$\mathcal{PT}$-symmetric quantum field theory

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Abstract. $\mathcal{PT}$-symmetric quantum theory began with an analysis of the strange-looking non-Hermitian Hamiltonian $H = p^2 + x^2(ix)^\varepsilon$. This Hamiltonian is $\mathcal{PT}$ symmetric and the eigenvalues Hamiltonian are *discrete, real, and positive* when $\varepsilon \geq 0$. In this talk we describe the properties of the corresponding quantum-field-theoretic Hamiltonian $H = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\phi^2(i\phi)^\varepsilon$ in D-dimensional spacetime, where ϕ is a pseudoscalar field. We show how to calculate all of the Green's functions as series in powers of ε directly from the Euclidean partition function. We derive exact finite expressions for the vacuum energy density, the renormalized mass, and the connected n-point Green's functions for all n $0 \leq D < 2$. For $D \geq 2$ the one-point Green's function and the renormalized mass become infinite, but perturbative renormalization can be performed. The beautiful spectral properties of $\mathcal{PT}$-symmetric quantum mechanics appear to persist in $\mathcal{PT}$-symmetric quantum field theory.

1. Introduction

We begin this talk by explaining how the early ideas of $\mathcal{PT}$-symmetric quantum theory came about. Suppose that we must find the *positive root* of the polynomial equation $P(x) = 0$, where $P(x) = x^5 + x - 1$. I am using a quintic polynomial here because there is no general formula for the roots of such a polynomial. Thus, we consider $P(x) = 0$ to be a "difficult" problem and we use perturbative methods to find the root rather than searching for an exact expression. If we use a *strong-coupling* expansion, we insert a small parameter ε in front of the lower power of x, $x^5 + \varepsilon x - 1 = 0$, and seek a perturbative expansion of the form $x(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n$, where $a_0 = 1$. The perturbation coefficients a_n satisfy a sequence of *linear* equations:

$$\begin{aligned} 5a_1 + 1 &= 0, \\ 5a_2 + 10a_1^2 + a_1 &= 0, \\ 5a_3 + 20a_1a_2 + a_2 + 10a_1^3 &= 0, \\ 5a_4 + 20a_1a_3 + a_3 + 10a_2^2 + 30a_1^2a_2 + 5a_1^4 &= 0, \end{aligned}$$

and so on. Solving these equations for the coefficients is easy. The perturbation series to sixth order in ε is (the coefficient of ε^4 is 0): $x(\varepsilon) = 1 - \frac{1}{5}\varepsilon - \frac{1}{25}\varepsilon^2 - \frac{1}{125}\varepsilon^3 + \frac{21}{15625}\varepsilon^5 + \frac{78}{78125}\varepsilon^6 + \cdots$.

To recover the positive root of $P(x)$ we set $\varepsilon = 1$ and sum the perturbation series. This procedure works because the radius of convergence of the series is $5/4^{4/5} = 1.64938\cdots$, which is greater than 1. To sixth order in powers of ε the sum of this series is $x(1) = 0.75434$ and the exact (numerical) answer is $x = 0.75488\cdots$. Thus, to this order in perturbation theory we get a relative error of 0.07%, which is quite impressive.

A high-energy physicist might use a *weak-coupling* expansion in which the small parameter ε is inserted as the coefficient of the *highest* power of x in $P(x)$: $\varepsilon x^5 + x - 1 = 0$. The resulting

perturbation series is $x(\varepsilon) = 1 - \varepsilon + 5\varepsilon^2 - 35\varepsilon^3 + 285\varepsilon^4 - 2530\varepsilon^5 + 23751\varepsilon^6 - \cdots$. Unfortunately, the radius of convergence of this series is $4^4/5^5 = 0.08192$, which is less than 1. If we sum this sixth-order perturbation series directly at $\varepsilon = 1$ we get the silly answer $x(1) = 21476$, but a (3,3)-Padé evaluated at $\varepsilon = 1$ is correct to 0.7%, which is not bad.

An unusual place to insert a small parameter ε into $P(x)$ is in the exponent: $x^{1+\varepsilon} + x - 1 = 0$. Now, ε plays an interesting role. Rather than acting as a coupling constant that measures the strength of an interaction, it measures the *degree of nonlinearity* of the problem; that is, how much it deviates from a linear problem. The first few terms in the perturbation series are

$$x(\varepsilon) = \tfrac{1}{2} + \tfrac{1}{4}\varepsilon \log 2 - \tfrac{1}{8}\varepsilon^2 \log 2 + \tfrac{1}{96}\varepsilon^3[-2(\log 2)^3 + 3(\log 2)^2 + 6\log 2] + \cdots.$$

The radius of convergence of this perturbation series is 1, but if we again construct a (3,3)-Padé and evaluate it at $\varepsilon = 4$, the error is 0.5%, which is better than the weak-coupling result.

