invariant quantities.

Given a theory in an arbitrary basis,

- How does one obtain basis invariant quantities?
- How many independent basis invariant quantities exist?
- How are these basis invariant quantities related to physical observables?

e.g. for 2HDM, [Ogreid '18]

Motivation

Existing basis invariant methods (for NHDM)

“Brute force”
[Davidson, Gunion, Haber, O’Neil, . . .]

- 👍 straightforward
- 👎 trial and error
- 👎 not systematic

“Orbit space”
[Ivanov, Maniatis, Manteuffel, Nachtmann, Nishi, . . .]

- 👍 elegant
- 👎 very specialized to NHDM
- 👎 not systematic

Why is this such a big deal? Shouldn’t this be a simple group theory problem?

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \otimes \mathbf{1}_0 \;\; \ldots etc. \ldots \text{done} \; \textbf{?}$$

. . . and there exist powerful codes to do that, e.g. **`Susyno`** [Fonseca ’11]

Motivation

However:

- What are the building blocks?
  i.e. what do we even have to contract to singlets?
- When can we stop contracting?
  i.e. which singlets are $independent$?
- What does $independent$ even mean in this context?

$\Rightarrow$ Simple group theory problem (construction of singlets)
turns into an **invariant theory**, algebra problem!

And we have to deal with both...

Fortunately: powerful methods exist for both, but we have to learn them...

Outline

- Short general synopsis on jargon of invariant theory
- General strategy, outline the systematic procedure

2HDM:

- Construction of the building blocks
  - CP properties of the building blocks
- Characterize full ring of basis invariants
- Construct full ring of basis invariants
- Necessary and sufficient conditions for CP conservation

Jargon

- $Algebraic$ independence

  An invariant $\mathcal{I}_1$, is algebraically dependent of a set of invariants, say $\mathcal{I}_{2,3,..}$, if and only if there is a polynomial P

$$P\left(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \ldots\right) = 0 \ .$$

- $Primary$ invariants

  A maximal set of algebraically independent invariants (not unique, but the number of invariants is).
  Turns out: # of algebraically independent invariants = # of physical parameters.

- $Generating\ set$ of invariants

  All invariants that *cannot* be written as a polynomial of other invariants,

$$\mathcal{I}_i \neq P\left(\mathcal{I}_j, \ldots\right) \ .$$

  Vice versa, all invariants in the ring can be written as a polynomial in the generating set of invariants,

$$\mathcal{I} = P\left(\mathcal{I}_1, \mathcal{I}_2 \ldots\right) \ .$$

Proceedure / Algorithm

- Construction of *basis covariant* objects: “building blocks”.
  - Determine CP transformation behavior of the building blocks.
- Derive Hilbert series & Plethystic logarithm.
  - $\Rightarrow$ # and order of primary invariants.
  - $\Rightarrow$ # and structure of generating set of invariants.
  - $\Rightarrow$ interrelations between invariants (syzygies).
- Derive all invariants and needed interrelations explicitly.
- Finally: use this to find necessary and sufficient conditions for CPC (here in 2HDM).

Sidenote

New also: In order to explicitly construct singlets out of

$$(\boldsymbol{r}_1 \otimes \boldsymbol{r}_2 \otimes \boldsymbol{r}_3 \otimes \ldots) \supset \mathbf{1}_0 \oplus \ldots \ ,$$

we use **hermitean** projection operators constructed from **Young tableaux** via the diagrammatic technique of **birdtrack** diagrams.
These projection operators correspond to an orthogonal basis for the different singlets.
This is in contrast to, for example, the “trace basis”
see e.g. [Keppeler and Sjödahl ’13]

$$\mathrm{tr}\left[\Lambda\right],\ \mathrm{tr}\left[\Lambda^2\right],\ \mathrm{tr}\left[\Lambda^3\right],\ \ldots$$

Using orthogonal basis for invariants makes them much shorter and easier to handle.

Construction of the building blocks

2HDM scalar potential:

($a, b = 1, 2$ Higgs-flavor index)

$$V = \Phi_a^\dagger \, Y^a_{\ b} \, \Phi^b + \Phi_a^\dagger \, \Phi_b^\dagger \, Z^{ab}_{\ \ cd} \, \Phi^c \, \Phi^d \, .$$

Hermiticity and $\mathrm{SU}(2)_\mathrm{L}$ invariance

$$Y^a_{\ b} = (Y^b_{\ a})^* \, , \qquad Z^{ab}_{\ \ cd} = (Z^{cd}_{\ \ ab})^* \, , \qquad Z^{ab}_{\ \ cd} = Z^{ba}_{\ \ dc} \, .$$

$\Rightarrow Y$ and Z have $4 + 10$ independent real d.o.f.
After using basis changes (which we don't): find 11 "physical" parameters.

Decomposition into **building blocks**

$$Y \; \widehat{=} \; \mathbf{2} \otimes \mathbf{2} \; \widehat{=} \; \mathbf{1} \oplus \mathbf{3} \, ,$$
$$Z \; \widehat{=} \; \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} \; \widehat{=} \; \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5} \, .$$

Construction of the building blocks

$$Y: \quad \boxed{a}_{\,\mathbf{2}} \otimes \boxed{b}_{\,\mathbf{2}} = \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array}_{\,\mathbf{1}} \oplus \begin{array}{|c|c|}\hline a & b \\ \hline \end{array}_{\,\mathbf{3}}\;,$$

$$\begin{aligned} Z: \quad \boxed{a}_{\,\mathbf{2}} \otimes \boxed{b}_{\,\mathbf{2}} \otimes \boxed{c}_{\,\mathbf{2}} \otimes \boxed{d}_{\,\mathbf{2}} = {} & \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array}_{\,\mathbf{1}_{(1)}} \oplus \begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline \end{array}_{\,\mathbf{1}_{(2)}} \oplus \\ & \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d \\ \cline{1-1} \end{array}_{\,\mathbf{3}_{(1)}} \oplus \begin{array}{|c|c|c|}\hline a & b & d \\ \hline c \\ \cline{1-1} \end{array}_{\,\mathbf{3}_{(2)}} \oplus \begin{array}{|c|c|c|}\hline a & c & d \\ \hline b \\ \cline{1-1} \end{array}_{\,\mathbf{3}_{(3)}} \oplus \\ & \begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline \end{array}_{\,\mathbf{5}}\;. \end{aligned}$$

Construction of the building blocks

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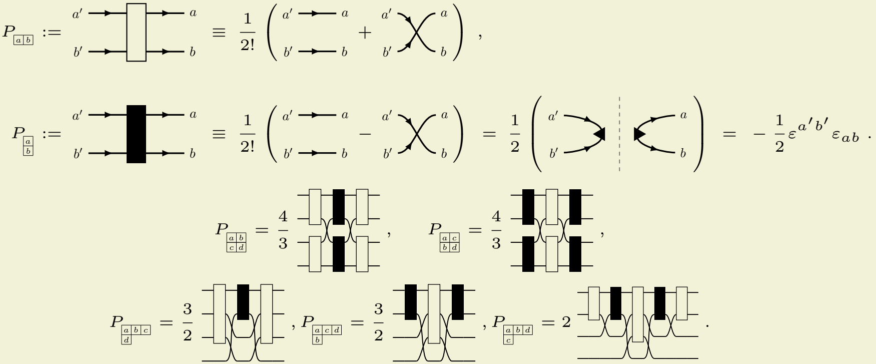
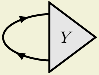
$$\begin{array}{|c|c|c|}\hline a & b & c\\\hline d \\\cline{1-1}\end{array}_{\,\mathbf{3}_{(1)}} \oplus \begin{array}{|c|c|c|}\hline a & b & d\\\hline c \\\cline{1-1}\end{array}_{\,\mathbf{3}_{(2)}} \oplus \begin{array}{|c|c|c|}\hline a & c & d\\\hline b \\\cline{1-1}\end{array}_{\,\mathbf{3}_{(3)}} \oplus$$

$$\begin{array}{|c|c|c|c|}\hline a & b & c & d\\\hline\end{array}_{\,\mathbf{5}}\,.$$

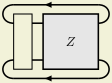


see e.g. [Alcock-Zeilinger and Weigert '17]

