On the CP-odd basis invariants of the 2HDM

Andreas Trautner

based on:
181x.xxxx, AT to appear

DISCRETE18
Wien
27.11.18
Motivation

Open physical question:

**Sufficient** conditions for CP conservation in NHDM?

note, solved in special cases [Nishi '07]

More general motivation:

Physical observables must not depend on basis and notation.
⇒ Can only depend on basis invariant quantities.

Given a theory in an arbitrary basis,

• How does one obtain basis invariant quantities?
• How many independent basis invariant quantities exist?
• How are these basis invariant quantities related to physical observables?

e.g. for 2HDM, [Ogreid '18]
Motivation

Existing basis invariant methods (for NHDM)

“Brute force”

[Davidson, Gunion, Haber, O’Neil, ...]

👍 straightforward

👎 trial and error

👎 not systematic

“Orbit space”

[Ivanov, Maniatis, Manteuffel, Nachtmann, Nishi, ...]

👍 elegant

👎 very specialized to NHDM

👎 not systematic

Why is this such a big deal? Shouldn’t this be a simple group theory problem?

\[ 3 \otimes \bar{3} = 8 \otimes 1_0 \ldots etc. \ldots done? \]

... and there exist powerful codes to do that, e.g. Susyno [Fonseca ’11]
Motivation

However:

- What are the building blocks? i.e. what do we even have to contract to singlets?
- When can we stop contracting? i.e. which singlets are independent?
- What does independent even mean in this context?

⇒ Simple group theory problem (construction of singlets) turns into an invariant theory, algebra problem!

And we have to deal with both…

Fortunately: powerful methods exist for both, but we have to learn them…
Outline

– Short general synopsis on jargon of invariant theory
– General strategy, outline the systematic procedure

2HDM:
– Construction of the building blocks
  • CP properties of the building blocks
– Characterize full ring of basis invariants
– Construct full ring of basis invariants
– Necessary and sufficient conditions for CP conservation
Jargon

• **Algebraic independence**

An invariant $\mathcal{I}_1$, is algebraically dependent of a set of invariants, say $\mathcal{I}_2, \ldots$, if and only if there is a polynomial $P$

$$P(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \ldots) = 0.$$

• **Primary invariants**

A maximal set of algebraically independent invariants (not unique, but the number of invariants is). Turns out: # of algebraically independent invariants $=$ # of physical parameters.

• **Generating set of invariants**

All invariants that *cannot* be written as a polynomial of other invariants,

$$\mathcal{I}_i \neq P(\mathcal{I}_j, \ldots).$$

Vice versa, all invariants in the ring can be written as a polynomial in the generating set of invariants,

$$\mathcal{I} = P(\mathcal{I}_1, \mathcal{I}_2 \ldots).$$
Procedure / Algorithm

- Construction of *basis covariant* objects: “building blocks”.
  - Determine CP transformation behavior of the building blocks.
- Derive Hilbert series & Plethystic logarithm.
  - # and order of primary invariants.
  - # and structure of generating set of invariants.
  - Interrelations between invariants (syzygies).
- Derive all invariants and needed interrelations explicitly.
- Finally: use this to find necessary and sufficient conditions for CPC (here in 2HDM).
New also: In order to explicitly construct singlets out of

\[(r_1 \otimes r_2 \otimes r_3 \otimes \ldots) \supset 1_0 \oplus \ldots ,\]

we use hermitean projection operators constructed from Young tableaux via the diagrammatic technique of birdtrack diagrams.

These projection operators correspond to an orthogonal basis for the different singlets.

This is in contrast to, for example, the “trace basis”

\[\text{tr} [\Lambda], \text{tr} [\Lambda^2], \text{tr} [\Lambda^3], \ldots\]

Using orthogonal basis for invariants makes them much shorter and easier to handle.
Construction of the building blocks

2HDM scalar potential: \((a, b = 1, 2\) Higgs-flavor index)\)

\[ V = \Phi_a^\dagger Y^a_{\ b} \Phi^b + \Phi_a^\dagger \Phi_b^\dagger Z^{ab}_{\ cd} \Phi^c \Phi^d. \]

Hermiticity and \(SU(2)_L\) invariance

\[ Y^a_{\ b} = (Y^b_{\ a})^*, \quad Z^{ab}_{\ cd} = (Z^{cd}_{\ ab})^*, \quad Z^{ab}_{\ cd} = Z^{ba}_{\ dc}. \]

\(\Rightarrow Y\) and \(Z\) have \(4 + 10\) independent real d.o.f.

After using basis changes (which we don’t): find 11 “physical” parameters.

Decomposition into building blocks

\[ Y \cong 2 \otimes 2 \cong 1 \oplus 3, \]
\[ Z \cong 2 \otimes 2 \otimes 2 \otimes 2 \cong 1 \oplus 1 \oplus 3 \oplus 5. \]
Construction of the building blocks

\[ Y : \quad \frac{a}{2} \otimes \frac{b}{2} = \frac{a}{1} \oplus \frac{a}{3} \]

\[ Z : \quad \frac{a}{2} \otimes \frac{b}{2} \otimes \frac{c}{2} \otimes \frac{d}{2} = \frac{a}{1(1)} \oplus \frac{a}{1(2)} \oplus \frac{a}{3(1)} \oplus \frac{a}{3(2)} \oplus \frac{a}{3(3)} \oplus \frac{a}{5} . \]
Construction of the building blocks

\( Y : \quad a_2 \otimes b_2 = a_1 \oplus a_2 b_3 \),

\( Z : \quad a_2 \otimes b_2 \otimes c_2 \otimes d_2 = a_1(1) \oplus a_2 b_3(1) \oplus a_2 c_3(2) \oplus a_2 d_3(3) \oplus a_1 b_3(2) \oplus a_1 c_3(3) \oplus a_1 d_3(3) \).