This procedure of introducing a perturbation parameter into the exponent so that it measures the nonlinearity of the problem rather than the strength of the coupling is a powerful new way to solve difficult nonlinear problems. It can be applied to both classical and quantum-mechanical problems. As a classical example, we consider the Thomas-Fermi equation,

$$y''(x) = [y(x)]^{3/2}/\sqrt{x}, \qquad y(0) = 1,\ y(\infty) = 0,$$

which is an approximate description of the charge density in a nucleus. The objective in solving this difficult and delicate boundary-value problem is to determine the initial condition $y'(0) = -1.5880710\cdots$. No exact analytical solution to this equation has been found but a novel way to attack this problem is to introduce the small parameter ε into the exponent $y''(x) = y(x)[y(x)/x]^{\varepsilon}$ and to seek a perturbative solution in powers of ε: $y(x) = y_0(x) + y_1(x)\varepsilon + y_2(x)\varepsilon^2 + y_3(x)\varepsilon^3 + \cdots$. Note that ε measures the departure of this problem from the trivial linear problem $y''(x) = y(x)$, so the zeroth term in the perturbation expansion is $y_0(x) = e^{-x}$. It is easy to calculate the higher coefficients in this expansion and a (2,1)-Padé approximation evaluated at $\varepsilon = 1/2$ gives an accuracy of 1.1% [1], which is quite impressive.

Another classical example is the Blasius equation, which describes the boundary layer in fluid flow across a flat surface:

$$y'''(x) = y(x)y''(x), \qquad y(0) = y'(0) = 0,\ y'(\infty) = 1.$$

This is a very difficult boundary value problem, but is is easy to find the first three coefficients in the perturbation expansion for the problem $y'''(x) = y''(x)[y(x)]^{\varepsilon}$. (Once again, the parameter ε measures the departure from linearity.) The (1,1)-Padé gives a numerical accuracy of 8.7%, which is impressive for such a difficult problem [1].

Two other classical equations for which this perturbation technique gives accurate results [1] are the Lane-Emden equation (stellar structure) $y''(x) + 2y'(x)/x + [y(x)]^n = 0$, which we solve by introducing ε as $y''(x) + 2y'(x)/x + [y(x)]^{1+\varepsilon} = 0$, and the Korteweg-de Vries equation $u_t + uu_x + u_{xxx} = 0$, which we solve by introducing ε as $u_t + u^{\varepsilon}u_x + u_{xxx} = 0$.

2. Quantum Mechanics

We have argued that introducing a small parameter ε into the exponent so that ε measures the departure of an equation from linearity is a powerful technique for finding accurate solutions to nonlinear problems. In this section we show how this technique applies in quantum mechanics and in the next section we extend it to quantum field theory.

Consider, for example, the anharmonic-oscillator Hamiltonian $H = p^2 + x^4$. This is a difficult problem; no analytical expression for the eigenvalues of H has been found. However, accurate numerical results can be obtained by studying the quantum-mechanical Hamiltonian

$H = p^2 + x^2(x^2)^\varepsilon$, where ε measures the departure of H from the quantum-harmonic-oscillator Hamiltonian. Similarly, we can study a ϕ^4 quantum field theory by replacing ϕ^4 with $\phi^2(\phi^2)^\varepsilon$ [2, 3]. In these cases we insert the small parameter ε so that x^2 and ϕ^2 and not x and ϕ that are raised to the power ε. This is because x and ϕ can be negative, and a negative number raised to a noninteger power is typically complex. It is important to avoid introducing complex numbers as an artifact of the perturbation procedure.

An extraordinary discovery was made in 1998. It was shown that while x^ε does indeed introduce spurious complex numbers as a consequence of the perturbation technique, one may avoid the appearance of complex numbers if one raises ix and not x to the power ε [4]. Indeed, as Fig. 1 shows, the spectrum of the non-Hermitian Hamiltonian $H = p^2 + x^2(ix)^\varepsilon$ with ε real is entirely real, positive, and discrete if $\varepsilon \geq 0$.