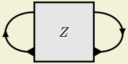
Building blocks

$$Y_{\mathbf{1}} :=$$ 

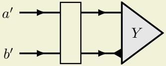
$$= Y^{a}{}_{a} = y_{11} + y_{22} \;,$$

$$Z_{\mathbf{1}_{(1)}} :=$$ 

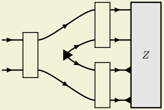
$$= \frac{1}{2}\left(Z^{ab}{}_{ab} + Z^{ab}{}_{ba}\right) \propto z_{1111} + z_{1212} + z_{1221} + z_{2222} \;,$$

$$Z_{\mathbf{1}_{(2)}} :=$$ 

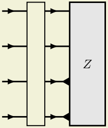
$$= \varepsilon_{ab}\varepsilon^{cd} Z^{ab}{}_{cd} \propto z_{1212} - z_{1221} \;,$$

$$Y_{\mathbf{3}}^{a'b'} :=$$ 

$$= \frac{1}{2}\left(Y^{a'b'} + Y^{b'a'}\right) \propto \begin{pmatrix} y_{12} & \frac{1}{2}\left(y_{22}-y_{11}\right) \\ \frac{1}{2}\left(y_{22}-y_{11}\right) & -y_{12}^{*} \end{pmatrix} ,$$

$$Z_{\mathbf{3}}^{ab} :=$$ 

$$\propto \begin{pmatrix} z_{1112}+z_{1222} & \frac{1}{2}\left(z_{2222}-z_{1111}\right) \\ \frac{1}{2}\left(z_{2222}-z_{1111}\right) & -\left(z_{1112}+z_{1222}\right)^{*} \end{pmatrix} ,$$

$$Z_{\mathbf{5}}^{abcd} :=$$ 

$$\propto \begin{pmatrix} \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & \zeta_3 \end{pmatrix} & \begin{pmatrix} \zeta_2 & \zeta_3 \\ \zeta_3 & -\zeta_2^{*} \end{pmatrix} \\ \begin{pmatrix} \zeta_2 & \zeta_3 \\ \zeta_3 & -\zeta_2^{*} \end{pmatrix} & \begin{pmatrix} \zeta_3 & -\zeta_2^{*} \\ -\zeta_2^{*} & \zeta_1^{*} \end{pmatrix} \end{pmatrix} .$$

$$\zeta_1 := z_{1122} \;,$$
$$\zeta_2 := \frac{1}{2}\left(z_{1222} - z_{1112}\right) \;,$$
$$\zeta_3 := \frac{1}{6}\left(z_{1111} - 2z_{1212} - 2z_{1221} + z_{2222}\right) \;.$$

CP trafo of the building blocks

$$Y_{\mathbf{1}} \mapsto Y_{\mathbf{1}} \,, \qquad Z_{\mathbf{1}_{(1)}} \mapsto Z_{\mathbf{1}_{(1)}} \,, \qquad Z_{\mathbf{1}_{(2)}} \mapsto Z_{\mathbf{1}_{(2)}} \,,$$
$$Y_{\mathbf{3}}^{ab} \mapsto -(Y_{\mathbf{3}})_{ab} \,, \qquad Z_{\mathbf{3}}^{ab} \mapsto -(Z_{\mathbf{3}})_{ab} \,,$$
$$Z_{\mathbf{5}}^{abcd} \mapsto (Z_{\mathbf{5}})_{abcd} \,.$$

For basis invariants ($\rightarrow$ all indices contracted):

A basis invariant is CP $\left\{ \begin{array}{c} \text{even} \\ \text{odd} \end{array} \right\}$ *iff* it contains an $\left\{ \begin{array}{c} \text{even} \\ \text{odd} \end{array} \right\}$ number of triplet building blocks ($Y_{\mathbf{3}}$, $Z_{\mathbf{3}}$).

Higher-order invariants

Building blocks as **input** for the ring:

$$q \mathrel{\widehat{=}} Z_{\mathbf{5}}\,, \quad y \mathrel{\widehat{=}} Y_{\mathbf{3}}\,, \quad \text{and} \quad t \mathrel{\widehat{=}} Z_{\mathbf{3}}\,.$$

From input, compute Hilbert series (HS) and Plethystic logarithm (PL):

$$\mathfrak{H}(q,y,t)\,, \qquad \mathrm{PL}\left[\mathfrak{H}\left(q,y,t\right)\right] := \sum_{k=1}^{\infty} \frac{\mu(k)\,\ln\mathfrak{H}\left(q^k,y^k,t^k\right)}{k}\,.$$

see e.g. [Benvenuti, Feng, Hanany, He '06]

$$\mathfrak{h}(z) \equiv \mathfrak{H}(z,z,z) = \frac{1+z^3+4\,z^4+2\,z^5+4\,z^6+z^7+z^{10}}{\left(1-z^2\right)^4\left(1-z^3\right)^3\left(1-z^4\right)}\,,$$

also found by [Bednyakov '18]

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also found by [Bednyakov '18]

$$\begin{aligned}\mathrm{PL}\left[\mathfrak{H}\left(q,y,t\right)\right] =\, & t^2+y^2+ty+q^2+q^3+qt^2+qy^2+qty \\ & +qt^2y+qty^2+q^2t^2+q^2y^2+q^2ty+q^2t^2y+q^2ty^2 \\ & +q^3t^3+q^3y^3+q^3t^2y+q^3ty^2-q^2t^2y^2-\mathcal{O}\left(\left[tyq\right]^7\right)\,.\end{aligned}$$

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also found by [Bednyakov '18]

$$\begin{aligned}\mathrm{PL}\left[\mathfrak{H}\left(q,y,t\right)\right] = {} & t^2+y^2+ty+q^2+q^3+qt^2+qy^2+qty \\ & + qt^2y+qty^2+q^2t^2+q^2y^2+q^2ty+q^2t^2y+q^2ty^2 \\ & + q^3t^3+q^3y^3+q^3t^2y+q^3ty^2-q^2t^2y^2-\mathcal{O}\left([tyq]^7\right)\,.\end{aligned}$$

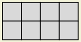
For example, the terms $\quad +1\,qyt \;\Rightarrow\; (Z_{\mathbf{5}}\otimes Y_{\mathbf{3}}\otimes Z_{\mathbf{3}}) \;\supset\; (1\,\times)\;\mathbf{1}_0$

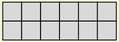
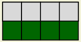
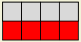
$-1\,q^2y^2t^2 \;\Rightarrow\; 1$ relation of $\mathcal{O}(q^2y^2t^2)$.

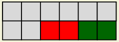
Explicit construction of primary invariants

$Z_{\mathbf{5}} \equiv$  , $Y_{\mathbf{3}} \equiv$  , $Z_{\mathbf{3}} \equiv$  .

$\mathcal{I}_{a,b,c} \;\widehat{=}$ Invariant with powers $q^a\, y^b\, t^c$, that is $Z_{\mathbf{5}}^{\otimes a} \otimes Y_{\mathbf{3}}^{\otimes b} \otimes Z_{\mathbf{3}}^{\otimes c}$.

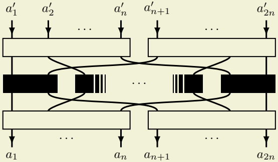
$\mathcal{I}_{2,0,0} :=$  , $\mathcal{I}_{0,2,0} :=$  , $\mathcal{I}_{0,1,1} :=$  , $\mathcal{I}_{0,0,2} :=$  ,

$\mathcal{I}_{3,0,0} :=$  , $\mathcal{I}_{1,2,0} :=$  , $\mathcal{I}_{1,0,2} :=$  , and

$\mathcal{I}_{2,1,1} :=$  .

Needed projection operators are simple! (always Sym – ASym – Sym)

$$P_{\begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n \\ \hline n+1 & n+2 & \cdots & 2n \\ \hline \end{array}} =$$



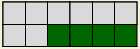
.

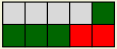
Together with the 3 linear invariants, these are 11 algebraically independent invariants.

Completing the generating set of invariants

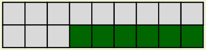
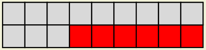
Find their structure from the graded plethystic logarithm. Then simply construct

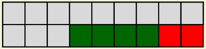
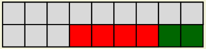
$\mathcal{I}_{1,1,1} :=$  ,

$\mathcal{I}_{2,2,0} :=$  , $\mathcal{I}_{2,0,2} :=$  ,

$\mathcal{J}_{1,2,1} :=$  , $\mathcal{J}_{1,1,2} :=$  ,

$\mathcal{J}_{2,2,1} :=$  , $\mathcal{J}_{2,1,2} :=$  ,

$\mathcal{J}_{3,3,0} :=$  , $\mathcal{J}_{3,0,3} :=$  ,

$\mathcal{J}_{3,2,1} :=$  , $\mathcal{J}_{3,1,2} :=$  .

Together with 3 linear + 8 invs. of the previous slide, these are *all* invariants of the generating set.

Odd total number of $Y_\mathbf{3}$ and $Z_\mathbf{3}$ $\Rightarrow$ CP-odd, denoted by $\mathcal{J}$ instead of $\mathcal{I}$.

In total 8 CP-odd invariants.

Curious: Note the $Y_\mathbf{3} \leftrightarrow Z_\mathbf{3}$ symmetry!

Systematic construction of syzygies

Leading negative terms in PL correspond to syzygies!