\[
P_{a'b'} := \quad \begin{array}{c}
\text{Diagram}
\end{array}
\equiv \frac{1}{2!} \left( \begin{array}{c}
\text{Diagram}
\end{array} + \begin{array}{c}
\text{Diagram}
\end{array} \right),
\]

\[
P \equiv \frac{1}{2!} \left( \begin{array}{c}
\text{Diagram}
\end{array} - \begin{array}{c}
\text{Diagram}
\end{array} \right) = \frac{1}{2} \begin{array}{c}
\text{Diagram}
\end{array} = -\frac{1}{2} \varepsilon_{a'b'} \varepsilon_{ab}.
\]

see e.g. [Alcock-Zeilinger and Weigert '17]
Building blocks

\[ Y_1 := Y^a_a = y_{11} + y_{22}, \]

\[ Z_{1(1)} := \frac{1}{2} \left( Z^{ab}_{ab} + Z^{ab}_{ba} \right) \propto z_{1111} + z_{1212} + z_{1221} + z_{2222}, \]

\[ Z_{1(2)} := \varepsilon_{ab} \varepsilon^{cd} Z^{ab}_{cd} \propto z_{1212} - z_{1221}, \]

\[ Y_{3}^{a'b'} := \frac{1}{2} \left( Y^{a'b'} + Y^{b'a'} \right) \propto \begin{pmatrix} y_{12} & \frac{1}{2} (y_{22} - y_{11}) \\ \frac{1}{2} (y_{22} - y_{11}) & -y_{12}^{*} \end{pmatrix}, \]

\[ Z_{3}^{ab} := \propto \begin{pmatrix} z_{1112} + z_{1222} & \frac{1}{2} (z_{2222} - z_{1111}) \\ \frac{1}{2} (z_{2222} - z_{1111}) & -(z_{1112} + z_{1222})^{*} \end{pmatrix}, \]

\[ Z_{5}^{abcd} := \propto \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & \zeta_3 \end{pmatrix} \begin{pmatrix} \zeta_2 & \zeta_3 \\ \zeta_3 & -\zeta_2^{*} \end{pmatrix} \propto \begin{pmatrix} z_{1122} \zeta_1 := & \frac{1}{2} \left( z_{1222} - z_{1112} \right), \\ \frac{1}{2} \left( z_{1222} - z_{1112} \right) & \zeta_2 := \frac{1}{6} \left( z_{1111} - 2z_{1212} - 2z_{1221} + z_{2222} \right), \end{pmatrix} \]

\[ \zeta_3 := \frac{1}{6} \left( z_{1111} - 2z_{1212} - 2z_{1221} + z_{2222} \right). \]
CP trafo of the building blocks

\[ Y_1 \mapsto Y_1 , \quad Z_{1(1)} \mapsto Z_{1(1)} , \quad Z_{1(2)} \mapsto Z_{1(2)} , \]
\[ Y_{3}^{ab} \mapsto -(Y_3)^{ab} , \quad Z_{3}^{ab} \mapsto -(Z_3)^{ab} , \]
\[ Z_{5}^{abcd} \mapsto (Z_5)^{abcd} . \]

For basis invariants (\(\mapsto\) all indices contracted):

A basis invariant is CP \(\{\text{even odd}\}\) iff it contains an \(\{\text{even odd}\}\) number of triplet building blocks \((Y_3, Z_3)\).
Higher-order invariants

Building blocks as input for the ring:

\[
q \cong \mathbb{Z}_5, \quad y \cong Y_3, \quad \text{and} \quad t \cong Z_3.
\]

From input, compute Hilbert series (HS) and Plethystic logarithm (PL):

\[
\mathcal{H}(q, y, t), \quad \text{PL} \left[ \mathcal{H}(q, y, t) \right] := \sum_{k=1}^{\infty} \frac{\mu(k) \ln \mathcal{H}(q^k, y^k, t^k)}{k}.
\]

see e.g. [Benvenuti, Feng, Hanany, He ’06]

\[
h(z) \equiv \mathcal{H}(z, z, z) = \frac{1 + z^3 + 4z^4 + 2z^5 + 4z^6 + z^7 + z^{10}}{(1 - z^2)^4 (1 - z^3)^3 (1 - z^4)},
\]

also found by [Bednyakov ’18]
Higher-order invariants

Building blocks as input for the ring:

\[ q \cong \mathbb{Z}_5, \quad y \cong Y_3, \quad \text{and} \quad t \cong \mathbb{Z}_3. \]

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\]

also found by [Bednyakov '18]

\[
\text{PL} \left[ \mathcal{H}(q, y, t) \right] = t^2 + y^2 + ty + q^2 + q^3 + qt^2 + qy^2 + qt
\]
\[
+ qt^2 y + qty^2 + q^2 t^2 + q^2 y^2 + q^2 ty + q^2 t^2 y + q^2 t y^2
\]
\[
+ q^3 t^3 + q^3 y^3 + q^3 t^2 y + q^3 t y^2 - q^2 t^2 y^2 - \mathcal{O} \left( [tyq]^7 \right).
\]
Higher-order invariants

Building blocks as \textbf{input} for the ring:

\[
\begin{align*}
q & \cong Z_5, &
y & \cong Y_3, &
t & \cong Z_3.
\end{align*}
\]

From input, compute Hilbert series (HS) and Plethystic logarithm (PL):

\[
\mathcal{H}(q, y, t), \quad \text{PL} \left[ \mathcal{H}(q, y, t) \right] := \sum_{k=1}^{\infty} \frac{\mu(k) \ln \mathcal{H}(q^k, y^k, t^k)}{k}.
\]

see e.g. [Benvenuti, Feng, Hanany, He '06]

\[
\mathcal{H}(z) \equiv \mathcal{H}(z, z, z) = \frac{1 + z^3 + 4 z^4 + 2 z^5 + 4 z^6 + z^7 + z^{10}}{(1 - z^2)^4 (1 - z^3)^3 (1 - z^4)},
\]

also found by [Bednyakov '18]

\[
\text{PL} \left[ \mathcal{H}(q, y, t) \right] = t^2 + y^2 + ty + q^2 + q^3 + qt^2 + qy^2 + qyt
\]

\[
+ qt^2y + qty^2 + q^2t^2 + q^2y^2 + q^2ty + q^2t^2y + q^2ty^2
\]

\[
+ q^3t^3 + q^3y^3 + q^3t^2y + q^3ty^2 - q^2t^2y^2 - \mathcal{O}(\left[tyq\right]^7).
\]