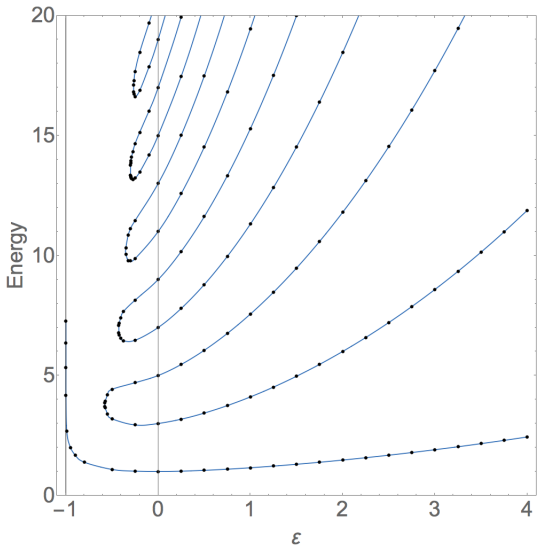

Figure 1. Energy levels of the parametric family of Hamiltonians $H = p^2 + x^2(ix)^\varepsilon$. Note that when $\varepsilon \geq 0$, the eigenvalues are all real and positive, and they increase with increasing ε. When ε decreases below 0, the eigenvalues disappear into the complex plane as complex conjugate pairs. Eventually, only one real eigenvalue remains when ε is less than about -0.57, and as ε approaches -1 from above, this eigenvalue becomes infinite. The region $\varepsilon \geq 0$ in which the eigenvalues are all real is called the *region of unbroken $\mathcal{PT}$ symmetry* and the region $\varepsilon < 0$ in which some of the eigenvalues are complex is called the *region of broken $\mathcal{PT}$ symmetry*. The transition point between these two regions is called the *$\mathcal{PT}$ phase transition*, and this transition has been observed in many published laboratory experiments.

In Ref. [4] it is argued that the reality of the spectrum is a consequence of the $\mathcal{PT}$ invariance ($\mathcal{PT}$ symmetry) of the Hamiltonian. (Under parity $\mathcal{P}$, $x \to -x$, and under time reversal $\mathcal{T}$, $i \to -i$, so the combination ix is $\mathcal{PT}$ invariant.) A proof that the eigenvalues are real was first formulated in Refs. [5]. The positivity of the spectrum is particularly remarkable for the case $\varepsilon = 2$ because in this case we have an "upside-down" $-x^4$ potential, and the conventional view is that such a potential could not have discrete positive bound-state energies!

In the region of unbroken $\mathcal{PT}$ symmetry ($\varepsilon \geq 0$), where the spectrum is entirely real, the Hamiltonian $H = p^2 + x^2(ix)^\varepsilon$ defines a fully consistent unitary quantum-mechanical theory [6]. One may think of a $\mathcal{PT}$-symmetric quantum mechanical theory as a complex deformation of a conventional quantum theory. Specifically, for H ε plays the role of a deformation parameter and the theory that is being deformed is the quantum harmonic oscillator.

There have been dozens of experimental confirmations published in *Nature*, *Science*, and *Physical Review Letters* in diverse research areas such as optics [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19], superconductivity [20, 21], microwave cavities [22, 23], lasers [24, 25], electronic circuits [26], chaos and noise [27, 28], and graphene [29, 30]. Some particularly interesting recent experimental work involves the study of $\mathcal{PT}$-symmetric whispering-gallery microcavities [31, 32]. New experiments involving Bose-Einstein condensates are in the planning stage [33] and new $\mathcal{PT}$-symmetric metamaterials are being developed [34, 35, 36, 37]. There are also beautiful experiments on $\mathcal{PT}$-symmetric wireless power transfer [38]. For a brief nontechnical description of $\mathcal{PT}$ symmetry see Ref. [39] and for detailed studies see [40, 41].

3. $\mathcal{PT}$-Symmetric Quantum Field Theory

The main point of this talk is to show how to solve a $\mathcal{PT}$-symmetric quantum field theory described by the Euclidean Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\phi^2(i\phi)^\varepsilon \quad (\varepsilon \geq 0) \tag{1}$$

as a series in powers of ε (see the recently published work with N. Hassanpour, S. P. Klevansky, and S. Sarkar [42]). We treat the parameter ε as small and calculate the vacuum energy density E_0, the connected n-point Green's functions G_n, and the renormalized mass $M_{\rm R}$ as series in powers of ε. We assume that $0 \leq D < 2$ to avoid the appearance of renormalization infinities and at the end of the talk we comment briefly on the perturbative renormalization procedure for the case $D \geq 2$.

