PT AND CPT SYMMETRY AS A DYNAMICS

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P. D. Mannheim, *Extension of the CPT Theorem to non-Hermitian Hamiltonians and Unstable States*, Phys. Lett. B 753, 286 (2016). (arXiv:1512.03736 [quant-ph]).

P. D. Mannheim, *Antilinearity Rather than Hermiticity as a Guiding Principle for Quantum Theory*, J. Phys. A 51, 315302 (2018). (arXiv:1512.04915 [hep-th]).

1 OUTLINE

(1). Interest in discrete symmetries in particle physics is mainly in determining the degree to which they hold.

(2). With the advent of the non-Hermitian, antilinear, discrete PT-symmetry program of Bender and collaborators it became apparent that quantum theory is richer than the standard Dirac Hermitian approach.

(3). Antilinear symmetry can be used as a dynamics that constrains what is allowed in physics. While interest has focused on PT itself, here we show that the fundamental antilinear symmetry is CPT.

(4). We show that if one imposes only two requirements, namely the time independence of inner products and invariance under the complex Lorentz group, it follows that the Hamiltonian must be CPT invariant. Since no Hermiticity requirement is imposed the CPT theorem is thus extended to non-Hermitian theories.

(5). Charge conjugation plays no role in non-relativistic physics where one is below the threshold for particle production. CPT then defaults to PT, to thus put the PT-symmetry program on a quite secure foundation.

2 ANTILINEARITY AND THE REALITY OF EIGENVALUES

Hermiticity of a Hamiltonian is only sufficient to yield real eigenvalues.

What is the necessary condition?

Hamiltonian must possess an antilinear symmetry.

What is the necessary and sufficient condition?

Hamiltonian must possess an antilinear symmetry, and the eigenstates of the Hamiltonian must be eigenstates of the antilinear operator.

For finite-dimensional systems that obey $[H, PT] = 0$, one can always construct a $\mathcal{C}$ operator that obeys $\mathcal{C}^2 = I$, $[H, \mathcal{C}] = 0$. Necessary and sufficient condition for real eigenvalues (Bender and Mannheim 2010) is that $[\mathcal{C}, PT] = 0$.

3 HOW ANTILINEAR SYMMETRY WORKS

Consider the eigenvector equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle = E|\psi(t)\rangle. \tag{1}$$

Replace the parameter t by $-t$ and then multiply by some general antilinear operator A:

$$i\frac{\partial}{\partial t}A|\psi(-t)\rangle = AHA^{-1}A|\psi(-t)\rangle = E^*A|\psi(-t)\rangle. \tag{2}$$

If H has an antilinear symmetry so that $AHA^{-1} = H$, then

$$HA|\psi(-t)\rangle = E^*A|\psi(-t)\rangle. \tag{3}$$

(1). (Wigner): Energies can be real and have eigenfunctions that obey $A|\psi(-t)\rangle = |\psi(t)\rangle$,

(2). or energies can appear in complex conjugate pairs that have conjugate eigenfunctions ($|\psi(t)\rangle \sim \exp(-iEt)$ and $A|\psi(-t)\rangle \sim \exp(-iE^*t)$).

As to the converse, suppose we are given that the energy eigenvalues are real or appear in complex conjugate pairs. In such a case not only would E be an eigenvalue but E^* would be too. Hence, we can set $HA|\psi(-t)\rangle = E^*A|\psi(-t)\rangle$ in (2), and obtain

$$(AHA^{-1} - H)A|\psi(-t)\rangle = 0. \tag{4}$$

Then if the eigenstates of H are complete, (4) must hold for every eigenstate, to yield $AHA^{-1} = H$ as an operator identity, with H thus having an antilinear symmetry.

4 A SIMPLE EXAMPLE

The matrix

$$M = \begin{pmatrix} 1+i & s \\ s & 1-i \end{pmatrix} \tag{5}$$

with real s is PT symmetric (set $P = \sigma_1$ and $T = K$ where K denotes complex conjugation).

Even though this M is not Hermitian, its eigenvalues are given by

$$E_{\pm} = 1 \pm (s^2 - 1)^{1/2}, \tag{6}$$

and both of these eigenvalues are real if s is greater than one.

Moreover, these eigenvalues come in complex conjugate pairs if s is less than one.

In addition, if $s = 1$ M is a non-diagonalizable Jordan-block Hamiltonian with only one eigenvector despite having two solutions to $|M - \lambda I| = 0$ (both with $\lambda = 1$), and cannot be diagonalized by a similarity transformation:

$$\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \begin{pmatrix} 1+i & 1 \\ 1 & 1-i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{7}$$

The Jordan-block situation is a case where the Hamiltonian is manifestly non-diagonalizable and thus manifestly non-Hermitian and yet all eigenvalues are real.

Three options for antilinear symmetry:

(1) Energies real and eigenvectors complete.

(2) Energies in complex pairs and eigenvectors complete.

(3) Energies real but eigenvectors incomplete.

Antilinearity richer than Hermiticity.

5 PROBABILITY CONSERVATION

Consider a right eigenstate of H in which H acts to the right as $i\partial_t|R(t)\rangle = H|R(t)\rangle$ with solution $|R(t)\rangle = \exp(-iHt)|R(0)\rangle$. The Dirac norm

$$\langle R(t)|R(t)\rangle = \langle R(0)|\exp(iH^\dagger t)\exp(-iHt)|R(0)\rangle \tag{8}$$

is not time independent if H is not Hermitian, and would not describe unitary time evolution. However, this only means that the Dirac norm is not unitary, not that no norm is unitary.