For example, the terms

\[
+1 qyt \Rightarrow (Z_5 \otimes Y_3 \otimes Z_3) \supset (1 \times) 1_0
\]

\[
-1 q^2y^2t^2 \Rightarrow 1 \text{ relation of } \mathcal{O}(q^2y^2t^2).
\]
Explicit construction of primary invariants

\[ Z_5 \equiv \begin{array}{ccc} \end{array}, \quad Y_3 \equiv \begin{array}{cc} \end{array}, \quad Z_3 \equiv \begin{array}{c} \end{array}. \]

\[ I_{a,b,c} \equiv \text{Invariant with powers } q^a y^b t^c, \text{ that is } Z_5^\otimes a \otimes Y_3^\otimes b \otimes Z_3^\otimes c. \]

\[ I_{2,0,0} := \begin{array}{ccc} \end{array}, \quad I_{0,2,0} := \begin{array}{cc} \end{array}, \quad I_{0,1,1} := \begin{array}{c} \end{array}, \quad I_{0,0,2} := \begin{array}{c} \end{array}, \quad I_{3,0,0} := \begin{array}{ccc} \end{array}, \quad I_{1,2,0} := \begin{array}{ccc} \end{array}, \quad I_{1,0,2} := \begin{array}{ccc} \end{array}, \quad \text{and} \quad I_{2,1,1} := \begin{array}{ccc} \end{array}. \]

Needed projection operators are simple! (always Sym – ASym – Sym)

\[ P = \begin{array}{cccc} a_1' & a_2' & \cdots & a_n' \end{array} \begin{array}{cccc} a_1' & a_2' & \cdots & a_n' \end{array}. \]

Together with the 3 linear invariants, these are 11 algebraically independent invariants. 
Completing the generating set of invariants
Find their structure from the graded plethystic logarithm. Then simply construct

\[ \mathcal{I}_{1,1,1} := \begin{array}{c|c|c} \text{green} & \text{red} & \text{green} \\ \hline \text{red} & \text{green} & \text{red} \end{array}, \]

\[ \mathcal{I}_{2,2,0} := \begin{array}{c|c|c} \text{green} & \text{green} & \text{green} \\ \hline \text{green} & \text{green} & \text{green} \end{array}, \quad \mathcal{I}_{2,0,2} := \begin{array}{c|c|c} \text{red} & \text{red} & \text{red} \\ \hline \text{red} & \text{red} & \text{red} \end{array}, \]

\[ \mathcal{I}_{1,2,1} := \begin{array}{c|c|c} \text{green} & \text{red} & \text{red} \\ \hline \text{red} & \text{green} & \text{red} \end{array}, \quad \mathcal{I}_{1,1,2} := \begin{array}{c|c|c} \text{red} & \text{green} & \text{red} \\ \hline \text{red} & \text{green} & \text{red} \end{array}, \]

\[ \mathcal{I}_{2,2,1} := \begin{array}{c|c|c} \text{green} & \text{green} & \text{green} \\ \hline \text{green} & \text{green} & \text{green} \end{array}, \quad \mathcal{I}_{2,1,2} := \begin{array}{c|c|c} \text{red} & \text{red} & \text{red} \\ \hline \text{red} & \text{red} & \text{red} \end{array}, \]

\[ \mathcal{I}_{3,3,0} := \begin{array}{c|c|c|c|c|c|c} \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} \\ \hline \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} \end{array}, \quad \mathcal{I}_{3,0,3} := \begin{array}{c|c|c|c|c|c|c} \text{red} & \text{red} & \text{red} & \text{red} & \text{red} & \text{red} & \text{red} \\ \hline \text{red} & \text{red} & \text{red} & \text{red} & \text{red} & \text{red} & \text{red} \end{array}, \]

\[ \mathcal{I}_{3,2,1} := \begin{array}{c|c|c|c|c|c|c} \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} \\ \hline \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} \end{array}, \quad \mathcal{I}_{3,1,2} := \begin{array}{c|c|c|c|c|c|c} \text{red} & \text{red} & \text{red} & \text{red} & \text{red} & \text{red} & \text{red} \\ \hline \text{red} & \text{red} & \text{red} & \text{red} & \text{red} & \text{red} & \text{red} \end{array}. \]

Together with 3 linear + 8 invs. of the previous slide, these are all invariants of the generating set.

Odd total number of $Y_3$ and $Z_3 \Rightarrow$ CP-odd, denoted by $\mathcal{J}$ instead of $\mathcal{I}$.

In total 8 CP-odd invariants.

Curious: Note the $Y_3 \leftrightarrow Z_3$ symmetry!
Systematic construction of syzygies

Leading negative terms in PL correspond to syzygies!
For example: $-q^2 t^2 y^2$

Matching power products of invariants of this structure:

\[
\mathcal{I}_{1,1,1}^2, \quad \mathcal{I}_{2,1,1} \mathcal{I}_{0,1,1}, \quad \mathcal{I}_{2,2,0} \mathcal{I}_{0,0,2}, \quad \mathcal{I}_{2,0,2} \mathcal{I}_{0,2,0},
\]
\[
\mathcal{I}_{1,2,0} \mathcal{I}_{1,0,2}, \quad \mathcal{I}_{2,0,0} \mathcal{I}_{0,2,0} \mathcal{I}_{0,0,2}, \quad \mathcal{I}_{2,0,0} \mathcal{I}_{0,1,1}^2.
\]

A simple linear ansatz then reveals the first syzygy:

\[
3 \mathcal{I}_{1,1,1}^2 = 2 \mathcal{I}_{2,1,1} \mathcal{I}_{0,1,1} - \mathcal{I}_{2,2,0} \mathcal{I}_{0,0,2} - \mathcal{I}_{2,0,2} \mathcal{I}_{0,2,0} + 3 \mathcal{I}_{1,2,0} \mathcal{I}_{1,0,2} + \mathcal{I}_{2,0,0} \mathcal{I}_{0,2,0} \mathcal{I}_{0,0,2} - \mathcal{I}_{2,0,0} \mathcal{I}_{0,1,1}^2.
\]
Necessary and sufficient conditions for CPC

Gunion and Haber: vanishing of four specific invariants is necessary and sufficient for explicit CPC.

\[ I_{2Y2Z} \sim J_{1,2,1}, \quad I_{Y3Z} \sim J_{1,1,2}, \]
\[ I_{3Y3Z} \sim J_{3,3,0}, \quad I_{6Z} \sim J_{3,0,3}. \]

8 CP-odd invariants? \( \Rightarrow \) 4 relations? Yes! (...actually, there are more)

\[ 3 J_{2,2,1} I_{1,2,0} - J_{3,2,1} I_{0,2,0} + 3 J_{3,3,0} I_{0,1,1} + J_{1,2,1} I_{2,2,0} = 0, \quad \text{and} \quad y \leftrightarrow t. \]

\[ 3 J_{2,2,1}^2 + 3 J_{3,3,0} J_{1,1,2} - J_{3,2,1} J_{1,2,1} - J_{1,2,1}^2 I_{2,0,0} = 0, \quad \text{and} \quad y \leftrightarrow t. \]

From these equations it is trivial to show:

\[ J_{1,2,1} = J_{1,1,2} = J_{3,3,0} = J_{3,0,3} = 0 \quad \Rightarrow \quad J_{3,2,1} = J_{3,1,2} = J_{2,2,1} = J_{2,1,2} = 0. \]
Summary

• Group theory + invariant theory: a new systematic way to construct basis invariants.