Treating ε as small ($\varepsilon << 1$), the unusual Lagrangian density $\mathcal{L}$ in (1) develops a *logarithmic* self-interaction term:

$$\mathcal{L} = \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\phi^2 + \tfrac{1}{2}\varepsilon\phi^2\log(i\phi) + \mathrm{O}(\varepsilon^2). \tag{2}$$

For a quantum field theory having a complex logarithmic interaction term it is not obvious whether one can find Feynman rules for performing perturbative diagramatic calculations. We will show how to construct such Feynman rules. To begin, we replace the complex logarithm with a real logarithm in such a way as to preserve $\mathcal{PT}$ symmetry. We define $\log(i\phi) \equiv \frac{1}{2}i\pi + \log(\phi)$ if $\phi > 0$ and we define $\log(i\phi) \equiv -\frac{1}{2}i\pi + \log(-\phi)$ if $\phi < 0$. Combining these two equations, we get

$$\log(i\phi) = \tfrac{1}{2}i\pi\,|\phi|/\phi + \tfrac{1}{2}\log(\phi^2). \tag{3}$$

Observe that the imaginary part of (3) is *odd* in ϕ and the real part of (3) is *even* in ϕ. Thus, (3) enforces the $\mathcal{PT}$ symmetry because the pseudoscalar field ϕ changes sign under parity $\mathcal{P}$ and i changes sign under time reversal $\mathcal{T}$.

To derive (3) we must assume that ϕ is *real*. This is justified because the functional integration

$$Z(\varepsilon) = \int\mathcal{D}\phi\,\exp\bigl(-\int d^Dx\,\mathcal{L}\bigr) \tag{4}$$

is performed along the *real-*ϕ axis and not in the complex-ϕ domain. Because we are treating ε as small we need not be concerned here with complex functional integration paths that terminate in complex Stokes sectors. This is because the functional integral (4) converges term-by-term in powers of ε. In $\mathcal{PT}$-symmetric quantum mechanics the boundary conditions on the Schrödinger equation associated with the Hamiltonian $H = p^2 + x^2(x^2)^\varepsilon$ are imposed in complex Stokes sectors [4]. However, in the context of quantum field theory it would be far too difficult to consider *functional* Stokes sectors. This is why we treat ε as small. This talk is concerned with calculating the coefficients in the ε series and we defer the mathematical issues involving the summation of such a series for large values of ε.

Graphical techniques were developed in Ref. [1] to handle real logarithmic interaction terms. These techniques are generalizations of the replica trick [43], which has been used in the study

of spin glasses. The idea of the replica trick is that a logarithmic term $\log A$ can be reformulated as the limit $\log A = \lim_{N\to 0} \frac{1}{N}(A^N - 1)$, or equivalently and slightly more simply as the limit

$$\log A = \lim_{N\to 0} \tfrac{d}{dN} A^N. \tag{5}$$

We regard N as an integer and treat A^N as an N-point vertex in a graphical expansion. This procedure is not rigorous because it requires taking the *continuous* limit $N \to 0$. The validity of this approach has not been established rigorously, but when we compare with exact results in low-dimensional theories, the replica trick gives the correct answer. In this talk we verify our field-theoretic results by comparing them with the exact answers for $D = 0$ (where the functional integral becomes an ordinary integral) and for $D = 1$ (quantum mechanics).

The graphical calculations in this talk are done in coordinate space. Once the vertices are identified, we only need the free propagator in D-dimensional Euclidean space $\Delta(x - y)$, which satisfies the differential equation $(-\nabla_x^2 + 1)\Delta(x - y) = \delta^{(D)}(x - y)$. The Fourier transform of this equation gives the amplitude for the free propagator of a particle of mass 1 in momentum space: $\tilde{\Delta}(p) = 1/(p^2 + 1)$. The D-dimensional inverse Fourier transform of this expression then gives the D-dimensional coordinate-space propagator in terms of an associated Bessel function:

$$\Delta(x_1 - x_2) = (2\pi)^{-D/2}|x_1 - x_2|^{1-D/2}\mathrm{K}_{1-D/2}(|x_1 - x_2|). \tag{6}$$

If we let $x_1 \to x_2$, we obtain $\Delta(0)$ for a *self loop*, that is, the amplitude for a line to originate from and return to the same point in coordinate space: $\Delta(0) = (4\pi)^{-D/2}\Gamma(1 - D/2)$. This expression is finite and nonsingular for $0 \le D < 2$.