Since $i\partial_t|R(t)\rangle = H|R(t)\rangle$ only involves ket vectors, there is some freedom in choosing bra vectors. So let us introduce a more general scalar product $\langle R(t)|V|R(t)\rangle$ with some time-independent linear operator V. We find

$$i\frac{\partial}{\partial t}\langle R_j(t)|V|R_i(t)\rangle = \langle R_j(t)|(VH - H^\dagger V)|R_i(t)\rangle. \tag{9}$$

Thus if we set

$$VH - H^\dagger V = 0, \tag{10}$$

then scalar products will be time independent and probability is conserved.

For the converse we note if we are given that all V scalar products are time independent, then if the set of all $|R_i(t)\rangle$ is complete we would obtain $VH - H^\dagger V = 0$ as an operator identity. The condition $VH - H^\dagger V = 0$ is thus both **necessary and sufficient** for the time independence of the V scalar products $\langle R(t)|V|R(t)\rangle$.

Now if $VH - H^\dagger V = 0$, we can set $VH|\psi\rangle = EV|\psi\rangle = H^\dagger V|\psi\rangle$. Consequently H and $H^\dagger$ have the same set of eigenvalues, i.e. for every E there is an E^*. (When V is invertible, this also follows from $H^\dagger = VHV^{-1}$, an isospectral similarity transformation.) Energy eigenvalues are thus either real or in complex conjugate pairs.

Consequently, H must have an antilinear symmetry.

Consider

$$-i\frac{\partial}{\partial t}\langle R|V = \langle R|H^\dagger V = \langle R|VH. \tag{11}$$

Can identify $\langle L| = \langle R|V$ as left-eigenvector, and thus inner product is $\langle L|R\rangle$, and operator matrix elements are $\langle L|\hat{O}|R\rangle$.

6 THE TAKEAWAY

Antilinear symmetry of a Hamiltonian implies that energies are real or in complex pairs.

Energies real or in complex pairs implies that Hamiltonian has an antilinear symmetry.

REALITY OF EIGENVALUES IMPLIES ANTILINEARITY NOT HERMITICITY

Probability conservation implies that Hamiltonian has an antilinear symmetry.

Antilinear symmetry of a Hamiltonian implies that probability is conserved.

CONSERVATION OF PROBABILITY IMPLIES ANTILINEARITY NOT HERMITICITY

Since a Hamiltonian cannot have more eigenvectors than right and left ones, $\langle L|R\rangle = \langle R|V|R\rangle$ is most general inner product one can use.

One never needs to impose Hermiticity. One only needs to impose antilinear symmetry.

But is any particular antilinear symmetry to be preferred. Study implications of Lorentz invariance.

7 COMPLEX LORENTZ INVARIANCE

Lorentz transformations are of the form $\Lambda = e^{iw^{\mu\nu}M_{\mu\nu}}$ with six angles $w^{\mu\nu} = -w^{\nu\mu}$ and six Lorentz generators $M_{\mu\nu} = -M_{\nu\mu}$ that obey

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(-\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\rho\nu} + \eta_{\nu\sigma}M_{\rho\mu}). \quad (12)$$

Under a Lorentz transformation the line element transforms as

$$x^\alpha \eta_{\alpha\beta} x^\beta \rightarrow x^\alpha \widetilde{\Lambda} \eta_{\alpha\beta} \Lambda x^\beta, \quad (13)$$

(tilde denotes transpose), with $\widetilde{\Lambda} = e^{iw^{\mu\nu}\widetilde{M}_{\mu\nu}}$. Given the Lorentz algebra one has $e^{iw^{\mu\nu}\widetilde{M}_{\mu\nu}}\eta_{\alpha\beta} = \eta_{\alpha\beta}e^{-iw^{\mu\nu}M_{\mu\nu}}$ (i.e. Minkowski metric orthogonal), with the line element thus being invariant. While this analysis familiarly holds for real $w^{\mu\nu}$, since $w^{\mu\nu}$ plays no explicit role in it, the analysis equally holds if $w^{\mu\nu}$ is **complex**.

For a general spin zero Lagrangian where $w^{\mu\nu}M_{\mu\nu}$ acts as

$$w^{\mu\nu}(x_\mu p_\nu - x_\nu p_\mu) = 2w^{\mu\nu}x_\mu p_\nu.$$

Under an infinitesimal Lorentz transformation the action $I = \int d^4x L(x)$ transforms as

$$\delta I = 2w^{\mu\nu}\int d^4x x_\mu \partial_\nu L(x) = 2w^{\mu\nu}\int d^4x \partial_\nu [x_\mu L(x)], \qquad (14)$$

to thus be a total derivative and thus be left invariant. However the change will be a total derivative even if $w^{\mu\nu}$ is complex. So again we see that we have invariance under **complex** Lorentz transformations.

For Majorana spinors ψ under a Lorentz transformation we have

$$\widetilde{\psi}\gamma^0\psi \to \tilde{\psi}e^{iw^{\mu\nu}\widetilde{M}_{\mu\nu}}\gamma^0 e^{iw^{\mu\nu}M_{\mu\nu}}\psi = \tilde{\psi}\gamma^0 e^{-iw^{\mu\nu}M_{\mu\nu}}e^{iw^{\mu\nu}M_{\mu\nu}}\psi = \widetilde{\psi}\gamma^0\psi. \tag{15}$$

So once again we see that we have invariance under **complex** Lorentz transforms and not just under real ones.

