Quantum Mechanics and Riemann Hypothesis

Dorje C. Brody

Department of Mathematics
University of Surrey

Discrete 2018: PT-symmetric Hamiltonians
Vienna, 26 – 30 November 2018

0. Hilbert-Pólya programme vs operator-valued zeta functions

In the Hilbert-Pólya programme one investigates operators whose eigenvalues correspond to the locations of the zeros of the zeta function.

For example, one might consider the properties of the operator

$$\tfrac{1}{2}(1 - \mathrm{i}\hat{h}_{\mathrm{BK}}),$$

where $\hat{h}_{\mathrm{BK}} = \hat{x}\hat{p} + \hat{p}\hat{x}$ denotes the Berry-Keating Hamiltonian.

The chances of proving the Riemann hypothesis via this route are slim...

As an alternative, we consider the operator

$$\zeta\left(\tfrac{1}{2}(1 - \mathrm{i}\hat{h}_{\mathrm{BK}})\right).$$

We investigate such an operator by letting it act on trigonometric functions.

We will find, for $x \in (0,\pi]$, that

$$\zeta\left(\tfrac{1}{2}(1-\mathrm{i}\hat{h}_{\rm BK})\right)\sin x = \frac{\sin x}{2(1-\cos x)}.$$

From this we can deduce that

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

for n a positive-odd integer, where $\{B_n\}$ are the Bernoulli numbers.

There are numerous similar relations of the kind.

Another example is

$$\zeta\left(\tfrac{1}{2}(3-\mathrm{i}\hat{h}_{\rm BK})\right)\sin x = \frac{\pi - x}{2}$$

for $x \in [0,\pi]$, from which we can deduce that $\zeta(s)$ vanishes for negative-even integers s without analytically continuing the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\tfrac{1}{2}\pi s\right)\Gamma(1-s)\zeta(1-s).$$

We can also deduce that $\zeta(0) = -\frac{1}{2}$, and that $\zeta(s)$ has a pole at $s=1$.

1. Hamiltonian for the Riemann zeros

Consider the 'Hamiltonian' operator

$$\hat{H} = \frac{\mathbb{1}}{\mathbb{1} - \mathrm{e}^{-\mathrm{i}\hat{p}}} \left(\hat{x}\hat{p} + \hat{p}\hat{x}\right) \left(\mathbb{1} - \mathrm{e}^{-\mathrm{i}\hat{p}}\right)$$

on $\mathbb{R}^+ = (0, \infty)$.

With the boundary condition

$$\psi(0) = 0,$$

required to ensure

$$\langle \varphi, \hat{H}\psi \rangle = \langle \hat{H}\varphi, \psi \rangle,$$

the eigenvalues $\{E_n\}$ of $\hat{H}$ satisfy the property that $\left\{\frac{1}{2}(1 - \mathrm{i}E_n)\right\}$ are the nontrivial zeros of the Riemann zeta function.

The eigenstates of $\hat{H}$ are given by the Hurwitz zeta function

$$\psi_n(x) = -\zeta(z_n, x + 1),$$

with eigenvalues

$$E_n = \mathrm{i}(2z_n - 1).$$

2. The shift operator and its inverse

Defining

$$\hat{\Delta} \equiv \mathbb{1} - \mathrm{e}^{-\mathrm{i}\hat{p}},$$

in units $\hbar = 1$ we have

$$\hat{p} = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$$

so that

$$\hat{\Delta} f(x) = f(x) - f(x-1).$$

As for $\hat{\Delta}^{-1}$ we have

$$\hat{\Delta}^{-1} = \frac{\mathbb{1}}{\mathbb{1} - \mathrm{e}^{-\mathrm{i}\hat{p}}} = \frac{\mathbb{1}}{\mathrm{i}\hat{p}} \frac{-\mathrm{i}\hat{p}}{\mathrm{e}^{-\mathrm{i}\hat{p}} - \mathbb{1}} = \frac{\mathbb{1}}{\mathrm{i}\hat{p}} \sum_{n=0}^{\infty} B_n \frac{(-\mathrm{i}\hat{p})^n}{n!}.$$

In particular, if $f(x) \to 0$ sufficiently fast, then we have

$$\hat{\Delta}^{-1} f(x) = -\sum_{k=1}^{\infty} f(k+x).$$

3. Uniqueness of $\hat{\Delta}\psi$

We multiply the eigenvalue equation

$$\hat{H}\psi = E\psi$$

on the left by $\hat{\Delta}$.