Ground-state energy density: The partition function is the exponential of the ground-state energy density E_0 multiplied by the volume of spacetime V: $Z = e^{-E_0 V}$. The *shift* in the ground-state energy density ΔE to order ε is given by $\Delta E = \frac{\varepsilon}{4Z(0)V}\int \mathcal{D}\phi\, e^{-\int d^D x\, \mathcal{L}_0} \int d^D y\, \phi^2(y) \log[\phi^2(y)]$, so from (5) we get

$$\Delta E = \lim_{N\to 1} \frac{\varepsilon}{4Z(0)V} \frac{d}{dN} \int \mathcal{D}\phi\, \exp\bigl(-\textstyle\int d^D x\, \mathcal{L}_0\bigr) \int d^D y\, \phi^{2N}(y).$$

This expression has a graphical interpretation as the product of N self loops from y back to y. There are $(2N - 1)!!$ ways to construct these self loops, so the expression for ΔE simplifies to $\Delta E = \lim_{N\to 1} \frac{\varepsilon}{4V} \frac{d}{dN} \int d^D y\, [\Delta(0)]^N (2N - 1)!!$. The integral $\int d^D y$ is the volume of spacetime V, so this formula simplifies further: $\Delta E = \lim_{N\to 1} \frac{\varepsilon}{4} \frac{d}{dN} [\Delta(0)]^N (2N - 1)!!$. Last, we use the duplication formula $(2N - 1)!! = 2^N \Gamma(N + \frac{1}{2})/\sqrt{\pi}$ and then differentiate with respect to N:

$$\Delta E = \tfrac{1}{4}\varepsilon\Delta(0)\{\log[2\Delta(0)] + \Gamma'(\tfrac{3}{2})/\Gamma(\tfrac{3}{2})\} = \tfrac{1}{4}\varepsilon\Delta(0)\{\log[\Delta(0)/2] + 2 - \gamma\}. \tag{7}$$

The final result in (7) may be verified for the special cases $D = 0$ and $D = 1$.

One-point Green's function: Let us now perform a more sophisticated calculation. The one-point Green's function G_1 in $\mathcal{PT}$-symmetric quantum theory is *nonperturbative* in character but it can be calculated by following the approach used above to calculate ΔE. We keep terms that do not vanish under $\phi \to -\phi$, and evaluate directly the functional-integral representation

$$G_1(a) = -\frac{\varepsilon}{4Z(0)} \int \mathcal{D}\phi\, \phi(a) e^{-\int d^D x\, \mathcal{L}_0} \int d^D y\, i\pi\phi(y)|\phi(y)|, \tag{8}$$

where $\mathcal{L}_0 = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\phi^2$ is the free Euclidean Lagrangian. We introduce the identity

$$\phi|\phi| = \tfrac{2}{\pi}\phi^2 \textstyle\int_0^\infty dt\, \sin(t\phi)/t \tag{9}$$

to replace $\phi(y)|\phi(y)|$ in (8) and then replace $\sin(t\phi)$ by its Taylor series in powers of t. This converts (8) to the product of an infinite sum over n, a one-dimensional integral in t, a D-dimensional integral in y, and a functional integral in ϕ:

$$G_1(a) = -\frac{i\varepsilon}{2}\int_{t=0}^{\infty} dt \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n+1)!}\int d^D y \frac{1}{Z(0)}\int \mathcal{D}\phi\, \exp\big(-\int d^D x\, \mathcal{L}_0\big)\phi(a)[\phi(y)]^{2n+3}. \quad (10)$$

The sum and multiple integrals in (10) are not difficult; like the calculation of the ground-state energy density, the functional integral in (10) has a graphical interpretation; it is the product of the free propagator $\Delta(a-y)$ representing a line from a to y multiplied by $n+1$ self loops from y to y, and this product is accompanied by the combinatorial factor $(2n+3)!!$. Thus, the functional integral in (10) reduces to $(2n+3)!!\Delta(y-a)\Delta^{n+1}(0)$.