For Dirac spinors written as a sum of two Majorana spinors $\psi(x) = \psi_1(x) + i\psi_2(x)$, we find that under $\hat{P}$, $\hat{T}$, and $\hat{C}\hat{P}\hat{T}$

$$\begin{aligned} &\hat{P}\psi(\vec{x},t)\hat{P}^{-1} = \gamma^0\psi(-\vec{x},t), \qquad \hat{T}\psi(\vec{x},t)\hat{T}^{-1} = \gamma^1\gamma^2\gamma^3\psi(\vec{x},-t), \\ &\hat{C}\hat{P}\hat{T}[\psi_1(x) + i\psi_2(x)]\hat{T}^{-1}\hat{P}^{-1}\hat{C}^{-1} = i\gamma^5[\psi_1(-x) - i\psi_2(-x)], \end{aligned} \tag{16}$$

The last of these relations is central to the derivation of the CPT theorem.

THE TAKEAWAY

Complex Lorentz invariance is just as natural as real Lorentz invariance.

8 RELATION OF PT AND CPT TO COMPLEX LORENTZ TRANSFORMATIONS

On coordinates PT implements $x^\mu \to -x^\mu$, and thus so does CPT since the coordinates are charge conjugation even. With a boost in the x_1-direction implementing $x_1' = x_1 \cosh\xi + t\sinh\xi$, $t' = t\cosh\xi + x_1 \sinh\xi$, with complex $\xi = i\pi$ we obtain

$$\begin{aligned} \Lambda^0{}_1(i\pi): &\quad x_1 \to -x_1, \quad t \to -t, \\ \Lambda^0{}_2(i\pi): &\quad x_2 \to -x_2, \quad t \to -t, \\ \Lambda^0{}_3(i\pi): &\quad x_3 \to -x_3, \quad t \to -t, \\ \pi\tau = \Lambda^0{}_3(i\pi)\Lambda^0{}_2(i\pi)\Lambda^0{}_1(i\pi): &\quad x^\mu \to -x^\mu. \end{aligned} \tag{17}$$

Complex $\pi\tau$ implements the linear part of a PT and CPT transformation on coordinates.

With $\Lambda^0{}_i(i\pi)$ implementing $e^{-i\pi\gamma^0\gamma_i/2} = -i\gamma^0\gamma_i$ for Dirac gamma matrices, on introducing

$$\hat{\pi}\hat{\tau} = \hat{\Lambda}^0{}_3(i\pi)\hat{\Lambda}^0{}_2(i\pi)\hat{\Lambda}^0{}_1(i\pi), \tag{18}$$

we obtain

$$\hat{\pi}\hat{\tau}\psi_1(x)\hat{\tau}^{-1}\hat{\pi}^{-1} = \gamma^5\psi_1(-x), \qquad \hat{\pi}\hat{\tau}\psi_2(x)\hat{\tau}^{-1}\hat{\pi}^{-1} = \gamma^5\psi_2(-x). \tag{19}$$

Thus up to an overall complex phase, quite remarkably we recognize this transformation as acting as none other than the **linear** part of a CPT transformation since $\hat{C}\hat{P}\hat{T}[\psi_1(x) + i\psi_2(x)]\hat{T}^{-1}\hat{P}^{-1}\hat{C}^{-1} = i\gamma^5[\psi_1(-x) - i\psi_2(-x)]$.

Thus CPT is naturally associated with the complex Lorentz group.

With the Lagrangian density $L(x)$ being spin zero, $\hat{\pi}\hat{\tau}$ effects $\hat{\pi}\hat{\tau}L(x)\hat{\tau}^{-1}\hat{\pi}^{-1} = L(-x)$ up to a phase. We will show below that the phase is one. Thus, with K denoting complex conjugation, when acting on a spin zero Lagrangian we can identify $\hat{C}\hat{P}\hat{T} = K\hat{\pi}\hat{\tau}$. On applying $\hat{\pi}\hat{\tau}$ we obtain

$$\begin{aligned} \hat{C}\hat{P}\hat{T}\int d^4x L(x)[\hat{C}\hat{P}\hat{T}]^{-1} &= K\hat{\pi}\hat{\tau}\int d^4x L(x)\hat{\tau}^{-1}\hat{\pi}^{-1}K \\ &= K\int d^4x L(-x)K = K\int d^4x L(x)K = \int d^4x L^*(x). \end{aligned} \quad (20)$$

Establishing the CPT theorem is thus reduced to showing that $L(x) = L^*(x)$.