For example: $-q^2t^2y^2$

Matching power products of invariants of this structure:

$$\mathcal{I}_{1,1,1}^2\,, \quad \mathcal{I}_{2,1,1}\,\mathcal{I}_{0,1,1}\,, \quad \mathcal{I}_{2,2,0}\,\mathcal{I}_{0,0,2}\,, \quad \mathcal{I}_{2,0,2}\,\mathcal{I}_{0,2,0}\,,$$
$$\mathcal{I}_{1,2,0}\,\mathcal{I}_{1,0,2}\,, \quad \mathcal{I}_{2,0,0}\,\mathcal{I}_{0,2,0}\,\mathcal{I}_{0,0,2}\,, \quad \mathcal{I}_{2,0,0}\,\mathcal{I}_{0,1,1}^2\,.$$

A simple linear ansatz then reveals the first syzygy:

$$\begin{aligned} 3\,\mathcal{I}_{1,1,1}^2 &= 2\,\mathcal{I}_{2,1,1}\,\mathcal{I}_{0,1,1} - \mathcal{I}_{2,2,0}\,\mathcal{I}_{0,0,2} - \mathcal{I}_{2,0,2}\,\mathcal{I}_{0,2,0} \\ &\quad + 3\,\mathcal{I}_{1,2,0}\,\mathcal{I}_{1,0,2} + \mathcal{I}_{2,0,0}\,\mathcal{I}_{0,2,0}\,\mathcal{I}_{0,0,2} - \mathcal{I}_{2,0,0}\,\mathcal{I}_{0,1,1}^2\,. \end{aligned}$$

Necessary and sufficient conditions for CPC

Gunion and Haber: vanishing of **four** specific invariants is necessary and sufficient for explicit CPC. [Gunion, Haber '05]

$$I_{2Y2Z} \sim \mathcal{J}_{1,2,1}\,, \qquad I_{Y3Z} \sim \mathcal{J}_{1,1,2}\,,$$
$$I_{3Y3Z} \sim \mathcal{J}_{3,3,0}\,, \qquad I_{6Z} \sim \mathcal{J}_{3,0,3}\,.$$

8 CP-odd invariants? $\Rightarrow$ 4 relations? Yes! (...actually, there are more)

$$3\,\mathcal{J}_{2,2,1}\mathcal{I}_{1,2,0}-\mathcal{J}_{3,2,1}\mathcal{I}_{0,2,0}+3\,\mathcal{J}_{3,3,0}\mathcal{I}_{0,1,1}+\mathcal{J}_{1,2,1}\mathcal{I}_{2,2,0} = 0\,, \quad \text{and} \quad y \leftrightarrow t\,.$$

$$3\,\mathcal{J}_{2,2,1}^2+3\,\mathcal{J}_{3,3,0}\mathcal{J}_{1,1,2}-\mathcal{J}_{3,2,1}\mathcal{J}_{1,2,1}-\mathcal{J}_{1,2,1}^2\mathcal{I}_{2,0,0} = 0\,, \quad \text{and} \quad y \leftrightarrow t\,.$$

From these equations it is trivial to show:

$$\mathcal{J}_{1,2,1}=\mathcal{J}_{1,1,2}=\mathcal{J}_{3,3,0}=\mathcal{J}_{3,0,3}=0 \quad \Rightarrow \quad \mathcal{J}_{3,2,1}=\mathcal{J}_{3,1,2}=\mathcal{J}_{2,2,1}=\mathcal{J}_{2,1,2}=0\,.$$

□

Summary

- Group theory + invariant theory: a new systematic way to construct basis invariants.
- Used(invariant theory): Hilbert series + Plethystic logarithm
  $\Rightarrow$ number and structure of invariants + relations (syzygies).
- Used(group theory): Young tableaux + birdtracks
  $\Rightarrow$ building blocks + higher-order invariants.
- Constructed full ring of 22 CP-even($3+8+3$) and CP-odd(8) basis invariants of 2HDM.
- Explicitly constructed syzygies.

$\Rightarrow$ Very simple proof of sufficient conditions for CPC.

Advantages:

- Shorter words for primary invariants.
- No choice of basis necessary whatsoever.
- CP trafo of invariants are clear.
- Direct access to the syzygies.

Outlook

- 3HDM, . . . , NHDM?
- Take into account Higgs VEVs.
- In principle, algorithm generalizes to all simple groups.
- Analysis of RGE running simplified.
  [Bednyakov '18][Bijnens, Oredsson, Rathsman '18]
- Effect of global symmetries?
- Formulate theories in terms of invariants?
  [Ogreid '18]

Thank You!

Backup slides

Explicit form of building blocks

Components of Y and Z: $[Y]^a{}_b = y_{ab}$ and $[Z]^{ab}{}_{cd} = z_{abcd}$, dropping irrelevant global prefactors

$$
\begin{aligned}
Y_{\mathbf{1}} &= y_{11} + y_{22} \;, \\
Z_{\mathbf{1}_{(1)}} &= z_{1111} + z_{1212} + z_{1221} + z_{2222} \;, \\
Z_{\mathbf{1}_{(2)}} &= z_{1212} - z_{1221} \;, \\
Y_{\mathbf{3}}^{ab} &= \begin{pmatrix} y_{12} & \frac{1}{2}\left(y_{22} - y_{11}\right) \\ \frac{1}{2}\left(y_{22} - y_{11}\right) & -y_{12}^* \end{pmatrix} , \\
Z_{\mathbf{3}}^{ab} &= \begin{pmatrix} z_{1112} + z_{1222} & \frac{1}{2}\left(z_{2222} - z_{1111}\right) \\ \frac{1}{2}\left(z_{2222} - z_{1111}\right) & -\left(z_{1112} + z_{1222}\right)^* \end{pmatrix} , \\
Z_{\mathbf{5}}^{abcd} &= \begin{pmatrix} \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & \zeta_3 \end{pmatrix} & \begin{pmatrix} \zeta_2 & \zeta_3 \\ \zeta_3 & -\zeta_2^* \end{pmatrix} \\ \begin{pmatrix} \zeta_2 & \zeta_3 \\ \zeta_3 & -\zeta_2^* \end{pmatrix} & \begin{pmatrix} \zeta_3 & -\zeta_2^* \\ -\zeta_2^* & \zeta_1^* \end{pmatrix} \end{pmatrix} .
\end{aligned}
$$

with

$$\zeta_1 := z_{1122} \;, \quad \zeta_2 := \frac{1}{2}\left(z_{1222} - z_{1112}\right) \;, \quad \zeta_3 := \frac{1}{6}\left(z_{1111} - 2z_{1212} - 2z_{1221} + z_{2222}\right) \;.$$

Here, $\{y_{11}, y_{22}, z_{1111}, z_{1212}, z_{1221}, z_{2222}\} \in \mathbb{R}$, $\{y_{12}, z_{1122}, z_{1222}, z_{1112}\} \in \mathbb{C}$.

Building blocks in conventional notation

$$
\begin{aligned}
Y_{\mathbf{1}} &= m_{11}^2 + m_{22}^2 \;, \\
Z_{\mathbf{1}_{(1)}} &= \frac{1}{2}\left(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\right) \;, \\
Z_{\mathbf{1}_{(2)}} &= \frac{1}{2}\left(\lambda_3 - \lambda_4\right) \;, \\
Y_{\mathbf{3}}^{ab} &= \begin{pmatrix} -m_{12}^2 & \frac{1}{2}\left(m_{22}^2 - m_{11}^2\right) \\ \frac{1}{2}\left(m_{22}^2 - m_{11}^2\right) & \left(m_{12}^2\right)^* \end{pmatrix} \;, \\
Z_{\mathbf{3}}^{ab} &= \frac{1}{2}\begin{pmatrix} \lambda_6 + \lambda_7 & \frac{1}{2}\left(\lambda_2 - \lambda_1\right) \\ \frac{1}{2}\left(\lambda_2 - \lambda_1\right) & -\left(\lambda_6 + \lambda_7\right)^* \end{pmatrix} \;, \\
Z_{\mathbf{5}}^{abcd} &= \begin{pmatrix} \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & \xi_3 \end{pmatrix} & \begin{pmatrix} \xi_2 & \xi_3 \\ \xi_3 & -\xi_2^* \end{pmatrix} \\ \begin{pmatrix} \xi_2 & \xi_3 \\ \xi_3 & -\xi_2^* \end{pmatrix} & \begin{pmatrix} \xi_3 & -\xi_2^* \\ -\xi_2^* & \xi_1^* \end{pmatrix} \end{pmatrix} \;,
\end{aligned}
$$

with

$$
\xi_1 := \frac{1}{2}\lambda_5 \;, \qquad \xi_2 := \frac{1}{4}\left(\lambda_7 - \lambda_6\right) \;, \qquad \xi_3 := \frac{1}{12}\left(\lambda_1 - 2\lambda_3 - 2\lambda_4 + \lambda_2\right) \;.
$$

As usual $\{m_{12}^2, \lambda_5, \lambda_6, \lambda_7\} \in \mathbb{C}$ while all others are real.

Hilbert Series

With character polynomials $\chi_{\boldsymbol{r}}(z)$ for $\mathrm{SU}(2)$ irreps $\boldsymbol{r}$

$$\chi_{\mathbf{3}}(z) = z^2 + 1 + \frac{1}{z^2} \,, \qquad \chi_{\mathbf{5}}(z) = z^4 + z^2 + 1 + \frac{1}{z^2} + \frac{1}{z^4} \,,$$

and plethystic exponential (PE)

$$\mathrm{PE}\left[z, x, \boldsymbol{r}\right] := \exp\left(\sum_{k=1}^{\infty} \frac{x^k \, \chi_{\boldsymbol{r}}(z^k)}{k}\right) \,.$$

the multi-graded Hilbert series is computed as

$$\mathfrak{H}(q,y,t) = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{\mathrm{d}z}{z} \left(1 - z^2\right) \mathrm{PE}\left[z, q, \mathbf{5}\right] \, \mathrm{PE}\left[z, y, \mathbf{3}\right] \, \mathrm{PE}\left[z, t, \mathbf{3}\right] \,.$$