This result simplifies further because the D-dimensional integral is trivial: $\int d^D y \Delta(y-a) = 1$. This establishes the translation invariance of G_1. The rest is straightforward:

$$\begin{aligned} G_1 &= -\frac{i\varepsilon}{2}\int_{t=0}^{\infty} dt \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}\Delta^{n+1}(0)(2n+3)!!}{(2n+1)!} \\ &= -\frac{i\varepsilon}{2}\int_{t=0}^{\infty} dt\, \Delta(0)[3-\Delta(0)t^2]e^{-\Delta(0)t^2/2} = -i\varepsilon\sqrt{\pi\Delta(0)/2}. \end{aligned} \quad (11)$$

This expression for G_1 is *exact* to order ε. Note that G_1 is a *negative imaginary* number. This is what we would expect based on studies of classical $\mathcal{PT}$-symmetric systems. The *classical* trajectories in complex coordinate space of a particle described by the Hamiltonian $H = p^2 + x^2(ix)^\varepsilon$ are left-right symmetric but they lie mostly in the lower-half plane [6]. Thus, the average value of the classical orbits is a negative-imaginary number.

Two-point Green's function: We wish to obtain the *connected* two-point Green's function $G_2(a,b)$, and to do so we subtract G_1^2 from the vacuum expectation value of $\phi(a)\phi(b)$. Since G_1 is of order ε, we see that to first order in ε we need only evaluate $Z(\varepsilon)$ in (4) with $\phi(a)\phi(b)$ inserted after $\mathcal{D}\phi$ and then divide this integral by $Z(\varepsilon)$. We neglect the imaginary terms in these integrals because they are odd under $\phi \to -\phi$. Thus, to obtain $G_2(a,b)$ we must calculate

$$\frac{\int \mathcal{D}\phi\, \phi(a)\phi(b)\exp\big(-\int d^D x\, \mathcal{L}_0\big)\{1-\frac{\varepsilon}{4}\int d^D y\, \phi^2(y)\log[\phi^2(y)]\}}{\int \mathcal{D}\phi\, \exp\big(-\int d^D x\, \mathcal{L}_0\big)\{1-\frac{\varepsilon}{4}\int d^D y\, \phi^2(y)\log[\phi^2(y)]\}}.$$

To first order in ε this is a sum of three functional integrals, $G_2(a,b) = A + B + C$, where

$$\begin{aligned} A &= \frac{1}{Z(0)}\int \mathcal{D}\phi\, \phi(a)\phi(b)e^{-\int d^D x\, \mathcal{L}_0}, \\ B &= -\frac{\varepsilon}{4Z(0)}\int \mathcal{D}\phi\, \phi(a)\phi(b)e^{-\int d^D x\, \mathcal{L}_0}\int d^D y\, \phi^2(y)\log[\phi^2(y)], \\ C &= \frac{\varepsilon}{4Z(0)}\int \mathcal{D}\phi\, \phi(a)\phi(b)e^{-\int d^D x\, \mathcal{L}_0}\frac{1}{Z(0)}\int \mathcal{D}\phi\, e^{-\int d^D x\, \mathcal{L}_0}\int d^D y\, \phi^2(y)\log[\phi^2(y)]. \end{aligned} \quad (12)$$

We then evaluate the three contributions A, B, and C. The functional integral A in (12) is just the free propagator $\Delta(a-b)$ and this verifies that if we set $\varepsilon = 0$ in the Lagrangian (1), we obtain a free field theory; the two-point Green's function for such a field theory $\Delta(a-b)$.

The double functional integral C is slightly more complicated. The first part of C is proportional to A and evaluates to $\varepsilon\Delta(a-b)/4$. However, we saw in our calculation of the ground-state energy that the next part C evaluates to $4V\Delta E/\varepsilon$, where ΔE is the first-order shift in the ground-state energy density (7) and V is the volume of Euclidean spacetime. Thus, $C = \Delta(a-b)V\Delta E$. This quantity is *divergent* because V is infinite.

To resolve this divergence problem we calculate B. We use the identity in (5) to express the B integral as a limit: $B = -\frac{\varepsilon}{4Z(0)} \lim_{N\to 1} \frac{d}{dN} \int \mathcal{D}\phi \, \exp\big(-\int d^D x \, \mathcal{L}_0\big) \int d^D y \, \phi(a)\phi(b)\phi^{2N}(y)$. The interpretation of this functional integral is that we must connect the set of $2N+2$ points consisting of a, b, and the $2N$ points y pairwise with the free propagator Δ in (6). We must consider two cases: In the first case a is connected to b and the remaining $2N$ points at y are connected in pairs. This reproduces the result above for C except with the opposite sign. Thus, the volume divergences exactly cancel!