• Used(invariant theory): Hilbert series + Plethystic logarithm ⇒ number and structure of invariants + relations (syzygies).

• Used(group theory): Young tableaux + birdtracks ⇒ building blocks + higher-order invariants.

• Constructed full ring of $22 \text{ CP-even}(3 + 8 + 3)$ and CP-odd($8$) basis invariants of 2HDM.

• Explicitly constructed syzygies.

⇒ Very simple proof of sufficient conditions for CPC.

Advantages:

• Shorter words for primary invariants.

• No choice of basis necessary whatsoever.

• CP trafo of invariants are clear.

• Direct access to the syzygies.
Outlook

• 3HDM, . . . , NHDM?
• Take into account Higgs VEVs.
• In principle, algorithm generalizes to all simple groups.
• Analysis of RGE running simplified. [Bednyakov ’18][Bijnens, Oredsson, Rathsman ’18]
• Effect of global symmetries?
• Formulate theories in terms of invariants? [Ogreid ’18]
Thank You!
Backup slides
Explicit form of building blocks

Components of $Y$ and $Z$: $[Y]_{ab}^a = y_{ab}$ and $[Z]_{cd}^{ab} = z_{abcd}$, dropping irrelevant global prefactors

$$Y_1 = y_{11} + y_{22},$$
$$Z_{1(1)} = z_{1111} + z_{1212} + z_{1221} + z_{2222},$$
$$Z_{1(2)} = z_{1212} - z_{1221},$$

$$Y_{3}^{ab} = \begin{pmatrix}
y_{12} & \frac{1}{2} (y_{22} - y_{11}) \\
\frac{1}{2} (y_{22} - y_{11}) & -y_{12}^*
\end{pmatrix},$$

$$Z_{3}^{ab} = \begin{pmatrix}
z_{1112} + z_{1222} & \frac{1}{2} (z_{2222} - z_{1111}) \\
\frac{1}{2} (z_{2222} - z_{1111}) & -(z_{1112} + z_{1222})^*
\end{pmatrix},$$

$$Z_{5}^{abcd} = \begin{pmatrix}
\zeta_1 & \zeta_2 & \zeta_3 & \zeta_2^* \\
\zeta_2 & \zeta_3 & \zeta_3^* & -\zeta_2^* \\
\zeta_3 & -\zeta_2^* & -\zeta_2^* & \zeta_1
\end{pmatrix}.$$

with

$$\zeta_1 := z_{1122}, \quad \zeta_2 := \frac{1}{2} (z_{1222} - z_{1112}), \quad \zeta_3 := \frac{1}{6} (z_{1111} - 2z_{1212} - 2z_{1221} + z_{2222}).$$

Here, $\{y_{11}, y_{22}, z_{1111}, z_{1212}, z_{1221}, z_{2222}\} \in \mathbb{R}$, $\{y_{12}, z_{1122}, z_{1222}, z_{1112}\} \in \mathbb{C}$. 

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Building blocks in conventional notation

\[ Y_1 = m_{11}^2 + m_{22}^2, \]
\[ Z_{1(1)} = \frac{1}{2} (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4), \]
\[ Z_{1(2)} = \frac{1}{2} (\lambda_3 - \lambda_4), \]
\[ Y_{3}^{ab} = \begin{pmatrix} -m_{12}^2 & \frac{1}{2} \left( m_{22}^2 - m_{11}^2 \right) \\ \frac{1}{2} \left( m_{22}^2 - m_{11}^2 \right) & (m_{12}^2)^* \end{pmatrix}, \]
\[ Z_{3}^{ab} = \frac{1}{2} \begin{pmatrix} \lambda_6 + \lambda_7 & \frac{1}{2} (\lambda_2 - \lambda_1) \\ \frac{1}{2} (\lambda_2 - \lambda_1) & -(\lambda_6 + \lambda_7)^* \end{pmatrix}, \]
\[ Z_{5}^{abcd} = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & \xi_3 \\ \xi_3 & -\xi_2^* \end{pmatrix} \begin{pmatrix} \xi_2 & \xi_3 \\ \xi_3 & -\xi_2^* \\ -\xi_2^* & \xi_1 \end{pmatrix}, \]

with

\[ \xi_1 := \frac{1}{2} \lambda_5, \quad \xi_2 := \frac{1}{4} (\lambda_7 - \lambda_6), \quad \xi_3 := \frac{1}{12} (\lambda_1 - 2\lambda_3 - 2\lambda_4 + \lambda_2). \]

As usual \( \{m_{12}^2, \lambda_5, \lambda_6, \lambda_7\} \in \mathbb{C} \) while all others are real.
Hilbert Series

With character polynomials $\chi_r(z)$ for SU(2) irreps $r$

$$\chi_3(z) = z^2 + 1 + \frac{1}{z^2}, \quad \chi_5(z) = z^4 + z^2 + 1 + \frac{1}{z^2} + \frac{1}{z^4},$$

and plethystic exponential (PE)

$$\text{PE}[z, x, r] := \exp\left(\sum_{k=1}^{\infty} \frac{x^k \chi_r(z^k)}{k}\right).$$

the multi-graded Hilbert series is computed as

$$\mathcal{H}(q, y, t) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} (1 - z^2) \text{PE}[z, q, 5] \text{PE}[z, y, 3] \text{PE}[z, t, 3].$$

Integration via residue theorem $\Rightarrow \mathcal{H}(q, y, t) = \frac{N(q, y, t)}{D(q, y, t)}$, with

$$N(q, y, t) = 1 + qt y + q^2 t y + q t^2 y + q t y^2 + q^2 t^2 y + q^2 t y^2 + q^3 t^3 + q^3 t^2 y + q^3 t y^2 + q^3 y^3 - q^3 t^4 y - q^3 t^3 y^2 - q^3 t^2 y^3 - q^3 t y^4 - q^4 t^3 y^2 - q^4 t^2 y^3 - q^5 t^3 y^2 - q^5 t^2 y^3 - q^5 t y^4 - q^6 t^4 y^4,$$

$$D(q, y, t) = (1 - t^2) (1 - y^2) (1 - t y) (1 - q^2) (1 - q^3) (1 - q t^2) (1 - q y^2) (1 - q^2 t^2) (1 - q^2 y^2).$$
Plethystic Logarithm

\[ \text{PL} \left[ \mathcal{H} (q, y, t) \right] := \sum_{k=1}^{\infty} \frac{\mu(k) \ln \mathcal{H} (q^k, y^k, t^k)}{k}, \]

where \( \mu(k) \) is the Möbius function.