Integration via residue theorem $\Rightarrow \mathfrak{H}(q,y,t) = \dfrac{N\left(q,y,t\right)}{D\left(q,y,t\right)}$, with

$$\begin{aligned} N\left(q,y,t\right) = 1 &+ qty + q^2ty + qt^2y + qty^2 + q^2t^2y + q^2ty^2 \\ &+ q^3t^3 + q^3t^2y + q^3ty^2 + q^3y^3 \\ &- q^3t^4y - q^3t^3y^2 - q^3t^2y^3 - q^3ty^4 - q^4t^3y^2 - q^4t^2y^3 \\ &- q^5t^3y^2 - q^5t^2y^3 - q^4t^3y^3 - q^5t^3y^3 - q^6t^4y^4 \,, \end{aligned}$$

$$\begin{aligned} D\left(q,y,t\right) = &\left(1-t^2\right)\left(1-y^2\right)\left(1-ty\right)\left(1-q^2\right)\left(1-q^3\right)\left(1-qt^2\right)\left(1-qy^2\right) \\ &\left(1-q^2t^2\right)\left(1-q^2y^2\right) \,. \end{aligned}$$

Plethystic Logarithm

$$\mathrm{PL}\left[\mathfrak{H}\left(q,y,t\right)\right] := \sum_{k=1}^{\infty} \frac{\mu(k)\, \ln \mathfrak{H}\left(q^k, y^k, t^k\right)}{k} \;,$$

where $\mu(k)$ is the Möbius function.

$$\begin{aligned}\mathrm{PL}\left[\mathfrak{H}\left(q,y,t\right)\right] = {} & t^2 + ty + y^2 + q^2 + qt^2 + qty + qy^2 + q^3 + qt^2y + q^2t^2 + qty^2 + q^2ty \\ & + q^2y^2 + q^2t^2y + q^2ty^2 + q^3t^3 + q^3t^2y + q^3ty^2 + q^3y^3 - q^2t^2y^2 \\ & - q^2t^3y^2 - q^2t^2y^3 - q^3t^2y^2 - q^2t^4y^2 - q^3t^4y - q^2t^3y^3 - 3q^3t^3y^2 \\ & - q^2t^2y^4 - 3q^3t^2y^3 - q^4t^2y^2 - q^3ty^4 - \mathcal{O}\left([tyq]^9\right) \;.\end{aligned}$$

$$\mathfrak{h}(z) \equiv \mathfrak{H}(z,z,z) = \frac{1 + z^3 + 4\,z^4 + 2\,z^5 + 4\,z^6 + z^7 + z^{10}}{(1-z^2)^4\,(1-z^3)^3\,(1-z^4)} \;.$$

$$\mathrm{PL}\left[\mathfrak{h}(z)\right] = 4\,z^2 + 4\,z^3 + 5\,z^4 + 2\,z^5 + 3\,z^6 - 3\,z^7 - \mathcal{O}\left(z^8\right) \;.$$

Jacobi criterion

Easy way to check algebraic independence of a set of polynomials (here, invariants) $\mathcal{I}_i$, depending on a number of variables x_j (here, the components y_{ab} and z_{abcd} of Y and Z). The number of algebraically independent invariants is the rank of the Jacobian matrix:

$$\text{number of algebraically independent invariants} \;=\; \mathrm{rank}\left[\frac{\partial\,\mathcal{I}_i}{\partial x_j}\right]\;.$$

Besides symbolic evaluation one can also use this criterion with all variables put to random numbers for a fast machine evaluation.

Relation to invariants of Gunion and Haber

Original set of necessary and sufficient CP-odd invariants of Gunion and Haber [Gunion, Haber '05]

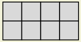
$$
\begin{aligned}
I_{Y3Z} &= \mathrm{Im}\left[Z^{ai}{}_{ic}Z^{ej}{}_{jb}Z^{bc}{}_{ed}Y^{d}{}_{a}\right] = -2\mathrm{i}\,\mathcal{J}_{1,1,2}\;, \\
I_{2Y2Z} &= \mathrm{Im}\left[Y^{a}{}_{b}\,Y^{c}{}_{d}\,Z^{bd}{}_{af}Z^{fi}{}_{ic}\right] = -2\mathrm{i}\,\mathcal{J}_{1,2,1}\;, \\
I_{6Z} &= \mathrm{Im}\left[Z^{ac}{}_{bd}Z^{b\ell}{}_{\ell f}Z^{dp}{}_{ph}Z^{fj}{}_{ak}Z^{km}{}_{jn}Z^{nh}{}_{mc}\right] = -2\mathrm{i}\,\mathcal{J}_{3,3,0}\;, \\
I_{3Y3Z} &= \mathrm{Im}\left[Z^{ab}{}_{cd}Z^{cd}{}_{eg}Z^{ef}{}_{hq}Y^{g}{}_{a}\,Y^{h}{}_{b}\,Y^{q}{}_{f}\right] = 2\mathrm{i}\,\mathcal{J}_{3,3,0} + 2\mathrm{i}\,\mathcal{J}_{1,2,1}\,\mathcal{I}_{0,1,1} + \mathrm{i}\,Y_{\mathbf{1}}^{2}\,\mathcal{J}_{1,1,2}\;.
\end{aligned}
$$

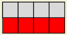
Invariants, explicitly

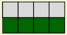
$\mathcal{I}_{0,2,0} :=$  $= \left(y^2 + y_\mathrm{i}^2 + y_\mathrm{r}^2\right)\,,$

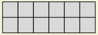
$\mathcal{I}_{0,1,1} :=$  $= \left(t y + t_\mathrm{i} y_\mathrm{i} + t_\mathrm{r} y_\mathrm{r}\right)\,,$

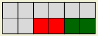
$\mathcal{I}_{0,0,2} :=$  $= \left(t^2 + t_\mathrm{i}^2 + t_\mathrm{r}^2\right)\,,$

$\mathcal{I}_{2,0,0} :=$  $= \left(3q_3^2 + q_{\mathrm{i}1}^2 + 4q_{\mathrm{i}2}^2 + q_{\mathrm{r}1}^2 + 4q_{\mathrm{r}2}^2\right)\,,$

$\mathcal{I}_{1,2,0} :=$  $= \left[q_3\left(2t^2 - t_\mathrm{i}^2 - t_\mathrm{r}^2\right) + 4t\left(q_{\mathrm{i}2} t_\mathrm{i} + q_{\mathrm{r}2} t_\mathrm{r}\right) + 2q_{\mathrm{i}1} t_\mathrm{i} t_\mathrm{r} + q_{\mathrm{r}1}\left(t_\mathrm{r}^2 - t_\mathrm{i}^2\right)\right]\,,$

$\mathcal{I}_{1,0,2} :=$  $= \left[q_3\left(2y^2 - y_\mathrm{i}^2 - y_\mathrm{r}^2\right) + 4y\left(q_{\mathrm{i}2} y_\mathrm{i} + q_{\mathrm{r}2} y_\mathrm{r}\right) + 2q_{\mathrm{i}1} y_\mathrm{i} y_\mathrm{r} + q_{\mathrm{r}1}\left(y_\mathrm{r}^2 - y_\mathrm{i}^2\right)\right]\,,$

$\mathcal{I}_{3,0,0} :=$  $= \left(-q_3\left(q_{\mathrm{i}1}^2 - 2q_{\mathrm{i}2}^2 + q_{\mathrm{r}1}^2 - 2q_{\mathrm{r}2}^2\right) + q_3^3 - 2q_{\mathrm{i}2}^2 q_{\mathrm{r}1} + 4q_{\mathrm{i}1} q_{\mathrm{i}2} q_{\mathrm{r}2} + 2q_{\mathrm{r}1} q_{\mathrm{r}2}^2\right)\,,$

$\mathcal{I}_{2,1,1} :=$  $= \{6q_3\left[-q_{\mathrm{i}2}(t y_\mathrm{i} + t_\mathrm{i} y) + q_{\mathrm{i}1}(t_\mathrm{i} y_\mathrm{r} + t_\mathrm{r} y_\mathrm{i}) + q_{\mathrm{r}1}(t_\mathrm{r} y_\mathrm{r} - t_\mathrm{i} y_\mathrm{i})\right]$

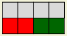
$\quad - 6q_{\mathrm{r}2}\left[q_3(t y_\mathrm{r} + t_\mathrm{r} y) + q_{\mathrm{i}1}(t y_\mathrm{i} + t_\mathrm{i} y) + 2q_{\mathrm{i}2}(t_\mathrm{i} y_\mathrm{r} + t_\mathrm{r} y_\mathrm{i}) + q_{\mathrm{r}1}(t y_\mathrm{r} + t_\mathrm{r} y)\right]$

$\quad + 3q_3^2(-2ty + t_\mathrm{i} y_\mathrm{i} + t_\mathrm{r} y_\mathrm{r}) + 6q_{\mathrm{i}2} q_{\mathrm{r}1}(t y_\mathrm{i} + t_\mathrm{i} y)$

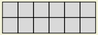
$\quad + q_{\mathrm{i}1}^2(2ty - t_\mathrm{i} y_\mathrm{i} - t_\mathrm{r} y_\mathrm{r}) - 4q_{\mathrm{i}2}^2(ty + t_\mathrm{i} y_\mathrm{i} - 2t_\mathrm{r} y_\mathrm{r}) - 6q_{\mathrm{i}1} q_{\mathrm{i}2}(t y_\mathrm{r} + t_\mathrm{r} y)$

$\quad + q_{\mathrm{r}1}^2(2ty - t_\mathrm{i} y_\mathrm{i} - t_\mathrm{r} y_\mathrm{r}) - 4q_{\mathrm{r}2}^2(ty - 2t_\mathrm{i} y_\mathrm{i} + t_\mathrm{r} y_\mathrm{r})\}\,.$