In the second case a is not connected to b. Instead, a connects to a point y ($2N$ ways to do this) and b connects to one of the remaining $2N-1$ points y ($2N-1$ ways to do this). The rest of the $2N-2$ points y are joined in pairs [$(2N-3)!!$ ways to do this]. Thus, the amplitude for this case is $-\frac{\varepsilon}{2} \int d^D y \, \Delta(a-y)\Delta(y-b) \lim_{N\to 1} \frac{d}{dN} N(2N-1)!!\Delta^{N-1}(0)$, which simplifies to $-\varepsilon K \int d^D y \, \Delta(a-y)\Delta(y-b)$, where $K = \frac{1}{2} + \frac{1}{2}\Gamma'(\frac{3}{2})/\Gamma(\frac{3}{2}) + \frac{1}{2}\log[2\Delta(0)] = \frac{3}{2} - \frac{1}{2}\gamma + \frac{1}{2}\log[\frac{1}{2}\Delta(0)]$.

Hence, our result for the coordinate-space two-point Green's function to order ε is

$$G_2(a-b) = \Delta(a-b) - \varepsilon K \int d^D y \, \Delta(a-y)\Delta(y-b), \tag{13}$$

which in momentum space becomes $\tilde{G}_2(p) = 1/(p^2+1) - \varepsilon K/(p^2+1)^2 + \mathrm{O}(\varepsilon^2)$. From $\tilde{G}_2(p)$ we construct the $(0,1)$-Padé approximant, which is merely the geometric sum of a chain of bubbles: $\tilde{G}_2(p) = 1/[p^2 + 1 + \varepsilon K + \mathrm{O}(\varepsilon^2)]$. We then read off the *square of the renormalized mass* to order ε: $M_{\mathrm{R}}^2 = 1 + K\varepsilon + \mathrm{O}(\varepsilon^2)$.

Higher Green's functions: The *connected* Green's functions are expressed as cumulants. Evaluating the functional integrals is tedious but not difficult, and we find an exact general formula for the nth Green's function that is valid for all $n \neq 2$:

$$G_n(x_1, x_2, \ldots x_n) = -\tfrac{1}{2}\varepsilon(-i)^n \Gamma(\tfrac{1}{2}n - 1)[\tfrac{1}{2}\Delta(0)]^{1-n/2} \int d^D u \, \textstyle\prod_{k=1}^n \Delta(x_k - u). \tag{14}$$

Note that all of the connected Green's functions are of order ε except for G_2 in (13), which is of order 1. Observe that (14) reduces to (11) for $n = 1$.

The formula (14) holds for both odd and even n. Our calculation of the odd-n Green's functions uses the integral representation in (9). By contrast, our calculation of the even-n Green's functions uses the identity in (5). It is quite amazing that these two extremely different techniques lead to the single universal formula in (14) for G_n.

4. Discussion and Future Work

In this talk we have presented the machinery necessary to calculate the Green's functions of the $\mathcal{PT}$-symmetric quantum field theory in (1) to any order in powers of ε. Thus, we have opened a vast area for future study and investigation; one can calculate the masses of the theory (poles of the Green's functions), scattering amplitudes, critical indices, and so on. The Green's-function calculations presented here are exact to first order in ε. However, the procedures discussed generalize to all orders in powers of ε.

This is useful because even in low orders the perturbation series in powers of ε is highly accurate and it continues to be accurate for large ε. To demonstrate this we calculate the one-point Green's function G_1 in $D = 0$ to *second* order in ε. In $D = 0$ this Green's function is a ratio of two *ordinary integrals* (not functional integrals):

$$G_1 = \textstyle\int_{-\infty}^{\infty} dx \, x \, \exp\big[-\frac{1}{2}x^2(1 + \varepsilon L + \frac{1}{2}\varepsilon^2 L^2)\big] / \int_{-\infty}^{\infty} dx \, \exp\big[-\frac{1}{2}x^2(1+\varepsilon L)\big],$$

where $L = \log(ix) = \frac{1}{2}i\pi|x|/x + \frac{1}{2}\log(x^2)$. Evaluating these integrals is easy and we get

$$G_1 = -i\varepsilon\sqrt{\pi/2}[1 + \tfrac{1}{4}\varepsilon(\gamma - 2 - 3\log 2) + \mathrm{O}(\varepsilon^2)] = -i\varepsilon\sqrt{\pi/2}[1 - 0.8756\varepsilon + \mathrm{O}(\varepsilon^2)]. \tag{15}$$

To check of the accuracy of (15) we calculate the one-point Green's function for a cubic theory ($\varepsilon = 1$). We convert (15) to a $[0,1]$ Padé approximant, $G_1 = -i\varepsilon\sqrt{\pi/2}[1/(1+0.8756\varepsilon)]$, and then set $\varepsilon = 1$ to obtain the approximate result $G_1 = -0.6682i$.