\[ \text{PL} \left[ \mathcal{H} (q, y, t) \right] = t^2 + ty + y^2 + q^2 + qt^2 + qty + qy^2 + q^3 + qt^2y + q^2t^2 + qty^2 + q^2ty \\
+ q^2y^2 + q^2t^2y + q^2ty^2 + q^3t^3 + q^3t^2y + q^3ty^2 + q^3y^3 - q^2t^2y^2 \\
- q^2t^3y^2 - q^2t^2y^3 - q^3t^2y^2 - q^2t^4y^2 - q^3t^4y - q^2t^3y^3 - 3q^3t^3y^2 \\
- q^2t^2y^4 - 3q^3t^2y^3 - q^4t^2y^2 - q^3ty^4 - \mathcal{O} \left( [tyq]^9 \right). \]

\( \mathcal{h}(z) \equiv \mathcal{H}(z, z, z) = \frac{1 + z^3 + 4z^4 + 2z^5 + 4z^6 + z^7 + z^{10}}{(1 - z^2)^4 (1 - z^3)^3 (1 - z^4)} \).

\[ \text{PL} \left[ \mathcal{h}(z) \right] = 4z^2 + 4z^3 + 5z^4 + 2z^5 + 3z^6 - 3z^7 - \mathcal{O} \left( z^8 \right). \]
Easy way to check algebraic independence of a set of polynomials (here, invariants) $I_i$, depending on a number of variables $x_j$ (here, the components $y_{ab}$ and $z_{abcd}$ of $Y$ and $Z$). The number of algebraically independent invariants is the rank of the Jacobian matrix:

$$\text{number of algebraically independent invariants} = \text{rank} \left[ \frac{\partial I_i}{\partial x_j} \right].$$

Besides symbolic evaluation one can also use this criterion with all variables put to random numbers for a fast machine evaluation.
Relation to invariants of Gunion and Haber

Original set of necessary and sufficient CP-odd invariants of Gunion and Haber

\[ I_{Y3Z} = \text{Im} \left[ Z^{ai}_{\ i} Z^{ej}_{\ jb} Z^{bc}_{\ ed} Y^d_{\ a} \right] = -2i \mathcal{J}_{1,1,2} , \]

\[ I_{2Y2Z} = \text{Im} \left[ Y^a_{\ b} Y^c_{\ d} Z^{bd}_{\ af} Z^{fi}_{\ ic} \right] = -2i \mathcal{J}_{1,2,1} , \]

\[ I_{6Z} = \text{Im} \left[ Z^{ac}_{\ bd} Z^{b\ell}_{\ df} Z^{dp}_{\ ph} Z^{fj}_{\ ak} Z^{km}_{\ jn} Z^{nh}_{\ mc} \right] = -2i \mathcal{J}_{3,3,0} , \]

\[ I_{3Y3Z} = \text{Im} \left[ Z^{ab}_{\ cd} Z^{cd}_{\ eg} Z^{ef}_{\ hq} Y^g_{\ a} Y^h_{\ b} Y^q_{\ f} \right] = 2i \mathcal{J}_{3,3,0} + 2i \mathcal{J}_{1,2,1} \mathcal{I}_{0,1,1} + i Y^2_{1} \mathcal{J}_{1,1,2} . \]
Invariants, explicitly

\[ I_{0,2,0} := (y^2 + y_i^2 + y_r^2), \]
\[ I_{0,1,1} := (ty + t_i y_i + t_r y_r), \]
\[ I_{0,0,2} := (t^2 + t_i^2 + t_r^2), \]
\[ I_{2,0,0} := (3q_3^2 + q_{i1}^2 + 4q_{i2}^2 + q_{r1}^2 + 4q_{r2}^2), \]
\[ I_{1,2,0} := [q_3 \left( 2t^2 - t_i^2 - t_r^2 \right) + 4t (q_{i2} t_i + q_{r2} t_r) + 2q_{i1} t_i t_r + q_{r1} \left( t_r^2 - t_i^2 \right)], \]
\[ I_{1,0,2} := [q_3 \left( 2y^2 - y_i^2 - y_r^2 \right) + 4y (q_{i2} y_i + q_{r2} y_r) + 2q_{i1} y_i y_r + q_{r1} \left( y_r^2 - y_i^2 \right)], \]
\[ I_{3,0,0} := \left( -q_3 \left( q_{i1}^2 - 2q_{i2}^2 + q_{r1}^2 - 2q_{r2}^2 \right) + q_3^3 - 2q_{i2} q_{r1} + 4q_{i1} q_{i2} q_{r2} + 2q_{r1} q_{r2}^2 \right), \]
\[ I_{2,1,1} := \left\{ 6q_3 \left[ -q_{i2} (ty_i + t_i y) + q_{i1} (t_i y_i + t_r y_i) + q_{r1} (t_r y_i - t_i y_i) \right] \\
- 6q_{r2} \left[ q_3 (ty_r + t_r y) + q_{i1} (ty_i + t_i y) + 2q_{i2} (t_i y_i + t_r y_i) + q_{r1} (t_r y + t_r y) \right] \\
+ 3q_3^2 (-2ty + t_i y_i + t_r y_r) + 6q_{i2} q_{r1} (ty_i + t_i y) \\
+ q_{i1}^2 (2ty - t_i y_i - t_r y_r) - 4q_{i2}^2 (ty + t_i y_i - 2t_r y_r) - 6q_{i1} q_{i2} (ty_r + t_r y) \\
+ q_{r1}^2 (2ty - t_i y_i - t_r y_r) - 4q_{r2}^2 (ty - 2t_i y_i + t_r y) \right\}. \]
Invariants, explicitly