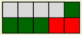
Invariants, explicitly



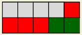
$$= \left[q_3\left(2ty - t_{\rm i}y_{\rm i} - t_{\rm r}y_{\rm r}\right) + 2q_{\rm i2}\left(ty_{\rm i} + t_{\rm i}y\right) + q_{\rm i1}\left(t_{\rm i}y_{\rm r} + t_{\rm r}y_{\rm i}\right) + 2q_{\rm r2}\left(ty_{\rm r} + t_{\rm r}y\right) + q_{\rm r1}\left(t_{\rm r}y_{\rm r} - t_{\rm i}y_{\rm i}\right)\right],$$



$$= \left(-q_3\left(q_{\rm i1}^2 - 2q_{\rm i2}^2 + q_{\rm r1}^2 - 2q_{\rm r2}^2\right) + q_3^3 - 2q_{\rm i2}^2q_{\rm r1} + 4q_{\rm i1}q_{\rm i2}q_{\rm r2} + 2q_{\rm r1}q_{\rm r2}^2\right),$$

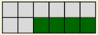


$$= {\rm i}\left[3q_3y(t_{\rm r}y_{\rm i} - t_{\rm i}y_{\rm r}) + 2q_{\rm i2}\left(tyy_{\rm r} - t_{\rm i}y_{\rm i}y_{\rm r} - t_{\rm r}y^2 + t_{\rm r}y_{\rm i}^2\right) + q_{\rm i1}\left(t\left(y_{\rm r}^2 - y_{\rm i}^2\right) + t_{\rm i}yy_{\rm i} - t_{\rm r}yy_{\rm r}\right) + q_{\rm r1}(-2ty_{\rm i}y_{\rm r} + t_{\rm i}yy_{\rm r} + t_{\rm r}yy_{\rm i}) + 2q_{\rm r2}(-tyy_{\rm i} + t_{\rm i}(y - y_{\rm r})(y + y_{\rm r}) + t_{\rm r}y_{\rm i}y_{\rm r})\right],$$

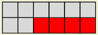


$$= -{\rm i}\left[3q_3t(t_{\rm r}y_{\rm i} - t_{\rm i}y_{\rm r}) + 2q_{\rm i2}\left(t^2y_{\rm r} - tt_{\rm r}y + t_{\rm i}(t_{\rm r}y_{\rm i} - t_{\rm i}y_{\rm r})\right) + q_{\rm i1}\left(-tt_{\rm i}y_{\rm i} + t_{\rm r}(ty_{\rm r} - t_{\rm r}y) + t_{\rm i}^2y\right) + 2q_{\rm r2}\left(t^2(-y_{\rm i}) + tt_{\rm i}y + t_{\rm r}(t_{\rm r}y_{\rm i} - t_{\rm i}y_{\rm r})\right) + q_{\rm r1}(-tt_{\rm i}y_{\rm r} - tt_{\rm r}y_{\rm i} + 2t_{\rm i}t_{\rm r}y)\right],$$

Invariants, explicitly



$$= \Big\{6q_{\mathrm{r}1}\left(q_3\left(y_{\mathrm{r}}^2-y_{\mathrm{i}}^2\right)+2q_{\mathrm{i}2}yy_{\mathrm{i}}\right)-12q_{\mathrm{r}2}\left(q_3yy_{\mathrm{r}}+q_{\mathrm{i}1}yy_{\mathrm{i}}+2q_{\mathrm{i}2}y_{\mathrm{i}}y_{\mathrm{r}}+q_{\mathrm{r}1}yy_{\mathrm{r}}\right)$$
$$+12q_3y_{\mathrm{i}}\left(q_{\mathrm{i}1}y_{\mathrm{r}}-q_{\mathrm{i}2}y\right)+3q_3^2\left(-2y^2+y_{\mathrm{i}}^2+y_{\mathrm{r}}^2\right)+q_{\mathrm{i}1}^2\left(2y^2-y_{\mathrm{i}}^2-y_{\mathrm{r}}^2\right)$$
$$-4q_{\mathrm{i}2}^2\left(y^2+y_{\mathrm{i}}^2-2y_{\mathrm{r}}^2\right)-12q_{\mathrm{i}1}q_{\mathrm{i}2}yy_{\mathrm{r}}+q_{\mathrm{r}1}^2\left(2y^2-y_{\mathrm{i}}^2-y_{\mathrm{r}}^2\right)-4q_{\mathrm{r}2}^2\left(y^2-2y_{\mathrm{i}}^2+y_{\mathrm{r}}^2\right)\Big\}$$



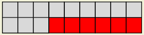
$$= \Big(6q_{\mathrm{r}1}\left(q_3\left(t_{\mathrm{r}}^2-t_{\mathrm{i}}^2\right)+2q_{\mathrm{i}2}tt_{\mathrm{i}}\right)-12q_{\mathrm{r}2}\left(q_3tt_{\mathrm{r}}+q_{\mathrm{i}1}tt_{\mathrm{i}}+2q_{\mathrm{i}2}t_{\mathrm{i}}t_{\mathrm{r}}+q_{\mathrm{r}1}tt_{\mathrm{r}}\right)$$
$$+12q_3t_{\mathrm{i}}\left(q_{\mathrm{i}1}t_{\mathrm{r}}-q_{\mathrm{i}2}t\right)+3q_3^2\left(-2t^2+t_{\mathrm{i}}^2+t_{\mathrm{r}}^2\right)+q_{\mathrm{i}1}^2\left(2t^2-t_{\mathrm{i}}^2-t_{\mathrm{r}}^2\right)$$
$$-4q_{\mathrm{i}2}^2\left(t^2+t_{\mathrm{i}}^2-2t_{\mathrm{r}}^2\right)-12q_{\mathrm{i}1}q_{\mathrm{i}2}tt_{\mathrm{r}}+q_{\mathrm{r}1}^2\left(2t^2-t_{\mathrm{i}}^2-t_{\mathrm{r}}^2\right)-4q_{\mathrm{r}2}^2\left(t^2-2t_{\mathrm{i}}^2+t_{\mathrm{r}}^2\right)\Big)$$

Invariants, explicitly

$$
\begin{aligned}
= \mathrm{i}\Big(&2q_3\Big(q_{\mathrm{i}1}\big(-ty_\mathrm{i}^2+ty_\mathrm{r}^2+t_\mathrm{i}yy_\mathrm{i}-t_\mathrm{r}yy_\mathrm{r}\big)+q_{\mathrm{i}2}(-tyy_\mathrm{r}+t_\mathrm{i}y_\mathrm{i}y_\mathrm{r}+t_\mathrm{r}(y-y_\mathrm{i})(y+y_\mathrm{i}))\\
&+q_{\mathrm{r}2}\big(tyy_\mathrm{i}-t_\mathrm{i}y^2+t_\mathrm{i}y_\mathrm{r}^2-t_\mathrm{r}y_\mathrm{i}y_\mathrm{r}\big)+q_{\mathrm{r}1}(-2ty_\mathrm{i}y_\mathrm{r}+t_\mathrm{i}yy_\mathrm{r}+t_\mathrm{r}yy_\mathrm{i})\Big)\\
&+3q_3^2y(t_\mathrm{i}y_\mathrm{r}-t_\mathrm{r}y_\mathrm{i})\\
&+2q_{\mathrm{i}1}\Big(q_{\mathrm{i}2}\big(tyy_\mathrm{i}-t_\mathrm{i}y^2+t_\mathrm{i}y_\mathrm{r}^2-t_\mathrm{r}y_\mathrm{i}y_\mathrm{r}\big)+q_{\mathrm{r}2}(-tyy_\mathrm{r}+t_\mathrm{i}y_\mathrm{i}y_\mathrm{r}+t_\mathrm{r}(y-y_\mathrm{i})(y+y_\mathrm{i}))\Big)\\
&+2q_{\mathrm{r}2}\Big(2q_{\mathrm{i}2}(t(y_\mathrm{i}-y_\mathrm{r})(y_\mathrm{i}+y_\mathrm{r})-t_\mathrm{i}yy_\mathrm{i}+t_\mathrm{r}yy_\mathrm{r})\\
&+q_{\mathrm{r}1}\big(tyy_\mathrm{i}-t_\mathrm{i}y^2+t_\mathrm{i}y_\mathrm{r}^2-t_\mathrm{r}y_\mathrm{i}y_\mathrm{r}\big)\Big)+2q_{\mathrm{i}2}q_{\mathrm{r}1}tyy_\mathrm{r}-2q_{\mathrm{i}2}q_{\mathrm{r}1}t_\mathrm{i}y_\mathrm{i}y_\mathrm{r}\\
&-2q_{\mathrm{i}2}q_{\mathrm{r}1}t_\mathrm{r}y^2+2q_{\mathrm{i}2}q_{\mathrm{r}1}t_\mathrm{r}y_\mathrm{i}^2-4q_{\mathrm{i}2}^2ty_\mathrm{i}y_\mathrm{r}+q_{\mathrm{i}1}^2y(t_\mathrm{r}y_\mathrm{i}-t_\mathrm{i}y_\mathrm{r})+4q_{\mathrm{i}2}^2t_\mathrm{i}yy_\mathrm{r}\\
&+4q_{\mathrm{r}2}^2y_\mathrm{i}(ty_\mathrm{r}-t_\mathrm{r}y)-q_{\mathrm{r}1}^2t_\mathrm{i}yy_\mathrm{r}+q_{\mathrm{r}1}^2t_\mathrm{r}yy_\mathrm{i}\Big)
\end{aligned}
$$