	C	P	T	CP	CT	PT	CPT
$\bar{\psi}\psi$	+	+	+	+	+	+	+
$\bar{\psi}i\gamma^5\psi$	+	-	-	-	-	+	+
$\bar{\psi}\gamma^0\psi$	-	+	+	-	-	+	-
$\bar{\psi}\gamma^i\psi$	-	-	-	+	+	+	-
$\bar{\psi}\gamma^0\gamma^5\psi$	+	-	+	-	+	-	-
$\bar{\psi}\gamma^i\gamma^5\psi$	+	+	-	+	-	-	-
$\bar{\psi}i[\gamma^0,\gamma^i]\psi$	-	-	+	+	-	-	+
$\bar{\psi}i[\gamma^i,\gamma^j]\psi$	-	+	-	-	+	-	+
$\bar{\psi}[\gamma^0,\gamma^i]\gamma^5\psi$	-	+	-	-	+	-	+
$\bar{\psi}[\gamma^i,\gamma^j]\gamma^5\psi$	-	-	+	+	-	-	+

Table 1: C, P, and T assignments for fermion bilinears

CPT phase alternates with spin. All spin zero quantities have even CPT. Also all are **real** (Mannheim 2018). Also, because of Lorentz invariance, Lagrangians have to be spin zero. And as we have seen, the action

$$I = \int d^4x L(x)$$

is invariant under complex Lorentz invariance.

	C	P	T	CP	CT	PT	CPT
$\bar{\psi}\psi$	+	+	+	+	+	+	+
$\bar{\psi}i\gamma^5\psi$	+	-	-	-	-	+	+
$\bar{\psi}\psi\bar{\psi}\psi$	+	+	+	+	+	+	+
$\bar{\psi}\psi\bar{\psi}i\gamma^5\psi$	+	-	-	-	-	+	+
$\bar{\psi}i\gamma^5\psi\bar{\psi}i\gamma^5\psi$	+	+	+	+	+	+	+
$\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi$	+	+	+	+	+	+	+
$\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\gamma^5\psi$	-	-	+	+	-	-	+
$\bar{\psi}\gamma^\mu\gamma^5\psi\bar{\psi}\gamma_\mu\gamma^5\psi$	+	+	+	+	+	+	+
$\bar{\psi}i[\gamma^\mu,\gamma^\nu]\psi\bar{\psi}i[\gamma_\mu,\gamma_\nu]\psi$	+	+	+	+	+	+	+
$\bar{\psi}i[\gamma^\mu,\gamma^\nu]\psi\bar{\psi}[\gamma_\mu,\gamma_\nu]\gamma^5\psi$	+	-	-	-	-	+	+
$\bar{\psi}i[\gamma^\mu,\gamma^\nu]\gamma^5\psi\bar{\psi}i[\gamma_\mu,\gamma_\nu]\gamma^5\psi$	+	+	+	+	+	+	+

Table 2: C, P, and T assignments for fermion bilinears and quadrilinears that have spin zero

All spin zero combinations have CPT even and real.

10 PROOF OF THE CPT THEOREM

Since probability conservation requires an antilinear symmetry, we have

$$\begin{aligned} K\hat{\pi}\hat{\tau}\int d^4xL(x)\hat{\tau}^{-1}\hat{\pi}^{-1}K &= K\int d^4xL(-x)K = K\int d^4xL(x)K = \int d^4xL^*(x) \\ &= K\int d^4xL(x)K = \int d^4xL(x), \end{aligned} \tag{21}$$

where we have used K as the antilinear symmetry needed for probability conservation. Thus infer that all the numerical coefficients in $L(x)$ are real, that $L(x) = L^*(x)$, and that $\int d^4xL(x)$ is CPT invariant, **with the CPT theorem thus being extended to non-Hermitian Hamiltonians.**

11 SOME IMPLICATIONS

(1) In the complex conjugate energy case time-independent transitions occur between decaying and growing states. A decay such as $K^+ \to \pi^+\pi^0$ can thus occur if the Hamiltonian has an antilinear symmetry, even though it would be forbidden if the Hamiltonian is Hermitian. Then the CPT theorem in the antilinear case ensures that its rate is equal to that of $K^- \to \pi^-\pi^0$. We thus extend the CPT theorem to unstable states.

(2) In those cases in which charge conjugation is separately conserved (in non-relativistic quantum mechanics C plays no role since one is below the threshold for particle production) CPT reduces to PT, even if the Hamiltonian is not Hermitian. (Even for non-Hermitian Hamiltonians CPT plus C implies PT.) In such cases we recover the non-Hermitian PT program of Bender and collaborators, and thus put the PT symmetry program on a quite firm theoretical foundation.

(3). Can extend Goldstone therem and Higgs mechanism to non-Hermitian case with CPT symmetry. (Alexandre, Ellis, Millington and Seynaeve, 2018; Mannheim, 2018)

(4) The conformal gravity theory with action $I_{\rm W} = -\alpha_g \int d^4x(-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa}$ where $C^{\lambda\mu\nu\kappa}$ is the Weyl conformal tensor falls into the non-Hermitian, CPT symmetric category (Bender and Mannheim 2008), and is able to be ghost free and unitary at the quantum level because of it (the $\langle L|R\rangle$ norm is positive definite), to thus provide a fully consistent quantum theory of gravity without any of the string theory need for supersymmetry or extra spacetime dimensions. It has been shown (Mannheim) that conformal gravity solves the dark matter, dark energy and quantum gravity problems. If conformal gravity can replace Einstein gravity then **one of the four fundamental forces is a Bender PT-type theory**.

12 SUMMARY

WE NEVER NEED TO POSTULATE HERMITICITY – ONLY NEED ANTILINEARITY.

HAMILTONIANS THAT HAVE AN ANTILINEAR SYMMETRY CAN BE HERMITIAN AS WELL. HERMITIAN ONLY IF PATH INTEGRAL EXISTS WITH REAL MEASURE.

