Bounding CPT and Lorentz symmetry violations through ultra-high-energy cosmic rays

Robertus Potting
University of the Algarve, Faro, Portugal
and CENTRA, IST, Lisbon, Portugal

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Introduction

Lorentz symmetry is a fundamental ingredient of both quantum field theory and General Relativity.

In the last two decades, there has been growing interest in the possibility that Lorentz symmetry may not be exact.

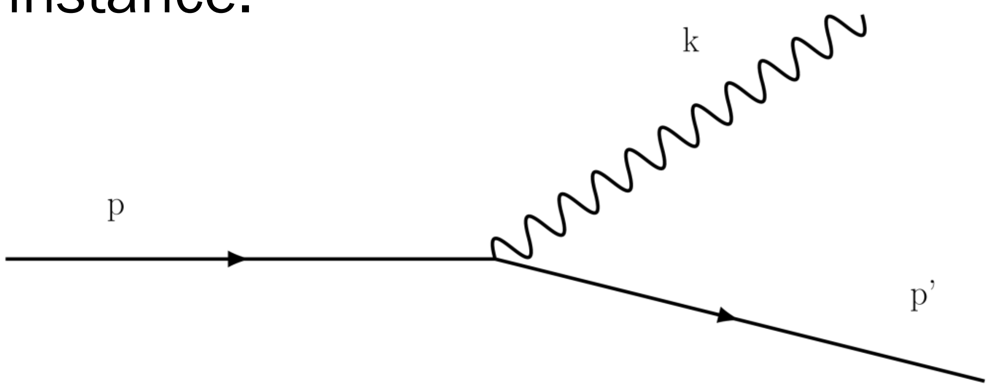
Some candidate theories of quantum gravity involve Lorentz Invariance Violation (LIV) as a possible effect, such as

- string field theory Kostelecky, RP '91
- loop quantum gravity Ashtekar '86
- space-time foam Wheeler '55
- Horava-Lifshitz gravity Horava '09

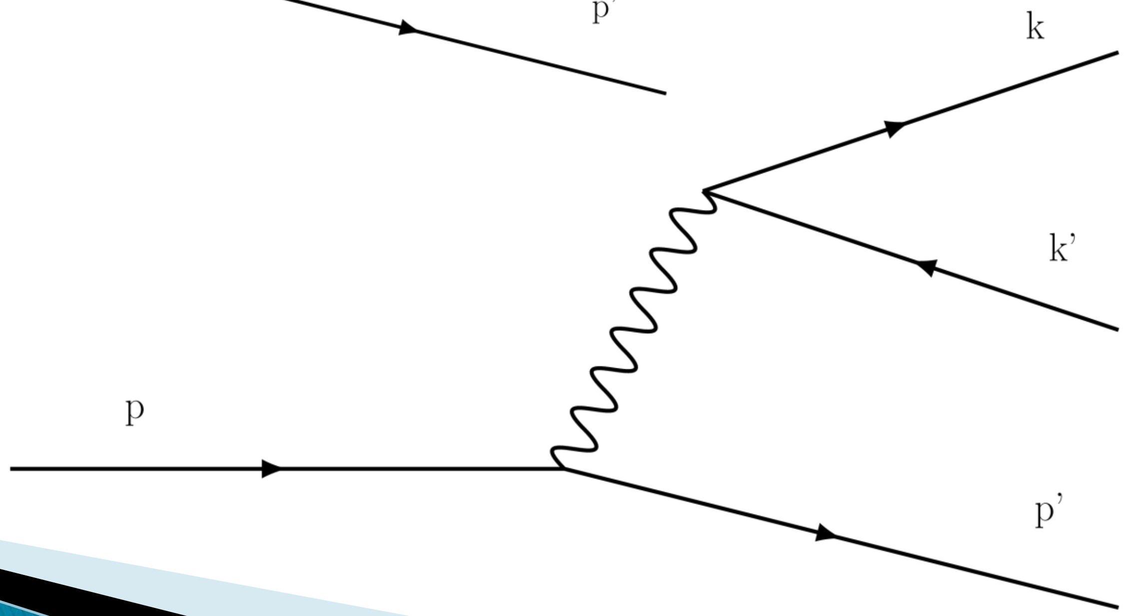
Combining energy-momentum conservation and Lorentz invariance forbids emission processes of the type:

fermion → (same mass) fermion + other particles

For instance:



or



However, in the presence of LIV such processes may become allowed!

Assume:

- LIV very suppressed at low energies
- LIV in the kinetic term of the emitted particles

The emission processes could then occur if one of the momenta of the emitted particles become spacelike at high energy.

We have applied this to protons that have been observed in Ultra-High-Energy cosmic rays (UHECR).

Knowing that these are stable at these observed energies permits bounding LIV for any candidate emitted particles.