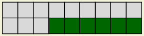
$$
\begin{aligned}
= -\mathrm{i}\Big(&2q_3\Big(q_{\mathrm{i}2}\big(t^2(-y_\mathrm{r})+tt_\mathrm{r}y+t_\mathrm{i}^2y_\mathrm{r}-t_\mathrm{i}t_\mathrm{r}y_\mathrm{i}\big)+q_{\mathrm{i}1}\big(-tt_\mathrm{i}y_\mathrm{i}+t_\mathrm{r}(ty_\mathrm{r}-t_\mathrm{r}y)+t_\mathrm{i}^2y\big)\\
&+q_{\mathrm{r}2}\big(t^2y_\mathrm{i}-tt_\mathrm{i}y+t_\mathrm{i}t_\mathrm{r}y_\mathrm{r}-t_\mathrm{r}^2y_\mathrm{i}\big)+q_{\mathrm{r}1}(-tt_\mathrm{i}y_\mathrm{r}-tt_\mathrm{r}y_\mathrm{i}+2t_\mathrm{i}t_\mathrm{r}y)\Big)\\
&+3q_3^2t(t_\mathrm{i}y_\mathrm{r}-t_\mathrm{r}y_\mathrm{i})\\
&+2q_{\mathrm{r}2}\Big(2q_{\mathrm{i}2}\big(tt_\mathrm{i}y_\mathrm{i}+t_\mathrm{r}(t_\mathrm{r}y-ty_\mathrm{r})+t_\mathrm{i}^2(-y)\big)+q_{\mathrm{r}1}\big(t^2y_\mathrm{i}-tt_\mathrm{i}y+t_\mathrm{r}(t_\mathrm{i}y_\mathrm{r}-t_\mathrm{r}y_\mathrm{i})\big)\Big)\\
&+2q_{\mathrm{i}1}\Big(q_{\mathrm{i}2}\big(t^2y_\mathrm{i}-tt_\mathrm{i}y+t_\mathrm{r}(t_\mathrm{i}y_\mathrm{r}-t_\mathrm{r}y_\mathrm{i})\big)+q_{\mathrm{r}2}\big(t^2(-y_\mathrm{r})+tt_\mathrm{r}y+t_\mathrm{i}(t_\mathrm{i}y_\mathrm{r}-t_\mathrm{r}y_\mathrm{i})\big)\Big)\\
&+2q_{\mathrm{i}2}q_{\mathrm{r}1}t^2y_\mathrm{r}-2q_{\mathrm{i}2}q_{\mathrm{r}1}tt_\mathrm{r}y-2q_{\mathrm{i}2}q_{\mathrm{r}1}t_\mathrm{i}^2y_\mathrm{r}+2q_{\mathrm{i}2}q_{\mathrm{r}1}t_\mathrm{i}t_\mathrm{r}y_\mathrm{i}+q_{\mathrm{i}1}^2t(t_\mathrm{r}y_\mathrm{i}-t_\mathrm{i}y_\mathrm{r})\\
&-4q_{\mathrm{i}2}^2tt_\mathrm{r}y_\mathrm{i}+4q_{\mathrm{i}2}^2t_\mathrm{i}t_\mathrm{r}y+4q_{\mathrm{r}2}^2t_\mathrm{i}(ty_\mathrm{r}-t_\mathrm{r}y)-q_{\mathrm{r}1}^2tt_\mathrm{i}y_\mathrm{r}+q_{\mathrm{r}1}^2tt_\mathrm{r}y_\mathrm{i}\Big)
\end{aligned}
$$

Invariants, explicitly



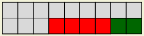
$$
\begin{aligned}
= \mathrm{i} \Big(6q_3 \Big(q_{\mathrm{r}2} \Big(q_{\mathrm{i}1} t_\mathrm{r} \left(2t^2 + t_\mathrm{i}^2 - t_\mathrm{r}^2\right) - 2q_{\mathrm{i}2} t (t_\mathrm{i} - t_\mathrm{r})(t_\mathrm{i} + t_\mathrm{r}) - 2q_{\mathrm{r}1} t_\mathrm{i} (t - t_\mathrm{r})(t + t_\mathrm{r}) \Big) \\
+ q_{\mathrm{i}2} \Big(2t_\mathrm{r} \Big(q_{\mathrm{i}2} t t_\mathrm{i} + q_{\mathrm{r}1} \left(t_\mathrm{i}^2 - t^2\right) \Big) + q_{\mathrm{i}1} t_\mathrm{i} \left(-2t^2 + t_\mathrm{i}^2 - t_\mathrm{r}^2\right) \Big) - 2q_{\mathrm{r}2}^2 t t_\mathrm{i} t_\mathrm{r} \Big) \\
+ 9q_3^2 t \left(q_{\mathrm{i}1} (t_\mathrm{i} - t_\mathrm{r})(t_\mathrm{i} + t_\mathrm{r}) + 2q_{\mathrm{r}1} t_\mathrm{i} t_\mathrm{r} \right) \\
- 2q_{\mathrm{i}1}^2 \Big(q_{\mathrm{i}2} t_\mathrm{r} \left(-2t^2 - t_\mathrm{i}^2 + t_\mathrm{r}^2\right) + q_{\mathrm{r}2} t_\mathrm{i} \left(2t^2 - t_\mathrm{i}^2 + t_\mathrm{r}^2\right) + q_{\mathrm{r}1} t t_\mathrm{i} t_\mathrm{r} \Big) \\
+ q_{\mathrm{i}1} \Big(-2q_{\mathrm{i}2} q_{\mathrm{r}1} t_\mathrm{i} \left(t_\mathrm{i}^2 - 3t_\mathrm{r}^2\right) + 4q_{\mathrm{i}2}^2 t \left(t^2 - t_\mathrm{i}^2 - 2t_\mathrm{r}^2\right) \\
+ 4q_{\mathrm{r}2}^2 t \left(-t^2 + 2t_\mathrm{i}^2 + t_\mathrm{r}^2\right) + q_{\mathrm{r}1}^2 t \left(t_\mathrm{r}^2 - t_\mathrm{i}^2\right) - 2q_{\mathrm{r}1} q_{\mathrm{r}2} t_\mathrm{r} \left(t_\mathrm{r}^2 - 3t_\mathrm{i}^2\right) \Big) \\
+ 4q_{\mathrm{r}2}^2 t_\mathrm{r} \Big(q_{\mathrm{r}1} t t_\mathrm{i} - 2q_{\mathrm{i}2} \left(t^2 + 2t_\mathrm{i}^2 - t_\mathrm{r}^2\right) \Big) \\
+ 4q_{\mathrm{r}2} \Big(q_{\mathrm{i}2} q_{\mathrm{r}1} t \Big(2t^2 - 3\left(t_\mathrm{i}^2 + t_\mathrm{r}^2\right)\Big) + 2q_{\mathrm{i}2}^2 t_\mathrm{i} \left(t^2 - t_\mathrm{i}^2 + 2t_\mathrm{r}^2\right) + q_{\mathrm{r}1}^2 t_\mathrm{i} \left(t_\mathrm{r}^2 - t^2\right) \Big) \\
- 2t_\mathrm{r} \left(2q_{\mathrm{i}2}^2 - q_{\mathrm{r}1}^2\right) \left(2q_{\mathrm{i}2} (t - t_\mathrm{i})(t + t_\mathrm{i}) - q_{\mathrm{r}1} t t_\mathrm{i}\right) + q_{\mathrm{i}1}^3 t \left(t_\mathrm{r}^2 - t_\mathrm{i}^2\right) \\
+ 8q_{\mathrm{r}2}^3 t_\mathrm{i} (t - t_\mathrm{r})(t + t_\mathrm{r}) \Big)
\end{aligned}
$$