ANTILINEAR SYMMETRY FOLLOWS FROM THE CONSERVATION OF PROBABILITY AND COMPLEX LORENTZ INVARIANCE ALONE.

ANTILINEAR CPT SYMMETRY HAS PRIMACY OVER HERMITICITY, AND IT IS CPT NOT HERMITICITY THAT SHOULD BE TAKEN AS A GUIDING PRINCIPLE FOR QUANTUM THEORY.

(3) Our derivation of the CPT theorem leads to $L = L^*$ and thus to $H = H^*$. In contrast, in the standard derivation of the CPT theorem $H = H^\dagger$ is input. Here $H = H^*$ is output, with it being probability conservation plus complex Lorentz invariance that is input. Now in one of the standard derivations of the CPT theorem (see e.g. Weinberg Quantum Field Theory I) one notes that all spin zero multilinears are Hermitian. Then a Hermiticity assumption requires all numerical coefficients be real and the CPT theorem follows. Remarkably then, both types of derivation lead to the very same functional form for the action, with real numerical coefficients in each case. So how can we tell them apart.

(4) So consider as an example

$$I_{\rm S}=\int d^4x\frac{1}{2}[\partial_\mu\phi\partial^\mu\phi-m^2\phi^2] \tag{22}$$

with Hamiltonian

$$H=\int d^3x\frac{1}{2}[\dot{\phi}^2+\vec{\nabla}\phi\cdot\vec{\nabla}\phi+m^2\phi^2]. \tag{23}$$

Solutions to the wave equation obey

$$\phi(\vec{x},t)=\sum[a(\vec{k})\exp(-i\omega_k t+i\vec{k}\cdot\vec{x})+a^\dagger(\vec{k})\exp(+i\omega_k t-i\vec{k}\cdot\vec{x})],\qquad \omega_k^2=\vec{k}^2+m^2 \tag{24}$$

and the Hamiltonian is given by

$$H=\sum\frac{1}{2}[\vec{k}^2+m^2]^{1/2}[a^\dagger(\vec{k})a(\vec{k})+a(\vec{k})a^\dagger(\vec{k})]. \tag{25}$$

If $m^2>0$ all energies are real, and both H and $\phi(\vec{x},t)$ are Hermitian. However, if $m^2=-n^2<0$, now

$$\omega_k^2=\vec{k}^2-n^2, \tag{26}$$

the $k<n$ energies come in complex conjugate pairs and neither H nor $\phi(\vec{x},t)$ is Hermitian. Instead H is CPT symmetric, and $\phi(\vec{x},t)$ is CPT even. Despite this, the standard derivation of the CPT theorem would have identified $I_{\rm S}=\int d^4x[\partial_\mu\phi\partial^\mu\phi+n^2\phi^2]/2$ as being a Hermitian theory. **But it is not, and one cannot tell by inspection.** One needs to solve the theory and get the solutions first. Nonetheless, in both the $m^2>0$ and $m^2<0$ cases $\phi(\vec{x},t)$ is a CPT even field and H is CPT invariant (since m^2 is real), and is something that one can tell by inspection. Thus CPT symmetry is input, and H and $\phi(\vec{x},t)$ will only be Hermitian for certain values of parameters (reminiscent of our two-dimensional example where $E_\pm=1\pm(s^2-1)^{1/2}$).

Another example: PU oscillator with $\omega_1=\alpha+i\beta$, $\omega_2=\alpha-i\beta$. Energies come in complex conjugate pairs and yet its non-Hermitian Hamiltonian is given by the seemingly Hermitian

$$H_{\rm PU}=\frac{p_x^2}{2\gamma}+p_zx+\gamma(\alpha^2-\beta^2)x^2-\frac{\gamma}{2}(\alpha^2+\beta^2)z^2. \tag{27}$$

Hermiticity never needs to be postulated, with it being output in those cases in which it is found to occur. Probability conservation and complex Lorentz invariance entail CPT invariance not Hermiticity.

13 The Hermiticity Puzzle – Where Does Hermiticity Come From?

If we introduce a path integral

$$W(J) = \int D[\phi]e^{i[I_{\rm S}(\phi)+J\phi]} \tag{28}$$

everything is classical. Thus no reference to any Hilbert space and no a priori justification for taking the quantum Hamiltonian to be Hermitian, since that is a quantum statement. However, one can implement CPT on every classical path, and thus if path integral is CPT invariant, then associated quantum theory will be CPT invariant too, regardless of whether or not the quantum Hamiltonian might be Hermitian.

Making the path integral CPT invariant is actually non-trivial for gauge theories, since need a rule to know to use combination $i\partial_\mu - eA_\mu$ in path integral rather than the in a sense more natural, but not viable, purely real derivative $\partial_\mu - eA_\mu$. Even though in quantum theory it is $i\partial_\mu$ that is Hermitian rather than ∂_μ, the path integral does not know this. However, classically it is $i\partial_\mu$ that is CPT even, just like eA_μ. It is thus CPT symmetry that forces $i\partial_\mu - eA_\mu$ in the path integral.

Now $W(J)$ generates the c-number quantum theory Green's functions, but how do we know that we can associate them with the matrix element of a q-number operator of the form $\langle\Omega_R|T(\phi(x)\phi(0))|\Omega_R\rangle$ rather than with $\langle\Omega_L|T(\phi(x)\phi(0))|\Omega_R\rangle = \langle\Omega_R|VT(\phi(x)\phi(0))|\Omega_R\rangle$ instead. So how does Hermiticity come into physics.