The Standard Model Extension

Colladay, Kostelecky ’97, ‘98

Effective Field Theory incorporating:

1. Standard Model of particle physics coupled to General Relativity;
2. Any scalar term formed by contracting operators for Lorentz violation with coefficients controlling size of the effects.
3. Possibly additional requirements like
   - gauge invariance,
   - locality,
   - stability,
   - renormalizability.

The SME includes, in principle, terms of any mass dimension (starting at dim 3).

Imposing power counting renormalizability limits one to terms of dimension ≤ 4. This is usually referred to as the minimal SME (mSME).

The mSME has a finite number of LV parameters, while the number of LV parameters in the full SME is in principle unlimited.

The SME leads not only to breaking of Lorentz symmetry, but also to that of CPT, for about half of its terms..

Example of particle with LIV:
Photon in Standard Model Extension (SME)

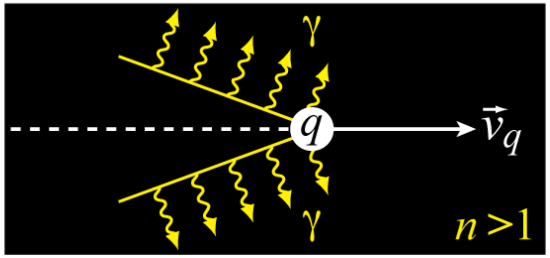
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}(k_F)_{\kappa\lambda\mu\nu}F^{\kappa\lambda}F^{\mu\nu} + \frac{1}{2}(k_{AF})^{\kappa}\epsilon_{\kappa\lambda\mu\nu}A^{\lambda}F^{\mu\nu}$$

CPT-violating k_{AF} term:

- Lagrangian can be consistently quantized
  Colladay, McDonald, Noordmans, RP '16
- Leads to *birefringence:* cosmological sources with known polarization permit searching for energy-dependent polarization changes either from distant sources or from CMB

  ⇨ $|(k_{AF})^{\mu}| \leq 10^{-43}$ GeV Carroll, Field'97

- Gives rise to vacuum Cherenkov radiation Lehnert, RP '04; Colladay, McDonald, RP '16



Case 1: Decay by emission of W bosons

Colladay, Noordmans, RP '17

mSME for SU(3) ×SU(2) ×U(1) gauge sector:

$$\mathcal{L}^{\mathrm{CPT-even}}_{\mathrm{gauge}} = -\frac{1}{2}(k_G)^{\kappa\lambda\mu\nu}\mathrm{Tr}(G_{\kappa\lambda}G_{\mu\nu}) - \frac{1}{2}(k_W)^{\kappa\lambda\mu\nu}\mathrm{Tr}(W_{\kappa\lambda}W_{\mu\nu}) - \frac{1}{2}(k_B)^{\kappa\lambda\mu\nu}B_{\kappa\lambda}B_{\mu\nu}$$

$$\begin{aligned}\mathcal{L}^{\mathrm{CPT-odd}}_{\mathrm{gauge}} = {} & (k_3)_\kappa\epsilon^{\kappa\lambda\mu\nu}\mathrm{Tr}(G_\lambda G_{\mu\nu} + \tfrac{2}{3}ig_3 G_\lambda G_\mu G_\nu) \\ & + (k_2)_\kappa\epsilon^{\kappa\lambda\mu\nu}\mathrm{Tr}(W_\lambda W_{\mu\nu} + \tfrac{2}{3}ig_3 W_\lambda W_\mu W_\nu) \\ & + (k_1)_\kappa\epsilon^{\kappa\lambda\mu\nu}B_\lambda B_{\mu\nu} + (k_0)_\kappa B^\kappa\end{aligned}$$

$$k^\mu_{AF} = 2\cos^2\theta_w\, k^\mu_1 + \sin^2\theta_w\, k^\mu_2$$

G_μ , W_μ, B_μ : SU(3), SU(2), U(1) gauge fields

$G_{\mu\nu}$, $W_{\mu\nu}$, $B_{\mu\nu}$: field strengths

W boson LV lagrangian:

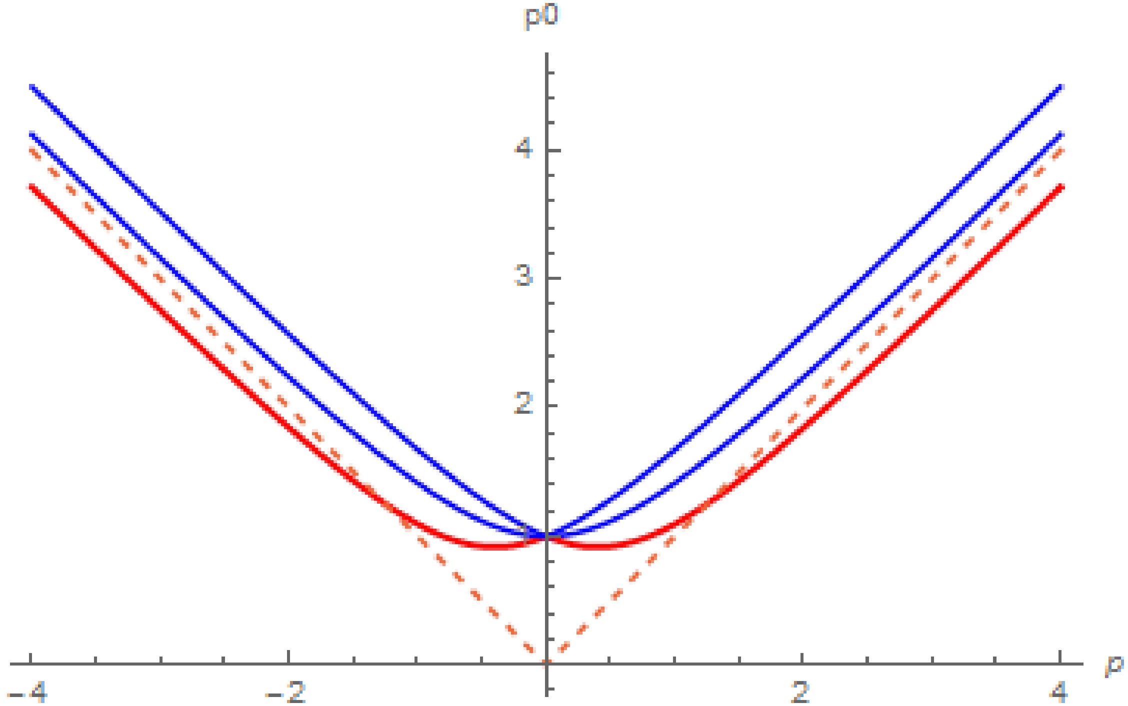
$$\mathcal{L}_W = -\frac{1}{4}W_{\mu\nu}^{+}W_{\mu\nu}^{-} - \frac{1}{2}M^2 W_{\mu}^{+}W_{\mu}^{-} + \frac{1}{2}(k_2)_{\kappa}\epsilon^{\kappa\lambda\mu\nu}(W_{\lambda}^{+}W_{\mu\nu}^{-} + W_{\lambda}^{-}W_{\mu\nu}^{+})$$

3 polarization modes have different dispersion relations:

$$\Lambda_0(p) = 0, \quad \Lambda_+(p) = 0, \quad \Lambda_-(p) = 0$$

$$\Lambda_0(p) = p^2 + M^2$$

$$\Lambda_{\pm}(p) = p^2 + M^2 \pm 2\sqrt{(p\cdot k)^2 - p^2 k^2}$$



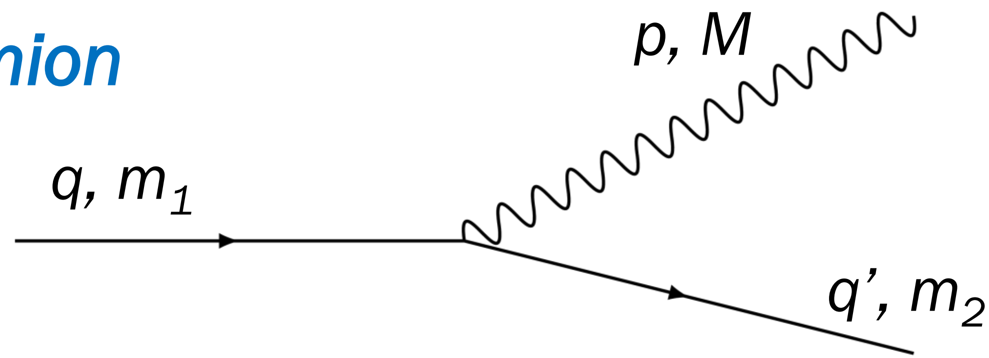
$$k^{\mu} = (k^0; \vec{0})$$

Λ_+ mode has spacelike four-momentum for sufficiently large energies.

- Decay by emission of on-shell W boson becomes possible for particles (electron, proton, …) with suitable momentum and regular dispersion relation.

Analysis of decay process

1. Elementary fermion



Differential decay rate:

$$d\Gamma = \frac{1}{2q^0}\frac{d^3p}{(2\pi)^3}\frac{1}{\Lambda'_+(p)}\frac{d^3q'}{(2\pi)^3}\frac{1}{2q'^0}\left(\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2\right)(2\pi)^4\delta^4(q-p-q')$$

$\Lambda'_+(p) = \partial\Lambda_+(p)/\partial p^0 \quad \longrightarrow \quad$ observer Lorentz-covariant measure

In this talk we will consider the case $m_1 = m_2$

Note that this process is prohibited by energy-momentum conservation in absence of Lorentz violation.