Invariants, explicitly



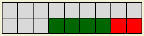
$$
\begin{aligned}
= \mathrm{i}\Big(6q_3\,\Big(q_{\mathrm{r}2}\,\Big(q_{\mathrm{i}1}y_{\mathrm{r}}\,\big(2y^2+y_{\mathrm{i}}^2-y_{\mathrm{r}}^2\big)-2q_{\mathrm{i}2}y(y_{\mathrm{i}}-y_{\mathrm{r}})(y_{\mathrm{i}}+y_{\mathrm{r}})-2q_{\mathrm{r}1}y_{\mathrm{i}}(y-y_{\mathrm{r}})(y+y_{\mathrm{r}})\Big)\\
+q_{\mathrm{i}2}\,\Big(2y_{\mathrm{r}}\,\Big(q_{\mathrm{i}2}yy_{\mathrm{i}}+q_{\mathrm{r}1}\,\big(y_{\mathrm{i}}^2-y^2\big)\Big)+q_{\mathrm{i}1}y_{\mathrm{i}}\,\big(-2y^2+y_{\mathrm{i}}^2-y_{\mathrm{r}}^2\big)\Big)-2q_{\mathrm{r}2}^2yy_{\mathrm{i}}y_{\mathrm{r}}\Big)\\
+9q_3^2y\,(q_{\mathrm{i}1}(y_{\mathrm{i}}-y_{\mathrm{r}})(y_{\mathrm{i}}+y_{\mathrm{r}})+2q_{\mathrm{r}1}y_{\mathrm{i}}y_{\mathrm{r}})\\
-2q_{\mathrm{i}1}^2\,\Big(q_{\mathrm{i}2}y_{\mathrm{r}}\,\big(-2y^2-y_{\mathrm{i}}^2+y_{\mathrm{r}}^2\big)+q_{\mathrm{r}2}y_{\mathrm{i}}\,\big(2y^2-y_{\mathrm{i}}^2+y_{\mathrm{r}}^2\big)+q_{\mathrm{r}1}yy_{\mathrm{i}}y_{\mathrm{r}}\Big)\\
+q_{\mathrm{i}1}\,\Big(-2q_{\mathrm{i}2}q_{\mathrm{r}1}y_{\mathrm{i}}\,\big(y_{\mathrm{i}}^2-3y_{\mathrm{r}}^2\big)+4q_{\mathrm{i}2}^2y\,\big(y^2-y_{\mathrm{i}}^2-2y_{\mathrm{r}}^2\big)+4q_{\mathrm{r}2}^2y\,\big(-y^2+2y_{\mathrm{i}}^2+y_{\mathrm{r}}^2\big)\\
+q_{\mathrm{r}1}^2y\,\big(y_{\mathrm{r}}^2-y_{\mathrm{i}}^2\big)-2q_{\mathrm{r}1}q_{\mathrm{r}2}y_{\mathrm{r}}\,\big(y_{\mathrm{r}}^2-3y_{\mathrm{i}}^2\big)\Big)+4q_{\mathrm{r}2}^2y_{\mathrm{r}}\,\Big(q_{\mathrm{r}1}yy_{\mathrm{i}}-2q_{\mathrm{i}2}\,\big(y^2+2y_{\mathrm{i}}^2-y_{\mathrm{r}}^2\big)\Big)\\
+4q_{\mathrm{r}2}\,\Big(q_{\mathrm{i}2}q_{\mathrm{r}1}y\,\Big(2y^2-3\,\big(y_{\mathrm{i}}^2+y_{\mathrm{r}}^2\big)\Big)+2q_{\mathrm{i}2}^2y_{\mathrm{i}}\,\big(y^2-y_{\mathrm{i}}^2+2y_{\mathrm{r}}^2\big)+q_{\mathrm{r}1}^2y_{\mathrm{i}}\,\big(y_{\mathrm{r}}^2-y^2\big)\Big)\\
-2y_{\mathrm{r}}\,\big(2q_{\mathrm{i}2}^2-q_{\mathrm{r}1}^2\big)\,(2q_{\mathrm{i}2}(y-y_{\mathrm{i}})(y+y_{\mathrm{i}})-q_{\mathrm{r}1}yy_{\mathrm{i}})+q_{\mathrm{i}1}^3y\,\big(y_{\mathrm{r}}^2-y_{\mathrm{i}}^2\big)+8q_{\mathrm{r}2}^3y_{\mathrm{i}}(y-y_{\mathrm{r}})(y+y_{\mathrm{r}})\Big)
\end{aligned}
$$

Invariants, explicitly



$$
\begin{aligned}
= \ \mathrm{i}\Big(&\Big(\big(-y t_{\mathrm{i}}^2 - 2t y_{\mathrm{i}} t_{\mathrm{i}} + t_{\mathrm{r}}(t_{\mathrm{r}} y + 2t y_{\mathrm{r}})\big)\, q_{\mathrm{i}1}^3 + 2\Big(\big(2y_{\mathrm{r}} t^2 + 4t_{\mathrm{r}} y t + 2t_{\mathrm{i}} t_{\mathrm{r}} y_{\mathrm{i}} + t_{\mathrm{i}}^2 y_{\mathrm{r}} - 3t_{\mathrm{r}}^2 y_{\mathrm{r}}\big)\, q_{\mathrm{i}2} \\
&- q_{\mathrm{r}1}(t_{\mathrm{i}} t_{\mathrm{r}} y + t t_{\mathrm{r}} y_{\mathrm{i}} + t t_{\mathrm{i}} y_{\mathrm{r}}) - q_{\mathrm{r}2}\big(2y_{\mathrm{i}} t^2 + 4t_{\mathrm{i}} y t - 3t_{\mathrm{i}}^2 y_{\mathrm{i}} + t_{\mathrm{r}}^2 y_{\mathrm{i}} + 2t_{\mathrm{i}} t_{\mathrm{r}} y_{\mathrm{r}}\big)\Big)\, q_{\mathrm{i}1}^2 \\
&+ \Big(4\big(3y t^2 - 2(t_{\mathrm{i}} y_{\mathrm{i}} + 2t_{\mathrm{r}} y_{\mathrm{r}})t - y\big(t_{\mathrm{i}}^2 + 2t_{\mathrm{r}}^2\big)\big)\, q_{\mathrm{i}2}^2 + 6\big(-y_{\mathrm{i}} t_{\mathrm{i}}^2 + 2t_{\mathrm{r}} y_{\mathrm{r}} t_{\mathrm{i}} + t_{\mathrm{r}}^2 y_{\mathrm{i}}\big)\, q_{\mathrm{r}1} q_{\mathrm{i}2} \\
&+ \big(-y t_{\mathrm{i}}^2 - 2t y_{\mathrm{i}} t_{\mathrm{i}} + t_{\mathrm{r}}^2 y + 2t t_{\mathrm{r}} y_{\mathrm{r}}\big)\, q_{\mathrm{r}1}^2 + 4\big(-3y t^2 + 4t_{\mathrm{i}} y_{\mathrm{i}} t + 2t_{\mathrm{r}} y_{\mathrm{r}} t + 2t_{\mathrm{i}}^2 y + t_{\mathrm{r}}^2 y\big)\, q_{\mathrm{r}2}^2 \\
&+ 6\big(y_{\mathrm{r}} t_{\mathrm{i}}^2 + 2t_{\mathrm{r}} y_{\mathrm{i}} t_{\mathrm{i}} - t_{\mathrm{r}}^2 y_{\mathrm{r}}\big)\, q_{\mathrm{r}1} q_{\mathrm{r}2}\Big)\, q_{\mathrm{i}1} \\
&+ 9\Big(\big(y t_{\mathrm{i}}^2 + 2t y_{\mathrm{i}} t_{\mathrm{i}} - t_{\mathrm{r}}(t_{\mathrm{r}} y + 2t y_{\mathrm{r}})\big)\, q_{\mathrm{i}1} + 2(t_{\mathrm{i}} t_{\mathrm{r}} y + t t_{\mathrm{r}} y_{\mathrm{i}} + t t_{\mathrm{i}} y_{\mathrm{r}}) q_{\mathrm{r}1}\Big)\, q_3^2 \\
&+ 4\Big((t_{\mathrm{i}} t_{\mathrm{r}} y + t t_{\mathrm{r}} y_{\mathrm{i}} + t t_{\mathrm{i}} y_{\mathrm{r}}) q_{\mathrm{r}1} + 2\big(y_{\mathrm{i}} t^2 + 2t_{\mathrm{i}} y t - 3t_{\mathrm{i}}^2 y_{\mathrm{i}} + 2t_{\mathrm{r}}^2 y_{\mathrm{i}} + 4t_{\mathrm{i}} t_{\mathrm{r}} y_{\mathrm{r}}\big)\, q_{\mathrm{r}2}\Big)\, q_{\mathrm{i}2}^2 \\
&- 2\big(q_{\mathrm{r}1}^2 - 2q_{\mathrm{r}2}^2\big)\Big((t_{\mathrm{i}} t_{\mathrm{r}} y + t t_{\mathrm{r}} y_{\mathrm{i}} + t t_{\mathrm{i}} y_{\mathrm{r}}) q_{\mathrm{r}1} + 2\big(y_{\mathrm{i}} t^2 + 2t_{\mathrm{i}} y t - t_{\mathrm{r}}(t_{\mathrm{r}} y_{\mathrm{i}} + 2t_{\mathrm{i}} y_{\mathrm{r}})\big)\, q_{\mathrm{r}2}\Big) \\
&-6q_3\Big(2\Big((t_{\mathrm{i}} t_{\mathrm{r}} y + t t_{\mathrm{r}} y_{\mathrm{i}} + t t_{\mathrm{i}} y_{\mathrm{r}}) q_{\mathrm{r}2}^2 + \Big(\big(y t_{\mathrm{i}}^2 + 2t y_{\mathrm{i}} t_{\mathrm{i}} - t_{\mathrm{r}}(t_{\mathrm{r}} y + 2t y_{\mathrm{r}})\big)\, q_{\mathrm{i}2} + \big(y_{\mathrm{i}} t^2 + 2t_{\mathrm{i}} y t - t_{\mathrm{r}}(t_{\mathrm{r}} y_{\mathrm{i}} + 2t_{\mathrm{i}} y_{\mathrm{r}})\big)\, q \\
&+ \Big(q_{\mathrm{i}2}\big(2y_{\mathrm{i}} t^2 + 4t_{\mathrm{i}} y t - 3t_{\mathrm{i}}^2 y_{\mathrm{i}} + t_{\mathrm{r}}^2 y_{\mathrm{i}} + 2t_{\mathrm{i}} t_{\mathrm{r}} y_{\mathrm{r}}\big) - q_{\mathrm{r}2}\big(2y_{\mathrm{r}} t^2 + 4t_{\mathrm{r}} y t + 2t_{\mathrm{i}} t_{\mathrm{r}} y_{\mathrm{i}} + t_{\mathrm{i}}^2 y_{\mathrm{r}} - 3t_{\mathrm{r}}^2 y_{\mathrm{r}}\big)\Big)\, q_{\mathrm{i}1}\Big) \\
&- 8q_{\mathrm{i}2}^3\big(y_{\mathrm{r}} t^2 + 2t_{\mathrm{r}} y t - t_{\mathrm{i}}(2t_{\mathrm{r}} y_{\mathrm{i}} + t_{\mathrm{i}} y_{\mathrm{r}})\big) \\
&+ 4\Big(\big(y_{\mathrm{r}} t^2 + 2t_{\mathrm{r}} y t - t_{\mathrm{i}}(2t_{\mathrm{r}} y_{\mathrm{i}} + t_{\mathrm{i}} y_{\mathrm{r}})\big)\, q_{\mathrm{r}1}^2 + 3\big(2y t^2 - 2(t_{\mathrm{i}} y_{\mathrm{i}} + t_{\mathrm{r}} y_{\mathrm{r}})t - y\big(t_{\mathrm{i}}^2 + t_{\mathrm{r}}^2\big)\big)\, q_{\mathrm{r}2} q_{\mathrm{r}1} \\
&- 2q_{\mathrm{r}2}^2\big(y_{\mathrm{r}} t^2 + 2t_{\mathrm{r}} y t + 4t_{\mathrm{i}} t_{\mathrm{r}} y_{\mathrm{i}} + 2t_{\mathrm{i}}^2 y_{\mathrm{r}} - 3t_{\mathrm{r}}^2 y_{\mathrm{r}}\big)\Big)\, q_{\mathrm{i}2}\Big)
\end{aligned}
$$