Answer: Path integral always exists with some real or complex measure if it is CPT invariant (analog of antilinearity implies self-adjointness). The underlying quantum Hamiltonian is Hermitian if path integral exists with a real measure, in which case the left vacuum is the conjugate of the right vacuum and $V = I$. However, if, in analog to wave functions, the path integral only exists if we need to continue the measure into the complex plane, then the underlying quantum theory is of the antilinear Bender type.

Antilinear CPT symmetry thus has primacy over Hermiticity, and it is antilinearity, not Hermiticity, that should be taken as a guiding principle for quantum theory.

For the two-dimensional matrix M above for instance, we have

$$M = \begin{pmatrix} 1+i & s \\ s & 1-i \end{pmatrix}, \qquad V = \frac{1}{(s^2-1)^{1/2}} \begin{pmatrix} s & -i \\ i & s \end{pmatrix}, \qquad VMV^{-1} = M^\dagger. \text{ V singular when s = 1.} \tag{29}$$

To reinforce the point we note that if $|R_i(t)\rangle$ is a right-eigenstate of H with energy eigenvalue $E_i = E_i^R + iE_i^I$, in general we can write

$$\langle R_j(t)|V|R_i(t)\rangle = \langle R_j(0)|V|R_i(0)\rangle e^{-i(E_i^R+iE_i^I)t+i(E_j^R-iE_j^I)t}. \tag{30}$$

Since V has been chosen so that the $\langle R_j(t)|V|R_i(t)\rangle$ scalar products are time independent, the only allowed non-zero norms are those that obey

$$E_i^R = E_j^R, \qquad E_i^I = -E_j^I, \tag{31}$$

with all other V-based scalar products having to obey $\langle R_j(0)|V|R_i(0)\rangle = 0$. We recognize (31) as being precisely none other than the requirement that eigenvalues be real or appear in complex conjugate pairs, with H thus possessing an antilinear symmetry.

Ordinarily in discussing decays one only keeps modes $e^{-i(E_R+iE_I)t}$ with negative imaginary part E_I. However now we keep both decaying and growing modes, with probability being conserved since the only transitions allowed by (31) are those in which the decaying mode couples to its growing partner, so that as the population of the decaying mode decreases, the population of the growing mode increases accordingly. Also with $U = e^{-iHt}$ obeying $U^{-1} = e^{iHt} = V^{-1}e^{iH^\dagger t}V = V^{-1}U^\dagger V$ unitarity is generalized to the non-Hermitian case.

With $VH - H^\dagger V = 0$ and $i\partial_t|R\rangle = H|R\rangle$, we obtain $-i\partial_t\langle R|V = \langle R|H^\dagger V = \langle R|VH$, and can thus identify left-eigenvectors $\langle L| = \langle R|V$, and can set $\langle R(t)|V|R(t)\rangle = \langle L(t)|R(t)\rangle = \langle L(0)|e^{iHt}e^{-iHt}|R(0)\rangle = \langle L(0)|R(0)\rangle$. Thus in the non-Hermitian case we should use the left-right norm, with this being the most general one possible if probability is to be conserved.

PROBABILITY CONSERVATION IMPLIES ANTILINEARITY NOT HERMITICITY

14 What do we Mean by Hermitian, and When is it Different from Self-adjoint.

In a given basis $H = H^\dagger$ means $H_{ij} = H^*_{ji}$. Apply a similarity transformation. $H' = SHS^{-1}$. Get

$$(H')^\dagger = (S^{-1})^\dagger H^\dagger S^\dagger = (S^{-1})^\dagger H S^\dagger. \tag{32}$$

Only equals H' if

$$(S^{-1})^\dagger H S^\dagger = SHS^{-1}, \tag{33}$$

i.e. if

$$S^{-1}(S^{-1})^\dagger H S^\dagger S = H. \tag{34}$$

Not obeyed in general if S is not unitary, i.e. if $S^\dagger \neq S^{-1}$. Arbitrary S thus transforms to a non-orthonormal skew basis, with $H_{ij} = H^*_{ji}$ being a nonlinear relation that only holds in certain bases. Thus what we mean by Hermitian is that we can find a basis in which $H_{ij} = H^*_{ji}$. **Basis independent definition: Hermitian means all eigenvalues real and eigenfunctions complete.**

In contrast, a commutation relation is preserved under a similarity transformation (even with antilinear operators), with $[A', B'] = [A, B]$. Antilinear symmetry is thus basis independent.

For a second-order differential operator D in the form $D = -p(x)d^2/dx^2 - p'(x)d/dx + q(x)$ that acts on wave functions $\phi(x)$, $\psi(x)$, one can show (Green's theorem) that

$$\int_a^b dx[\phi^* D\psi - (\psi^* D\phi)^*] = \int_a^b dx[\phi^* D\psi - [D\phi]^*\psi] = \left[p\psi\phi^{*\prime} - p\phi\psi^{*\prime}\right]_a^b \tag{35}$$

Self-adjointness requires the vanishing of the surface term. Then get standard definition of Hermitian.