Recall that

$$\hat{H} = \hat{\Delta}^{-1}(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta}.$$

This gives a first-order linear differential equation

$$(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta}\psi = -\mathrm{i}\left(2x\frac{\mathrm{d}}{\mathrm{d}x} + 1\right)\hat{\Delta}\psi = E\,\hat{\Delta}\psi$$

for the function $\hat{\Delta}\psi$, whose solution is unique and is given by

$$\hat{\Delta}\psi = x^{-z}$$

up to a multiplicative constant.

Therefore,

$$\psi(x) = \hat{\Delta}^{-1}x^{-z}.$$

4. Eigenstates and eigenvalues

To see $\hat{\Delta}^{-1}x^{-z} = -\zeta(z, x+1)$, observe that

$$\begin{aligned}\hat{\Delta}^{-1}x^{-z} &= \frac{1}{\mathrm{i}\hat{p}}\sum_{n=0}^{\infty} B_n \frac{(-\mathrm{i}\hat{p})^n}{n!} x^{-z} \\ &= \frac{1}{1-z}\sum_{n=0}^{\infty} B_n \frac{(-\mathrm{i}\hat{p})^n}{n!} x^{1-z}.\end{aligned}$$

Because $\mathrm{i}\hat{p} = \partial_x$ and

$$\partial_x^n \, x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)} \, x^{\mu-n},$$

setting $\mu = 1-z$ we find

$$\hat{\Delta}^{-1}x^{-z} = \frac{\Gamma(2-z)}{1-z}\sum_{n=0}^{\infty} B_n \frac{(-1)^n}{n!} \frac{x^{1-z-n}}{\Gamma(2-z-n)},$$

but we have $\Gamma(2-z) = (1-z)\Gamma(1-z)$ and

$$\frac{1}{\Gamma(2-z-n)} = \frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}u \, \mathrm{e}^u \, u^{n+z-2},$$

so

$$\hat{\Delta}^{-1}x^{-z} = \frac{\Gamma(1-z)}{2\pi \mathrm{i}} \int_C \mathrm{d}t\, \frac{\mathrm{e}^{xt}\, t^{z-1}}{1-\mathrm{e}^{-t}} = -\zeta(z, x+1).$$

As for the eigenvalues, we have

$$\hat{H}\psi_z(x) = \hat{\Delta}^{-1}\left(\hat{x}\hat{p} + \hat{p}\hat{x}\right)\hat{\Delta}\hat{\Delta}^{-1}x^{-z} = \mathrm{i}(2z-1)\psi_z(x).$$

Because $\psi(0) = -\zeta(z,1) = -\zeta(z) = 0$, z is a zero of $\zeta(z)$.

5. Quantisation condition for the Berry-Keating Hamiltonian

Hamiltonian $\hat{H}$ is similar to the Berry-Keating Hamiltonian

$$\hat{h}_{\rm BK} = \hat{x}\,\hat{p} + \hat{p}\,\hat{x},$$

whose eigenstates and eigenvalues are

$$\phi_z^{\rm BK}(x) = x^{-z} \quad \text{and} \quad E = \text{i}(2z - 1).$$

The boundary condition $\psi(0) = 0$ then translates into the quantisation condition for the Berry-Keating Hamiltonian (as a boundary condition):

$$\varphi_z(0) = 0,$$

where

$$\varphi_z(x) := \phi_z^{\rm BK}(x) - \zeta(z, x).$$

Because on $\mathcal{L}^2(\mathbb{R}^+)$, $\hat{p}$ has a strictly positive imaginary part, both $\hat{\Delta}$ and $\hat{\Delta}^{-1}$ are bounded and invertible. $\Rightarrow$ $\hat{h}_{\rm BK}$ and $\hat{H}$ are isospectral.

But $\hat{h}_{\rm BK}$ is self-adjoint on $\mathbb{R}^+$, so that E is real ... ??

6. The dilation generator and the Riemann dilation operator

Another ingredient we require is the notion of the dilation operator.