Invariants, explicitly



$$
\begin{aligned}
= \mathrm{i}\Big(&\Big(t y_\mathrm{r}^2 + 2t_\mathrm{r} y y_\mathrm{r} - y_\mathrm{i}(2t_\mathrm{i} y + t y_\mathrm{i})\Big) q_{\mathrm{i}1}^3 + 2\Big(\Big(2t_\mathrm{r} y^2 + 4t y_\mathrm{r} y + t_\mathrm{r} y_\mathrm{i}^2 - 3t_\mathrm{r} y_\mathrm{r}^2 + 2t_\mathrm{i} y_\mathrm{i} y_\mathrm{r}\Big) q_{\mathrm{i}2} \\
&- q_{\mathrm{r}1}(t_\mathrm{r} y y_\mathrm{i} + t y_\mathrm{r} y_\mathrm{i} + t_\mathrm{i} y y_\mathrm{r}) - q_{\mathrm{r}2}\Big(2t_\mathrm{i} y^2 + 4t y_\mathrm{i} y - 3t_\mathrm{i} y_\mathrm{i}^2 + t_\mathrm{i} y_\mathrm{r}^2 + 2t_\mathrm{r} y_\mathrm{i} y_\mathrm{r}\Big)\Big) q_{\mathrm{i}1}^2 \\
&+ \Big(4\Big(t\Big(3y^2 - y_\mathrm{i}^2 - 2y_\mathrm{r}^2\Big) - 2y(t_\mathrm{i} y_\mathrm{i} + 2t_\mathrm{r} y_\mathrm{r})\Big) q_{\mathrm{i}2}^2 + 6\Big(t_\mathrm{i}\Big(y_\mathrm{r}^2 - y_\mathrm{i}^2\Big) + 2t_\mathrm{r} y_\mathrm{i} y_\mathrm{r}\Big) q_{\mathrm{r}1} q_{\mathrm{i}2} \\
&+ \Big(t y_\mathrm{r}^2 + 2t_\mathrm{r} y y_\mathrm{r} - y_\mathrm{i}(2t_\mathrm{i} y + t y_\mathrm{i})\Big) q_{\mathrm{r}1}^2 + 4\Big(2y(2t_\mathrm{i} y_\mathrm{i} + t_\mathrm{r} y_\mathrm{r}) + t\Big(-3y^2 + 2y_\mathrm{i}^2 + y_\mathrm{r}^2\Big)\Big) q_{\mathrm{r}2}^2 \\
&+ 6(2t_\mathrm{i} y_\mathrm{i} y_\mathrm{r} + t_\mathrm{r}(y_\mathrm{i} - y_\mathrm{r})(y_\mathrm{i} + y_\mathrm{r})) q_{\mathrm{r}1} q_{\mathrm{r}2}\Big) q_{\mathrm{i}1} + 8(t_\mathrm{i}(y - y_\mathrm{r})(y + y_\mathrm{r}) + 2y_\mathrm{i}(t y - t_\mathrm{r} y_\mathrm{r})) q_{\mathrm{r}2}^3 \\
&+ 9\left((2t_\mathrm{i} y y_\mathrm{i} - 2t_\mathrm{r} y y_\mathrm{r} + t(y_\mathrm{i} - y_\mathrm{r})(y_\mathrm{i} + y_\mathrm{r})) q_{\mathrm{i}1} + 2(t_\mathrm{r} y y_\mathrm{i} + t y_\mathrm{r} y_\mathrm{i} + t_\mathrm{i} y y_\mathrm{r}) q_{\mathrm{r}1}\right) q_3^2 \\
&- 2\Big(2q_{\mathrm{i}2}^2 - q_{\mathrm{r}1}^2\Big)\Big(2q_{\mathrm{i}2}\Big(t_\mathrm{r} y^2 + 2t y_\mathrm{r} y - t_\mathrm{r} y_\mathrm{i}^2 - 2t_\mathrm{i} y_\mathrm{i} y_\mathrm{r}\Big) - q_{\mathrm{r}1}(t_\mathrm{r} y y_\mathrm{i} + t y_\mathrm{r} y_\mathrm{i} + t_\mathrm{i} y y_\mathrm{r})\Big) \\
&- 4q_{\mathrm{r}2}^2\Big(2q_{\mathrm{i}2}\Big(t_\mathrm{r} y^2 + 2t y_\mathrm{r} y + 2t_\mathrm{r} y_\mathrm{i}^2 - 3t_\mathrm{r} y_\mathrm{r}^2 + 4t_\mathrm{i} y_\mathrm{i} y_\mathrm{r}\Big) - q_{\mathrm{r}1}(t_\mathrm{r} y y_\mathrm{i} + t y_\mathrm{r} y_\mathrm{i} + t_\mathrm{i} y y_\mathrm{r})\Big) \\
&+ 6\Big(-2(t_\mathrm{r} y y_\mathrm{i} + t y_\mathrm{r} y_\mathrm{i} + t_\mathrm{i} y y_\mathrm{r}) q_{\mathrm{r}2}^2 \\
&+ \Big(\Big(2t_\mathrm{r} y^2 + 4t y_\mathrm{r} y + t_\mathrm{r} y_\mathrm{i}^2 - 3t_\mathrm{r} y_\mathrm{r}^2 + 2t_\mathrm{i} y_\mathrm{i} y_\mathrm{r}\Big) q_{\mathrm{i}1} + 2\Big(t y_\mathrm{r}^2 + 2t_\mathrm{r} y y_\mathrm{r} - y_\mathrm{i}(2t_\mathrm{i} y + t y_\mathrm{i})\Big) q_{\mathrm{i}2} + 2\Big(t_\mathrm{i} y_\mathrm{r}^2 + 2t_\mathrm{r} y_\mathrm{i} y_\mathrm{r} - \\
&+ \Big(-\Big(2t_\mathrm{i} y^2 + 4t y_\mathrm{i} y - 3t_\mathrm{i} y_\mathrm{i}^2 + t_\mathrm{i} y_\mathrm{r}^2 + 2t_\mathrm{r} y_\mathrm{i} y_\mathrm{r}\Big) q_{\mathrm{i}1} + 2(t_\mathrm{r} y y_\mathrm{i} + t y_\mathrm{r} y_\mathrm{i} + t_\mathrm{i} y y_\mathrm{r}) q_{\mathrm{i}2} + 2\Big(-t_\mathrm{r} y^2 - 2t y_\mathrm{r} y + t_\mathrm{r} y_\mathrm{i}^2 + 2 \\
&+ 4\Big(2\Big(2y_\mathrm{i}(t y + 2t_\mathrm{r} y_\mathrm{r}) + t_\mathrm{i}\Big(y^2 - 3y_\mathrm{i}^2 + 2y_\mathrm{r}^2\Big)\Big)\Big) q_{\mathrm{i}2}^2 \\
&+ 3\Big(t\Big(2y^2 - y_\mathrm{i}^2 - y_\mathrm{r}^2\Big) - 2y(t_\mathrm{i} y_\mathrm{i} + t_\mathrm{r} y_\mathrm{r})\Big) q_{\mathrm{r}1} q_{\mathrm{i}2} + \Big(t_\mathrm{i} y_\mathrm{r}^2 + 2t_\mathrm{r} y_\mathrm{i} y_\mathrm{r} - y(t_\mathrm{i} y + 2t y_\mathrm{i})\Big) q_{\mathrm{r}1}^2\Big) q_{\mathrm{r}2}\Big)
\end{aligned}
$$

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