Transition amplitude:

$$\mathcal{M} = \frac{ig}{2\sqrt{2}}\bar{u}(q')\gamma^{\mu}(1-\gamma^5)u(q)e_{\mu}^{(+)*}(p)$$

Emission rate:

- non-zero only if W-boson momentum is spacelike

$$\Longrightarrow |\vec{p}| > \frac{M^2}{2\kappa} \qquad (\kappa \sim |k^{\mu}|)$$

- incoming momentum $\vec{q}$ must be larger than a threshold value: $|\vec{q}| > |\vec{q}|_{th}$

$$|\vec{q}|_{th} = \frac{M(M+2m)}{2\kappa}$$

- W bosons are emitted inside a small forward cone

Formula for total decay rate:

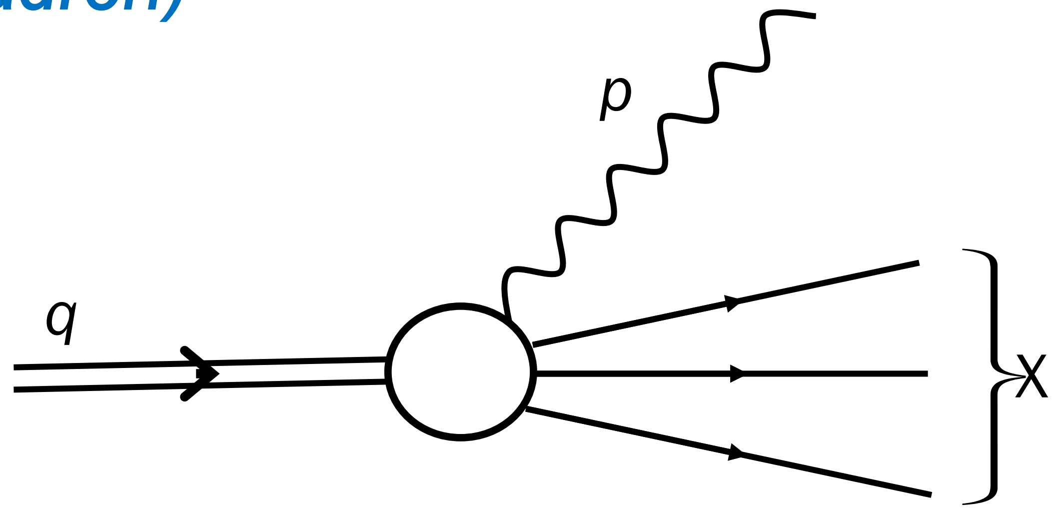
$$\Gamma = \frac{g^2\kappa}{16\pi}\theta(a-1)\,G(a)$$

$$a = \frac{|\vec{q}|}{|\vec{q}|_{th}}; \qquad G(a) = \ln a - 1 + \frac{1}{a} + \mathcal{O}\left(\frac{m}{M}, \frac{\kappa^2}{M^2}\right)$$

Example:

If $\kappa = \mathcal{O}(10^{-7}$ GeV), typical decay time is of order 10^{-15} s.

2. Proton (hadron)



Proton disintegrates upon emission of W boson by one of the quarks

$$\Gamma = \frac{1}{2q^0}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{4\pi}{\Lambda'_+(p)}e_\mu^{(+)}(p)e_\nu^{(+)*}(p)W^{\mu\nu}$$

Use parton model to calculate hadronic part $W^{\mu\nu}$.

Final result for decay rate:

$$\Gamma = \frac{g^2\kappa}{64\pi}\sum_q \int_0^1 \left(f_q(x) + \bar{f}_q(x)\right) G_q(ax)\theta(ax-1)dx$$

$f_q(x), \bar{f}_q(x)$: parton distribution functions

$$G_q(ax) = \ln(ax) - 1 + \frac{1}{ax} + \mathcal{O}\left(\frac{m_q}{M}, \frac{\kappa^2}{M^2}\right)$$

- Proton will decay whenever its momentum exceeds the threshold value $|\vec{q}|_{th} = \frac{M(M+2m)}{2\kappa}$
- We know that UHE cosmic ray protons with energies above 57 EeV $\equiv |\vec{q}|_{obs}$ have been observed
- Taking this as a lower bound for $|\vec{q}|_{th}$, it follows

$$\kappa < \frac{M(M+2m)}{|\vec{q}|_{obs}} \approx 1.5 \times 10^{-7}\text{GeV} = \kappa_0$$

- Conservative estimate for mean free path of protons:

$$L \simeq ct_p \sim (hc/\kappa_0) \times 10^{15} \approx 10^3 \text{ km}$$

- Evidently, L is much below the distance to UHECR sources!