\[\begin{align*}
&= \left[ q_3 (2ty - t_1yi - tryr) + 2q_{i2} (ty_i + tiy) \\
&
+ q_{i1} (t_1y_r + tryr) + 2q_{r2} (ty_r + tryr) + qr_1 (tryr - tiy_i) \right], \\
&= \left( -q_3 \left( q_{i1}^2 - 2q_{i2}^2 + q_{r1}^2 - 2q_{r2}^2 \right) + q_3^2 - 2q_{i2}qr_1 + 4q_{i1}q_{i2}q_{r2} + 2qr_1q_{r2}^2 \right), \\
&= i \left[ 3q_3y(t_r y_i - t_i y_r) \\
&
+ 2q_{i2} \left( tyy_r - t_i y_i y_r - t_r y_i^2 + t_r y_i^2 \right) + q_{i1} \left( t \left( y_r^2 - y_i^2 \right) + t_i y y_i - t_r y y_r \right) \\
&
+ qr_1 \left( -2tiy_i y_r + t_i y y_r + tr y y_i \right) \\
&
+ 2q_{r2} \left( -ty y_i + t_i (y - y_r)(y + y_r) + t_r y_i y_r \right) \right], \\
&= -i \left[ 3q_3y(t_r y_i - t_i y_r) \\
&
+ 2q_{i2} \left( t^2 y_r - ttr y + t_i (t_r y_i - t_i y_r) \right) + q_{i1} \left( -tti y_i + tr(ty_r - tr y) + t_i^2 y \right) \\
&
+ 2q_{r2} \left( t^2(-y_i) + tti y + tr(t_r y_i - t_i y_r) \right) \\
&
+ qr_1 \left( -tti y_r - ttr y_i + 2t_i t_r y \right) \right],
\end{align*}\]
\[ \{6q_{r1} \left( q_3 \left( y_r^2 - y_i^2 \right) + 2q_{i2} yy_i \right) - 12q_{r2} \left( q_3 yy_r + q_{i1} yy_i + 2q_{i2} y_i y_r + q_{r1} yy_r \right) \\
+ 12q_3 y_i \left( q_{i1} y_r - q_{i2} y \right) + 3q_3^2 \left( -2y^2 + y_i^2 + y_r^2 \right) + q_{i1}^2 \left( 2y^2 - y_i^2 - y_r^2 \right) \\
- 4q_{i2}^2 \left( y^2 + y_i^2 - 2y_r^2 \right) - 12q_{i1} q_{i2} yy_r + q_{r1}^2 \left( 2y^2 - y_i^2 - y_r^2 \right) - 4q_{r2}^2 \left( y^2 - 2y_i^2 + y_r^2 \right) \} \]

\[ = \left( 6q_{r1} \left( q_3 \left( t_r^2 - t_i^2 \right) + 2q_{i2} tt_i \right) - 12q_{r2} \left( q_3 tt_r + q_{i1} tt_i + 2q_{i2} t_i t_r + q_{r1} tt_r \right) \\
+ 12q_3 t_i \left( q_{i1} t_r - q_{i2} t \right) + 3q_3^2 \left( -2t^2 + t_i^2 + t_r^2 \right) + q_{i1}^2 \left( 2t^2 - t_i^2 - t_r^2 \right) \\
- 4q_{i2}^2 \left( t^2 + t_i^2 - 2t_r^2 \right) - 12q_{i1} q_{i2} tt_r + q_{r1}^2 \left( 2t^2 - t_i^2 - t_r^2 \right) - 4q_{r2}^2 \left( t^2 - 2t_i^2 + t_r^2 \right) \right) \]
\[
\begin{aligned}
= \ & i \left(2q_3 \left(q_{i1} \left(-ty_i^2 + ty_r^2 + t_i y y_i - t_r y y_r\right) + q_{i2}(-ty y_r + t_i y_i y_r + t_r(y - y_i)(y + y_i)) \right.ight.
\left.\right.
\left.\right. + q_{r2} \left(t y y_i - t_i y^2 + t_i y_r^2 - t_r y_i y_r\right) + q_{r1}(-2t y_i y_r + t_i y y_r + t_r y y_i)\right)
\left.\right. + 3q_{r3}^2 y(t_i y_r - t_r y_i)
\left.\right. + 2q_{i1} \left(q_{i2} \left(t y y_i - t_i y^2 + t_i y_r^2 - t_r y_i y_r\right) + q_{r2}(-t y y_r + t_i y_i y_r + t_r(y - y_i)(y + y_i)) \right.
\left.\right. + q_{r1} \left(t y y_i - t_i y^2 + t_i y_r^2 - t_r y_i y_r\right)\right)
\left.\right. + 2q_{i2} q_{r1} t y y_r - 2q_{i2} q_{r1} t y_i y_r
\left.\right. - 2q_{i2} q_{r1} t_r y^2 + 2q_{i2} q_{r1} t_r y_i^2 - 4q_{i2}^2 t y_i y_r + q_{i1}^2 y(t_r y_i - t_i y_r) + 4q_{i2}^2 t_i y y_r
\left.\right. + 4q_{r2}^2 y_i(t y_r - t_r y) - q_{r1}^2 t_i y y_r + q_{r1}^2 t_r y y_i\right)
\end{aligned}
\]