The generator of the dilation is $\hat{x}\hat{p}$, where $\hat{p} = -\mathrm{i}\,\mathrm{d}/\mathrm{d}x$, so

$$\mathrm{e}^{\mathrm{i}\lambda\hat{x}\hat{p}} f(x) = f(\mathrm{e}^{\lambda} x).$$

It follows that

$$\sin(nx) = n^{\mathrm{i}\hat{x}\hat{p}} \sin x.$$

Therefore, ignoring for now the question of the convergence of the sum, we find

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \sum_{n=1}^{\infty} \frac{n^{\mathrm{i}\hat{x}\hat{p}}}{n} \sin x = \zeta\left(1 - \mathrm{i}\hat{x}\hat{p}\right) \sin x.$$

Thus, the action of the Riemann dilation operator $\zeta(1 - \mathrm{i}\hat{x}\hat{p})$ on a trigonometric function generates a Fourier series.

From $\hat{h}_{\mathrm{BK}} = 2\hat{x}\hat{p} - \mathrm{i}$ we have $1 - \mathrm{i}\hat{x}\hat{p} = \frac{1}{2}(3 - \mathrm{i}\hat{h}_{\mathrm{BK}})$.

It follows that

$$\zeta\left(1-\mathrm{i}\hat{x}\hat{p}\right)\,\sin x=\zeta\left(\tfrac{1}{2}(3-\mathrm{i}\hat{h}_{\rm BK})\right)\sin x=\frac{\pi-x}{2},$$

from which we infer that

$$\zeta(0)=-\frac{1}{2}\quad\text{and}\quad\zeta(-2)=\zeta(-4)=\cdots=0.$$

An similar line of argument leads to the observation that

$$\zeta\left(2-\mathrm{i}\hat{x}\hat{p}\right)\,\cos x=\frac{\pi^2}{6}-\frac{\pi x}{2}+\frac{x^2}{4}$$

and that

$$\zeta\left(3-\mathrm{i}\hat{x}\hat{p}\right)\,\sin x=\frac{\pi^2 x}{6}-\frac{\pi x^2}{4}+\frac{x^3}{12},$$

and so on.

Now if the operator $\zeta\left(1-\mathrm{i}\hat{x}\hat{p}\right)$ were invertible, then

$$\frac{1}{\zeta\left(1-\mathrm{i}\hat{x}\hat{p}\right)}\,\frac{\pi-x}{2}=\sin x\qquad\cdots\qquad ???$$

7. Discussion

We have only considered one class of operator-valued zeta functions, namely, zeta functions evaluated at a linear function of the dilation operator.

The matrix elements of, for example, $\zeta(1 - \mathrm{i}\hat{x}\hat{p})$, in the standard sine basis $\{\sqrt{2/\pi}\,\sin(nx)\}$, is given by

$$\zeta_{mn} = \begin{cases} n/m & \text{if } n \text{ divides } m, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the matrix $\{\zeta_{mn}\}$ encodes the information about factorisation of integers.

This suggests that it might be possible to extract more information by studying further properties of the class of operator-valued zeta functions considered here.

To conclude, we have shown that by studying the action of Riemann dilation operators on trigonometric functions, we are able to infer some properties of the Riemann zeta function.

Of course, the properties of $\zeta(s)$ inferred here are already known.

Nevertheless, we were able to determine, for example, the locations of the trivial zeros from elementary Fourier analysis without relying explicitly on the analytic continuation of the zeta function.

This suggests that further research into actions of operator-valued zeta functions may yield interesting new results.

8. References

Bender, C.M., Brody, D.C. & Müller, M.P. "Hamiltonian for the zeros of the Riemann zeta function" Physical Review Letters, **118**, 130201 (2017) .

Bender, C.M. & Brody, D.C. "Asymptotic analysis on a pseudo-Hermitian Riemann-zeta Hamiltonian" Journal of Physics A**51**, 135203 (2018) .

Brody, D.C. "Biorthogonal systems on unit interval and zeta dilation operators" Journal of Physics A**51**, 285202 (2018) .

Bender, C.M. & Brody, D.C. "Operator-valued zeta functions and Fourier analysis" arxiv:1810.01821 (2018) .