We conclude the following bound:

$$\left|k_2^\mu\right| < \kappa_0$$

Using also the extremely tight bound on k_{AF}^μ, one deduces the following bounds on k_1^μ and k_2^μ:

$$\left|k_1^\mu\right| < 1.7 \times 10^{-8} \text{ GeV}$$

$$\left|k_2^\mu\right| < 1.1 \times 10^{-7} \text{ GeV}$$

Case 2: Decay by emission of pair tau–antitau

Escobar, Noordmans, RP ’18

CPT and LV Lagrangian for tau lepton from mSME:

$$\mathcal{L}_\tau = \bar{\psi}\left(i\gamma^\mu(D_\mu + \gamma^5 b_\mu\right) - m_\tau)\psi$$

Careful nontrivial analysis yields spin eigenstates of tau and antitau.

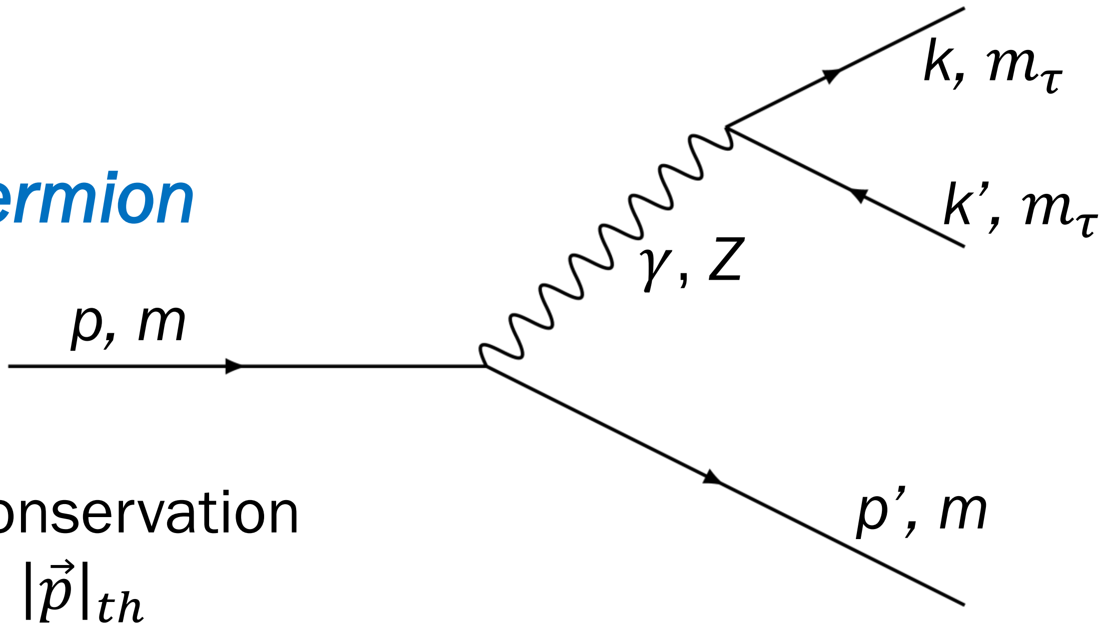
Different dispersion relations are given by the relations:

$$\Lambda_+(\mathrm{p}) = 0, \quad \Lambda_-(\mathrm{p}) = 0$$

$$\Lambda_\pm(p) = p^2 - m_\tau^2 - b^2 \pm 2\sqrt{(p \cdot b)^2 - p^2 b^2}$$

Analogous to dispersion relation of W boson!

Analysis of decay process



1. Elementary fermion

Energy-momentum conservation
only possible if $|\vec{p}| > |\vec{p}|_{th}$

$$|\vec{p}|_{th} = \frac{m_\tau(m_\tau+m)}{\xi_{b,p}} \; ; \qquad \xi_{b,p} = \left|b_0 - |\vec{b}|\cos\theta_{bp}\right|$$

At threshold: $\vec{k} = \overrightarrow{k'}$, $\overrightarrow{p'}$ and $\vec{p}$ are colinear

$$\left|\vec{k}\right| = \left|\vec{k}'\right| = \frac{m_\tau}{m+2m_\tau}|\vec{p}|_{th}, \qquad |\vec{p}'| = \frac{m}{m+2m_\tau}|\vec{p}|_{th}$$

Differential decay rate:

$$d\Gamma = \frac{1}{2p^0}\frac{d^3k}{(2\pi)^3}\frac{1}{\Lambda'_+(k)}\frac{d^3k'}{(2\pi)^3}\frac{1}{\Lambda'_+(k')}\frac{8\pi e^4}{q^4}L_\tau^{\mu\nu}W_{\mu\nu}$$