\[
\begin{aligned}
= \ & -i \left(2q_3 \left(q_{i2} \left(t^2(y_r - y) + t t_r y + t_i^2 y_r - t_i t_r y_i\right) + q_{i1}(-t t_i y_i + t_r(t y_r - t_r y) + t_i^2 y\right)
\left.\right. + q_{r2} \left(t^2 y_i - t t_i y + t i t r y_r - t_r^2 y_i\right) + q_{r1}(-t t_i y_r - t t_r y_i + 2t_i t_r y)\right)
\left.\right. + 3q_{r3}^2 t(t_i y_r - t_r y_i)
\left.\right. + 2q_{r2} \left(q_{i2} \left(t t_i y_i + t_r(t_y - y) + t_i^2(-y)\right) + q_{r1} \left(t^2 y_i - t t_i y + t_r(t_i y_r - t_r y_i)\right)\right)
\left.\right. + 2q_{i1} \left(q_{i2} \left(t^2 y_i - t t_i y + t_r(t_i y_r - t_r y_i)\right) + q_{r2} \left(t^2(-y_r) + t t_r y + t_i(t_i y_r - t_r y_i)\right)\right)
\left.\right. + 2q_{i2} q_{r1} t^2 y_r - 2q_{i2} q_{r1} t t r y - 2q_{i2} q_{r1} t_i^2 y_r + 2q_{i2} q_{r1} t_i t_r y_i + q_{i1}^2 t(t_r y_i - t_i y_r)
\left.\right. - 4q_{i2}^2 t t r y_i + 4q_{i2}^2 t_i t_r y + 4q_{r2}^2 t_i (t y_r - t r y) - q_{r1}^2 t t_i y_r + q_{r1}^2 t t_r y_i\right)
\end{aligned}
\]
\[\begin{align*}
&= i \left( 6q_3 \left( q_{i2} \left( q_{i1} t_r \left( 2t_i^2 + t_r^2 \right) - 2q_{i2} t(t_i - t_r)(t_i + t_r) - 2q_{r1} t_i(t - t_r)(t + t_r) \right) \\
&\quad + q_{i2} \left( 2t_r \left( q_{i2} t t_i + q_{r1} \left( t_i^2 - t_r^2 \right) \right) \right) + q_{i1} t_i \left( -2t_i^2 + t_r^2 \right) - 2q_{r2} t_i t_i t_r \right) \\
&\quad + 9q_3^2 t \left( q_{i1} (t_i - t_r)(t_i + t_r) + 2q_{r1} t_i t_r \right) \\
&\quad - 2q_{i1}^2 \left( q_{i2} t_r \left( -2t_i^2 + t_r^2 \right) + q_{r2} t_i \left( 2t_i^2 - t_r^2 \right) + q_{r1} t t_i t_r \right) \\
&\quad + q_{i1} \left( -2q_{i2} q_{r1} t_i \left( t_i^2 - 3t_r^2 \right) + 4q_{i2} t \left( t_i^2 - t_r^2 - 2t_r^2 \right) \right) \\
&\quad + 4q_{r2} t_i \left( t_i^2 - 2t_r^2 + t_i t_r \right) + q_{r1} t \left( t_r^2 - t_i^2 \right) - 2q_{r1} q_{r2} t_r \left( t_r^2 - 3t_i^2 \right) \right) \\
&\quad + 4q_{r2} t_r \left( q_{r1} t t_i - 2q_{i2} \left( t_r^2 + 2t_i^2 - t_r^2 \right) \right) \\
&\quad + 4q_{r2} \left( q_{i2} q_{r1} t \left( 2t_i^2 - 3 \left( t_i^2 + t_r^2 \right) \right) + 2q_{i2} t_i \left( t_i^2 - t_r^2 + 2t_r^2 \right) + q_{r1} t_i \left( t_r^2 - t_i^2 \right) \right) \\
&\quad - 2t_r \left( 2q_{i2} - q_{r1} \right) \left( 2q_{i2} (t - t_i)(t + t_i) - q_{r1} t t_i \right) + q_{i1} t \left( t_r^2 - t_i^2 \right) \right) \\
&\quad + 8q_{r2}^2 t_i (t - t_r)(t + t_r) \right).\end{align*}\]
\begin{align*}
&= i \left( 6q_3 \left( q_{r2} \left( q_{i1} y_r \left( 2y^2 + y_i^2 - y_r^2 \right) - 2q_{i2} y(y_i - y_r)(y_i + y_r) \right) - 2q_{r1} y_i(y - y_r)(y + y_r) \right) \\
&\quad + q_{i2} \left( 2y_r \left( q_{i2} y y_i + q_{r1} \left( y_i^2 - y_r^2 \right) \right) + q_{i1} y_i \left( -2y^2 + y_i^2 - y_r^2 \right) \right) \right) - 2q_{r2} y y_i y_r \\
&\quad + 9q_3 y \left( q_{i1} (y_i - y_r)(y_i + y_r) + 2q_{r1} y y_i y_r \right) \\
&\quad - 2q_{i1}^2 \left( q_{i2} y_r \left( -2y^2 - y_i^2 + y_r^2 \right) + q_{r2} y_i \left( 2y^2 - y_i^2 + y_r^2 \right) + q_{r1} y y_i y_r \right) \\
&\quad + q_{i1} \left( -2q_{i2} q_{r1} y_i \left( y_i^2 - 3y_r^2 \right) + 4q_{i2} y \left( y^2 - y_i^2 - 2y_r^2 \right) + 4q_{r2} y \left( -y^2 + 2y_i^2 + y_r^2 \right) \right) \\
&\quad + q_{r1} y \left( y_r^2 - y_i^2 \right) - 2q_{r1} q_{r2} y y_i \left( y_r^2 - 3y_i^2 \right) \right) + 4q_{r2} y \left( q_{r1} y y_i - 2q_{i2} \left( y^2 + 2y_i^2 - y_r^2 \right) \right) \\
&\quad + 4q_{r2} \left( q_{i2} q_{r1} y \left( 2y^2 - 3 \left( y_i^2 + y_r^2 \right) \right) + 2q_{i2} y_i \left( y^2 - y_i^2 + 2y_r^2 \right) + q_{r1} y_i \left( y_r^2 - y_i^2 \right) \right) \\
&\quad - 2y_r \left( 2q_{i2}^2 - q_{r1}^2 \right) \left( 2q_{i2} (y - y_i)(y + y_i) - q_{r1} y y_i \right) + q_{i1} y \left( y_r^2 - y_i^2 \right) + 8q_{r2} y y_i (y - y_r)(y + y_r) \right)
\end{align*}
= \ i \left( (-yt_i^2 - 2ty_i t_i + t_r(t_r y + 2t y_r)) q_{i1}^3 + 2 \left( (2y_r t_r^2 + 4t_r y t + 2t_i t_r y_i + t_i^2 y_r - 3t_r^2 y_r) q_{i2}
ight.
\ - q_{r1} (t_i t_r y + t t_i y r + t t_i y r) - q_{r2} \left( 2y_i t_i^2 + 4t_i y t - 3t_i^2 y_i + t_r^2 y_i + 2t_i t_r y_r \right) \right) q_{i1}^2
\ + (4 \left( 3yt_r^2 - 2(t_i y_i + 2t_r y_r) t - y \left( t_i^2 + 2t_r^2 \right) \right) q_{i2}^2 + 6 \left( -y_i t_i^2 + 2t_r y_r t_i + t_r^2 y_i \right) q_{r1} q_{i2}
\ + (\left( -yt_i^2 - 2ty_i t_i + t_r^2 y + 2t t_r y_r \right) q_{r1}^2 + 4 \left( -3yt_r^2 + 4t_i y_t + 2t_r y_r t + 2t_i^2 y + t_r^2 y \right) q_{r2}^2
\ + 6 \left( y_r t_i^2 + 2t_r y_i t_i - t_r^2 y_r \right) q_{r1} q_{r2}) q_{i1}
\ + 9 \left( (yt_i^2 + 2ty_i t_i - t_r(t_r y + 2t y_r)) q_{i1} + 2(t_i t_r y + t t_i y r + t t_i y r) q_{r1} \right) q_{i2}^2
\ + 4 \left( (t_i t_r y + t t_i y r + t t_i y r) q_{r1} + 2 \left( y_i t_i^2 + 2t_i y t - 3t_i^2 y_i + 2t_r^2 y_i + 4t_i t_r y_r \right) q_{r2} \right) q_{i2}^2
\ - 2 \left( q_{r1}^2 - 2q_{r2} \right) \left( \left( t_i t_r y + t t_i y r + t t_i y r \right) q_{r1} + 2 \left( y_i t_i^2 + 2t_i y t - t_r(t_r y_i + 2t_i y_r) \right) q_{r2} \right)
\ - 6q_3 \left( 2 \left( (t_i t_r y + t t_i y r + t t_i y r) q_{r2} + \left( (yt_i^2 + 2ty_i t_i - t_r(t_r y + 2t y_r) \right) q_{i2} + \left( y_i t_i^2 + 2t_i y t - t_r(t_r y_i + 2t_i y_r) \right) \right) \right)
\ + \left( q_{i2} \left( 2y_i t_i^2 + 4t_i y t - 3t_i^2 y_i + t_r^2 y_i + 2t_i t_r y_r \right) - q_{r2} \left( 2y_r t_r^2 + 4t_r y t + 2t_i t_r y_i + t_i^2 y_r - 3t_r^2 y_r \right) \right) q_{i1}
\ - 8q_{i2}^3 \left( y_r t_i^2 + 2t_r y t - t_i(2t_r y_i + t_i y_r) \right)
\ + 4 \left( \left( y_r t_r^2 + 2t_r y t - t_i(2t_r y_i + t_i y_r) \right) q_{r1}^2 + 3 \left( 2yt_r^2 - 2(t_i y_i + t_r y_r) t - y \left( t_i^2 + t_r^2 \right) \right) q_{r2} q_{r1}
\ - 2q_{r2} \left( y_r t_i^2 + 2t_r y t + 4t_i t_r y_i + 2t_i^2 y_r - 3t_r^2 y_r \right) \right) q_{i2}\)
Invariants, explicitly