To define a commutator $[\hat{x},\hat{p}]=i\hbar$, we need to specify a basis on which it acts. Can set $\hat{p}=-i\hbar d/dx$ only when acting on a good, i.e. normalizable, test function according to

$$\left[\hat{x},-i\hbar\frac{d}{dx}\right]\psi(x)=i\hbar\psi(x). \tag{36}$$

Thus for a harmonic oscillator $\hat{H}=\hat{p}^2+\hat{x}^2$ for instance we have the following two solutions:

$$\left[-\frac{d^2}{dx^2}+x^2\right]e^{-x^2/2}=e^{-x^2/2},\qquad \left[-\frac{d^2}{dx^2}+x^2\right]e^{+x^2/2}=-e^{+x^2/2} \tag{37}$$

Of them only the $e^{-x^2/2}$ wave function is normalizable (cf. vanishing of the surface term), with $\int dx\psi^*(x)\psi(x)$ being finite. And when acting on it we can indeed represent $\hat{p}$ as $\hat{p}=-i\hbar d/dx$. Here x is real and we are working in the coordinate basis in which $\hat{x}$ is Hermitian, has real eigenvalues x, and is diagonal in this basis.

But what of the $e^{+x^2/2}$ solution. It is not normalizable and we cannot represent $\hat{p}=-i\hbar d/dx$ when acting on it since cannot throw away the surface term in an integration by parts. However suppose we make x pure imaginary. Then $e^{+x^2/2}$ is normalizable on the imaginary axis. Thus we can take both $\hat{x}$ and $\hat{p}$ to be anti-Hermitian and represent $[\hat{x},\hat{p}]=i\hbar$ as $[-i\hat{x},i\hat{p}]=i\hbar$. This is equivalent to the similarity transformation $\hat{S}=\exp(-\pi\hat{p}\hat{x}/2)$ that effects

$$\hat{S}\hat{p}\hat{S}^{-1}=i\hat{p}=\hat{q},\quad \hat{S}\hat{x}\hat{S}^{-1}=-i\hat{x}=\hat{y}, \tag{38}$$

while preserving both the commutation relation $[\hat{x},\hat{p}]=[\hat{y},\hat{q}]=i$ and the eigenvalues of a Hamiltonian $\hat{H}(\hat{x},\hat{p})$ that is built out of $\hat{x}$ and $\hat{p}$. We thus have

$$\left[\hat{y},\hbar\frac{d}{dy}\right]\psi(y)=i\hbar\psi(y), \tag{39}$$

and now $e^{+x^2/2}=e^{-y^2/2}$ is a good test function. Thus $e^{-x^2/2}$ is a good test function when x is real, while $e^{+x^2/2}$ is a good test function when x is pure imaginary. When x is pure imaginary we can set

$$[\hat{p}^2+\hat{x}^2]e^{+x^2/2}=-[\hat{q}^2+\hat{y}^2]e^{-y^2/2}=\left[\frac{d^2}{dy^2}-y^2\right]e^{-y^2/2}=-e^{-y^2/2}. \tag{40}$$

Thus while the eigenvalues of $\hat{p}^2+\hat{x}^2$ would be positive if $\hat{p}$ and $\hat{x}$ are both Hermitian, the eigenvalues of $\hat{p}^2+\hat{x}^2$ would be negative if $\hat{p}$ and $\hat{x}$ are both anti-Hermitian.

Now we can always make a similarity transformation through any angle such as $\hat{S} = \exp(\theta\hat{p}\hat{x})$ that effects $\hat{S}\hat{p}\hat{S}^{-1} = \hat{p}\exp(-i\theta)$, $\hat{S}\hat{x}\hat{S}^{-1} = \hat{x}\exp(i\theta)$. Ordinarily this is not of any significance since we work with Hermitian operators that have normalizable wave functions on the real axis, and we have no need to go into the complex plane. But if the wave functions are not normalizable on the real axis, we may be able to continue into a so-called "Stokes wedge" in the complex plane where they then are normalizable, and cross over a "Stokes line" that divides the two regions ($\theta = \pi/4$ in the harmonic oscillator case). This is what happens with $\hat{H} = \hat{p}^2 + i\hat{x}^3$.

However, independent of whether or not a Hamiltonian might be Hermitian, if it has an antilinear symmetry it must be self-adjoint in some Stokes wedge in the complex plane. And in such wedges one must use the $\langle L|R\rangle$ norm.

Now this is true no matter whether energies are all real, whether some or all energies come in complex conjugate pairs, or whether the Hamiltonian is a non-diagonalizable Jordan-block Hamiltonian. These latter two cases represent Hamiltonians that are not Hermitian but are self-adjoint.

The art of the PT symmetry program is to find the appropriate Stokes wedges in the complex plane.

Theorem: Antilinear symmetry implies self-adjointness, while self-adjointness implies antilinearity.

Thus as with self-adjointness, Hermiticity is determined not by the form of the operators (i.e. not by inspection) but by the boundary conditions.