$\Lambda'_+(p) = \partial\Lambda_+(p)/\partial p^0 \;\rightarrow\;$ observer Lorentz-covariant measure

$$L_\tau^{\mu\nu} = \mathrm{Tr}[\bar{u}_\tau(k')\gamma^\mu\, v_\tau(k)\bar{v}_\tau(k)\gamma^\nu u_\tau(k')]$$

$$W_{\mu\nu} = \frac{1}{8\pi}\frac{d^3p'}{(2\pi)^3}\frac{1}{2p'^0}L^f_{\mu\nu}(2\pi)^4\delta^4(p-p'-k-k')$$

$$L^f_{\mu\nu} = \frac{1}{2}\sum_{\text{spins}}\mathrm{Tr}\left[\bar{u}(p')\gamma_\mu u(p)\bar{u}(p)\gamma_\nu u(p')\right]$$

Note that the τ spinors are modified by Lorentz violation!

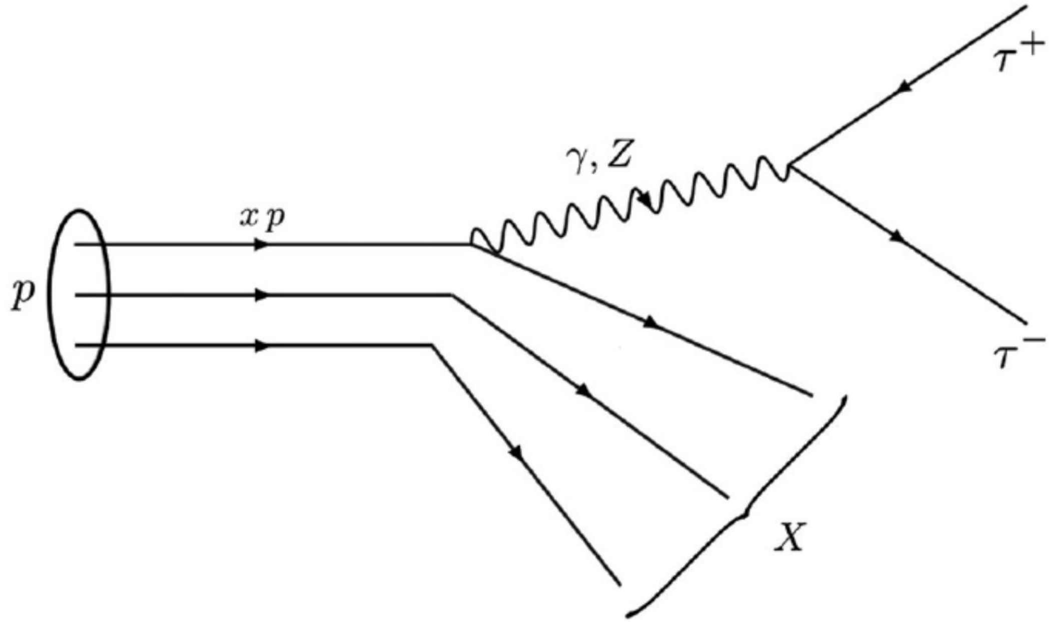
Can evaluate decay rate in limiting cases:

$$\Gamma = \frac{e^4 |\xi_{b,p}|}{2\pi^2} G(a)\theta(a-1)$$

$$G(a) \approx \begin{cases} \dfrac{2m_\tau^2}{\sqrt{m(m+m_\tau)^3}}(a-1)^2 & \text{if } \ a-1 \ll 1; \\ A_1 + A_2 \ln a + \cdots & \text{if } \ a \gg 1; \end{cases}$$

A_1 and A_2 are constants of order 1 that depend on m and m_τ

2. Proton



We have to adapt the hadronic part $W_{\mu\nu}$:

$$W_{\mu\nu} = \frac{1}{8\pi} \sum_{\sigma, X} \langle \mathrm{p}(p,\sigma)|J^{\nu}(-q)|X\rangle\langle X|J^{\mu}(q)|\mathrm{p}(p,\sigma)\rangle$$

It can be shown that for ultra-high momenta $|\vec{p}| \gg |\vec{p}|_{th}$, the rate formula for the elementar fermion applies.
The momentum transfer in the proton's rest-frame is tiny.

For small $|\vec{p}|$, can use parton model calculation, treating decay by taking integral over elementary quarks with momentum fraction *x*.
The momentum transfer in the protón's rest frame is large, leading to desintegration of the protón.

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- We know that UHE cosmic ray protons with energies above 57 EeV $\equiv |\vec{p}|_{obs}$ have been observed
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$$\xi_{b,p} \approx |b^\mu| < \frac{m_\tau(m_\tau + m)}{|\vec{p}|_{obs}} \approx 8.5 \times 10^{-11} \text{GeV} = \xi_0$$

- Conservative estimate for mean free path of protons:

$$L \simeq ct_p \sim 3 \times 10^9 \text{ km}$$

- Evidently, L is much below the distance to UHECR sources!