\[
\begin{align*}
&= \, i \left( \left( t y_r^2 + 2 t_r y y_r - y_i (2 t_i y + t y_i) \right) q_{i1}^3 + 2 \left( \left( 2 t_r y^2 + 4 t y_r y + t_r y_i^2 - 3 t_r y_r^2 + 2 t_i y_i y_r \right) q_{i2} \\
&\quad - q_{r1} (t_r y y_i + t y_r y_i + t_i y y_r) - q_{r2} \left( 2 t_i y^2 + 4 t y_i y - 3 t_i y_i^2 + t_i y_r^2 + 2 t_r y_i y_r \right) \right) q_{i1}^2 \\
&\quad + \left( 4 \left( t \left( 3 y^2 - y_i^2 - 2 y_r^2 \right) - 2 y (t_i y_i + 2 t_r y_r) \right) q_{i2}^2 + 6 \left( t_i \left( y_r^2 - y_i^2 \right) + 2 t_r y_i y_r \right) q_{r1} q_{i2} \\
&\quad + \left( t y_r^2 + 2 t_r y y_r - y_i (2 t_i y + t y_i) \right) q_{r1}^2 + 4 \left( 2 y (2 t_i y_i + t y_r) + t \left( -3 y^2 - 2 y_i^2 + y_r^2 \right) \right) q_{r2}^2 \\
&\quad + 6 (2 t_i y_i y_r + t_r (y_i - y_r) (y_i + y_r)) q_{r1} q_{r2} \right) q_{i1} + 8 (t_i (y - y_r) (y + y_r) + 2 y_i (t y - t_r y_r)) q_{r2}^3 \\
&\quad + 9 (\left( 2 t_i y y_i - 2 t_r y y_r + t (y_i - y_r) (y_i + y_r) \right) q_{i1} + 2 (t_r y y_i + t y_r y_i + t_i y y_r) q_{r1}) q_{r1}^2 \\
&\quad - 2 \left( 2 q_{i2}^2 - q_{r1}^2 \right) \left( 2 q_{i2} \left( t r y^2 + 2 t y_r y - t_i y_i y_r \right) - q_{r1} (t_r y y_i + t y_r y_i + t_i y y_r) \right) \\
&\quad - 4 q_{r2}^2 \left( 2 q_{i2} \left( t_r y^2 + 2 t y_r y + 2 t_r y_i^2 - 3 t_r y_r^2 + 4 t_i y_i y_r \right) - q_{r1} (t_r y y_i + t y_r y_i + t_i y y_r) \right) \\
&\quad + 6 \left( -2 (t_r y y_i + t y_r y_i + t_i y y_r) q_{r2}^2 \\
&\quad + \left( \left( 2 t_r y^2 + 4 t y_r y + t_r y_i^2 - 3 t_y r^2 + 2 t_i y_i y_r \right) q_{i1} + 2 \left( t y_r^2 + 2 t y_r y - y_i (2 t_i y + t y_i) \right) q_{i2} + 2 \left( t_i y_r^2 + 2 t_r y_i y_r - y_i (2 t_i y + t y_i) \right) q_{r2} \right) \\
&\quad + \left( - \left( 2 t_i y^2 + 4 t y_i y - 3 t_i y_i^2 + t_i y_r^2 + 2 t_r y_i y_r \right) q_{i1} + 2 (t_r y y_i + t y_r y_i + t_i y y_r) q_{i2} + 2 \left( -t_r y^2 - 2 t y_r y + t_r y_i^2 + 2 t_i y_i y_r \right) q_{r2} \right) \\
&\quad + 4 \left( 2 t_i (t y + 2 t_r y_r) + t_i \left( y^2 - 3 y_i^2 + 2 y_r^2 \right) \right) q_{i2}^2 \\
&\quad + 3 \left( t \left( 2 y^2 - y_i^2 - y_r^2 \right) - 2 y (t_i y_i + t y_r) \right) q_{r1} q_{i2} + \left( t_i y_r^2 + 2 t r y_i y r - y (t i y + 2 t y i) \right) q_{r1}^2 q_{r2} \right) \\
\end{align*}
\]


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