15 Conformal Gravity and Pais-Uhlenbeck Oscillator

Conformal gravity is a fourth-order derivative theory of gravity with action $I_{\rm W}=-\alpha_g\int d^4x(-g)^{1/2}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa}$, where $C_{\lambda\mu\nu\kappa}$ is the Weyl tensor. When coupled to Einstein gravity, and linearized around flat spacetime according to $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, one can find a gauge (the conformal gauge) in which the fluctuation equations are diagonal in the (μ,ν) indices and can be associated with the generic scalar field action I_S, propagator $D(k^2)$, and Hamiltonian $H=\int d^3xT_{00}$, where

$$
\begin{aligned}
I_S &= \frac{1}{2}\int d^4x\Big[\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi-(M_1^2+M_2^2)\partial_\mu\phi\partial^\mu\phi+M_1^2M_2^2\phi^2\Big],\\
D(k^2) &= \frac{1}{(k^2-M_1^2)(k^2-M_2^2)}=\frac{1}{M_1^2-M_2^2}\Big(\frac{1}{k^2-M_1^2}-\frac{1}{k^2-M_2^2}\Big),
\end{aligned}
\qquad (41)
$$

$$
\begin{aligned}
T_{00} &= \pi_0\dot{\phi}+\frac{1}{2}\pi_{00}^2+\frac{1}{2}(M_1^2+M_2^2)\dot{\phi}^2-\frac{1}{2}M_1^2M_2^2\phi^2-\frac{1}{2}\pi_{ij}\pi^{ij}+\frac{1}{2}(M_1^2+M_2^2)\phi_{,i}\phi^{,i},\\
\pi^\mu &= \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}-\partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu,\lambda}}\right),\qquad \pi_{\mu\lambda}=\frac{\partial\mathcal{L}}{\partial\phi_{,\mu,\lambda}}.
\end{aligned}
\qquad (42)
$$

The relative minus sign in $D(k^2)$ suggests that the theory contains states of negative norm. To find out whether or not this is the case, following Bender and Mannheim (2008) we explicitly construct the Hilbert space.

To see what is involved we note that on setting $\omega_1=(\bar{k}^2+M_1^2)^{1/2}$, $\omega_2=(\bar{k}^2+M_2^2)^{1/2}$ and dropping the spatial dependence, the action reduces to the quantum-mechanical Pais-Uhlenbeck oscillator model action

$$
I_{\rm PU}=\frac{\gamma}{2}\int dt\left[\ddot{z}^2-\left(\omega_1^2+\omega_2^2\right)\dot{z}^2+\omega_1^2\omega_2^2z^2\right], \qquad (43)
$$

and with $x=\dot{z}$, $[z,p_z]=i$, $[x,p_x]=i$, the Hamiltonian is given by (Mannheim and Davidson (2000))

$$
H_{\rm PU}=\frac{p_x^2}{2\gamma}+p_zx+\frac{\gamma}{2}\left(\omega_1^2+\omega_2^2\right)x^2-\frac{\gamma}{2}\omega_1^2\omega_2^2z^2. \qquad (44)
$$

When ω_1 and ω_2 are both taken to be real and positive, all the eigenvalues of $H_{\rm PU}$ are real. When $M_1^2 = M_2^2$ and $\omega_1 = \omega_2$, $H_{\rm PU}$, and thus the Hamiltonian associated with the pure conformal gravity $I_{\rm W}$, are Jordan-block (the partial fraction decomposition of $D(k^2)$ in (41) becomes undefined, just as in our simple example in (29)). When $\omega_1 = \alpha + i\beta$, $\omega_1 = \alpha - i\beta$, the energy eigenvalues appear in complex conjugate pairs. In all the three cases $H_{\rm PU}$ has an antilinear symmetry.

The wave function associated with $E = (\omega_1 + \omega_2)$ is of the form $(x = \dot{z})$

$$\psi(z,x) = \exp\Big[\frac{\gamma}{2}(\omega_1+\omega_2)\omega_1\omega_2 z^2 + i\gamma\omega_1\omega_2 zx - \frac{\gamma}{2}(\omega_1+\omega_2)x^2\Big]. \tag{45}$$

The wave function associated with $E = 2\omega$ where $\omega_1 = \omega_2 = \omega$ is of the form

$$\psi(z,x) = \exp\Big[\gamma\omega^3 z^2 + i\gamma\omega^2 zx - \gamma\omega x^2\Big]. \tag{46}$$

The wave function associated with $E = 2\alpha$ where $\omega_1 = \alpha + i\beta$, $\omega_1 = \alpha - i\beta$ $(\alpha > 0)$ is of the form

$$\psi(z,x) = \exp\Big[\gamma\alpha(\alpha^2+\beta^2)z^2 + i\gamma(\alpha^2+\beta^2)zx - \gamma\alpha x^2\Big]. \tag{47}$$

None of these wave functions is normalizable on the real z axis, but all are normalizable on the pure imaginary z axis. Thus all are self-adjoint in appropriate Stokes wedges that contain the imaginary z axis. We cannot use the Dirac norm in these wedges. Instead we must use the $\langle L|R\rangle$ norm, and it is found (Bender and Mannheim) to never be negative.

The propagator is not given by $\langle\Omega_R|T(\phi(x)\phi(0))|\Omega_R\rangle$ but by $\langle\Omega_L|T(\phi(x)\phi(0))|\Omega_R\rangle = \langle\Omega_R|VT(\phi(x)\phi(0))|\Omega_R\rangle$ instead. Thus one cannot identify a c-number propagator such $D(k^2) = 1/k^4$ with a quantum field theory matrix element until one first constructs the appropriate Hilbert space.

For conformal gravity, it is the V operator that generates the relative minus sign in $D(k^2)$ in (41) and not any Hilbert space negative norm structure. In consequence conformal gravity is ghost free and unitary. With it also being renormalizable (its coupling constant α_g being dimensionless), it provides a consistent quantum gravitational theory, one constructed in the four spacetime dimensions for which there is evidence.