Conclusions

- In the presence of LIV certain, normally forbidden, decay processes can become allowed for sufficiently large momenta.
- Standard Model Extension makes it possible to do precise analysis of the associated processes. Threshold momenta for decay are typically inversely proportional to the associated SME parameters.
- Analysis of protons in UHECR's allows for deriving bounds on the SME parameters:
  - decay into putative *W* bosons: new bound on SME parameters k_1 and k_2.
  - decay into putative tau-antitau pair: new bound on SME parameters b^μ.

Thank you for your attention!

The Standard Model Extension

Colladay, Kostelecky ’97, ‘98

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1. Standard Model of particle physics coupled to General Relativity;
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The mSME has a finite number of LV parameters, while the number of LV parameters in the full SME is in principle unlimited.

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Construction of the mSME

1. SU(3)*SU(2)*U(1) Standard Model

Leptons: $L_A = \begin{pmatrix} \nu_A \\ l_A \end{pmatrix}_L$, $R_A = (l_A)_R$

Quarks: $Q_A = \begin{pmatrix} u_A \\ d_A \end{pmatrix}_L$, $U_A = (u_A)_R$, $D_A = (d_A)_R$

$$l_A = (e, \mu, \tau), \quad \nu_A = (\nu_e, \nu_\mu, \nu_\tau), \quad u_A = (u, c, t), \quad d_A = (d, s, b).$$

Gauge fields: G_μ, W_μ, B_μ

Higgs doublet: ϕ

Gauge couplings: g_3, g, g'

Yukawa couplings: G_L, G_U, G_D

$$\mathcal{L}_{\text{lepton}} = \frac{1}{2} i \bar{L}_A \gamma^\mu \overleftrightarrow{D}_\mu L_A + \frac{1}{2} i \bar{R}_A \gamma^\mu \overleftrightarrow{D}_\mu R_A$$

$$\mathcal{L}_{\text{quark}} = \frac{1}{2} i \bar{Q}_A \gamma^\mu \overleftrightarrow{D}_\mu Q_A + \frac{1}{2} i \bar{U}_A \gamma^\mu \overleftrightarrow{D}_\mu U_A + \frac{1}{2} i \bar{D}_A \gamma^\mu \overleftrightarrow{D}_\mu U_A$$

$$\mathcal{L}_{\text{Yukawa}} = -(G_L)_{AB} \bar{L}_A \phi R_B - (G_U)_{AB} \bar{Q}_A \phi^c U_B - (G_D)_{AB} \bar{Q}_A \phi D_B$$

$$\mathcal{L}_{\text{Higgs}} = (D_\mu \phi)^\dagger D^\mu \phi + \mu^2 \phi^\dagger \phi - \frac{\lambda}{6} (\phi^\dagger \phi)^2$$

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2} \mathrm{Tr}(G_{\mu\nu} G^{\mu\nu}) - \frac{1}{2} \mathrm{Tr}(W_{\mu\nu} W^{\mu\nu}) - \frac{1}{4} (B_{\mu\nu} B^{\mu\nu})$$

2. mSME Lagrangian

a. Fermions

$$\mathcal{L}_{\text{lepton}}^{\text{CPT-even}} = \frac{1}{2} i (c_L)_{\mu\nu\,AB} \bar{L}_A \gamma^\mu \overleftrightarrow{D}^\nu L_B + \frac{1}{2} i (c_R)_{\mu\nu\,AB} \bar{R}_A \gamma^\mu \overleftrightarrow{D}^\nu R_B$$

$$\mathcal{L}_{\text{lepton}}^{\text{CPT-odd}} = -(a_L)_{\mu\,AB} \bar{L}_A \gamma^\mu L_B - (a_R)_{\mu\,AB} \bar{R}_A \gamma^\mu R_B$$

$$\begin{aligned}\mathcal{L}_{\text{quark}}^{\text{CPT-even}} &= \frac{1}{2} i (c_Q)_{\mu\nu\,AB} \bar{Q}_A \gamma^\mu \overleftrightarrow{D}^\nu Q_B + \frac{1}{2} i (c_U)_{\mu\nu\,AB} \bar{U}_A \gamma^\mu \overleftrightarrow{D}^\nu U_B \\ &\quad + \frac{1}{2} i (c_D)_{\mu\nu\,AB} \bar{D}_A \gamma^\mu \overleftrightarrow{D}^\nu D_B\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{\text{quark}}^{\text{CPT-odd}} &= -(a_Q)_{\mu\,AB} \bar{L}_A \gamma^\mu Q_B - (a_R)_{\mu\,AB} \bar{U}_A \gamma^\mu U_B \\ &\quad - (a_D)_{\mu\,AB} \bar{D}_A \gamma^\mu D_B\end{aligned}$$

b. Higgs sector

$$\mathcal{L}^{\mathrm{CPT-even}}_{\mathrm{Higgs}} = \frac{1}{2}(k_{\phi\phi})^{\mu\nu}(D_\mu\phi)^\dagger D_\nu\phi - \frac{1}{2}(k_{\phi B})^{\mu\nu}\phi^\dagger\phi B_{\mu\nu} - \frac{1}{2}(k_{\phi W})^{\mu\nu}\phi^\dagger W_{\mu\nu}\phi$$

$$\mathcal{L}^{\mathrm{CPT-even}}_{\mathrm{Yukawa}} = -\frac{1}{2}(H_L)_{\mu\nu\,AB}\bar{L}_A\phi\sigma^{\mu\nu}R_B - \frac{1}{2}(H_U)_{\mu\nu\,AB}\bar{Q}_A\phi\sigma^{\mu\nu}U_B - \frac{1}{2}(H_D)_{\mu\nu\,AB}\bar{Q}_A\phi\sigma^{\mu\nu}D_B$$

c. Gauge sector

$$\mathcal{L}^{\mathrm{CPT-even}}_{\mathrm{gauge}} = -\frac{1}{2}(k_G)^{\kappa\lambda\mu\nu}\mathrm{Tr}(G_{\kappa\lambda}G_{\mu\nu}) - \frac{1}{2}(k_W)^{\kappa\lambda\mu\nu}\mathrm{Tr}(W_{\kappa\lambda}W_{\mu\nu}) - \frac{1}{2}(k_B)^{\kappa\lambda\mu\nu}B_{\kappa\lambda}B_{\mu\nu}$$

$$\mathcal{L}^{\mathrm{CPT-odd}}_{\mathrm{gauge}} = (k_3)_\kappa\epsilon^{\kappa\lambda\mu\nu}\mathrm{Tr}(G_\lambda G_{\mu\nu} + \tfrac{2}{3}ig_3 G_\lambda G_\mu G_\nu) + (k_2)_\kappa\epsilon^{\kappa\lambda\mu\nu}\mathrm{Tr}(W_\lambda W_{\mu\nu} + \tfrac{2}{3}ig_3 W_\lambda W_\mu W_\nu) + (k_1)_\kappa\epsilon^{\kappa\lambda\mu\nu}B_\lambda B_{\mu\nu} + (k_0)_\kappa B^\kappa$$

3. *Inclusion of Gravity*

Example: lepton sector

Standard model Lagrangian density coupled to gravity:

$$\mathcal{L}_{\text{lepton}} = \tfrac{1}{2} i e\, e^{\mu}{}_{a} \bar{L}_A \gamma^a \overleftrightarrow{D}_{\mu} L_A + \tfrac{1}{2} i e\, e^{\mu}{}_{a} \bar{R}_A \gamma^a \overleftrightarrow{D}_{\mu} R_A$$

$e^{\mu}{}_{a}$: vierbein, used to convert local Lorentz indices to spacetime indices: $b^{\mu} = e^{\mu}{}_{a} b^{a}$

Flat-space LIV sectors can be coupled to gravity using vierbein, for example:

$$\mathcal{L}_{\text{lepton}}^{\text{CPT-even}} = -\tfrac{1}{2} i (c_L)_{\mu\nu\, AB}\, e e^{\mu}{}_{a} \bar{L}_A \gamma^a \overleftrightarrow{D}^{\nu} L_B - \tfrac{1}{2} i (c_R)_{\mu\nu\, AB}\, e e^{\mu}{}_{a} \bar{R}_A \gamma^a \overleftrightarrow{D}^{\nu} R_B$$

Pure gravity sector:

LIV Lagrangian terms are built of the vierbein, spin connection and derivatives. They can be converted to curvature and torsion. Minimal sector:

$$\mathcal{L}^{\mathrm{LV}}_{e,\omega} = e(k_T)^{\lambda\mu\nu} T_{\lambda\mu\nu} + e(k_R)^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} + e(k_{TT})^{\alpha\beta\gamma\lambda\mu\nu} T_{\alpha\beta\gamma} T_{\lambda\mu\nu} + e(k_{DT})^{\kappa\lambda\mu\nu} D_\kappa T_{\lambda\mu\nu}$$

Riemannian limit of minimal SME gravity sector:

$$S_{e,\omega,\Lambda} = \frac{1}{2\kappa} \int d^4x\, e\left[(1-u)R - 2\Lambda + s^{\mu\nu} R_{\mu\nu} + t^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu}\right]$$