The exact value of G_1 for the zero-dimensional cubic theory $\varepsilon = 1$ is the ratio of integrals

$$G_1 = \int_{-\infty}^{\infty} dx\, x \exp\,(\,-\tfrac{1}{2}ix^3)/\int_{-\infty}^{\infty} dx\, \exp\,(\,-\tfrac{1}{2}ix^3) = -i2^{\frac{1}{3}}\Gamma(2/3)/\Gamma(1/3) = -0.6369i.$$

Thus, the *two*-term ε expansion (15) has an accuracy of 5%, which is impressive for such a large value of ε. This good result is consistent with the results found in previous studies of the accuracy of the ε expansion for various classical equations (see Ref. [1]).

The third-order version of (15) is $G_1 = -i\varepsilon\sqrt{\pi/2}[1-0.8756\varepsilon+0.6447\varepsilon^2+\mathrm{O}(\varepsilon^3)]$. Converting this expansion above to a $(1,1)$-Padé approximant and setting $\varepsilon = 1$, we obtain $G_1 = -0.6213i$, which differs from the exact result by -2%. These numerical results motivate us to extend the ε expansion of $\mathcal{PT}$-symmetric quantum field theories to higher order.

A second issue is renormalization. Since $\Delta(0)$ becomes singular when the dimension D of Euclidean spacetime reaches 2, the one-point Green's function G_1 and the renormalized mass M_{R} become singular. (To order ε the higher Green's functions do not become infinite when $D = 2$.) Thus, for $D \geq 2$ we must undertake a perturbative renormalization procedure.

In this talk we have worked with dimensionless quantities. However, to do perturbative renormalization one must use the Lagrangian $\mathcal{L} = \frac{1}{2}(\nabla\phi)^2+\frac{1}{2}\mu^2\phi^2+\frac{1}{2}g\mu_0^2\phi^2(i\mu_0^{1-D/2}\phi)^{\varepsilon}$ for which the dimensional parameters are explicit: μ is the unrenormalized mass, μ_0 is a fixed parameter having dimensions of mass, and g is a dimensionless unrenormalized coupling constant. The mass renormalization procedure consists of expressing the renormalized mass M_{R} in terms of these Lagrangian parameters and absorbing the divergence that arises when $D \geq 2$ into parameter μ. Coupling-constant renormalization is similar; we define the renormalized coupling constant G_{R} as the value of the three-point or four-point Green's functions at particular values of the external momentum and again absorb the divergence that arises into the Lagrangian parameter g. One must then verify that all higher Green's functions are finite when expressed in terms of M_{R} and G_{R}. This program will be explored in detail in the future.

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References

[1] Bender C M *et al.* 1989 A new perturbative approach to nonlinear problems *J. Math. Phys.* **30**, 1447
[2] Bender C M *et al.* 1987 Logarithmic approximations to polynomial Lagrangians *Phys. Rev. Lett.* **58**, 2615
[3] Bender C M *et al.* 1988 Novel perturbative scheme in quantum field theory *Phys. Rev. D* **37**, 1472
[4] Bender C M and Boettcher S 1998 Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry *Phys. Rev. Lett.* **80** 5243
[5] Dorey P E, Dunning C and Tateo R 2007 The ODE/IM correspondence *J. Phys. A: Math. Gen.* **40** R205
[6] Bender C M 2007 Making sense of non-Hermitian Hamiltonians *Rept. Prog. Phys.* **70** 947
[7] Guo A *et al.* 2009 Observation of $\mathcal{PT}$-symmetry breaking in complex optical potentials *Phys. Rev. Lett.* **103** 093902
[8] Longhi S 2009 Bloch oscillations in complex crystals with $\mathcal{PT}$ symmetry *Phys. Rev. Lett.* **103** 123601
[9] Rüter C E *et al.* 2010 Observation of $\mathcal{PT}$ symmetry in optics *Nat. Phys.* **6** 192
[10] Longhi S 2010 Optical realization of relativistic non-Hermitian quantum mechanics *Phys. Rev. Lett.* **105** 013903
[11] Lin Z *et al.* 2011 Unidirectional invisibility induced by $\mathcal{PT}$-symmetric periodic structures *Phys. Rev. Lett.* **106** 213901
[12] Feng L *et al.* 2011 Nonreciprocal light propagation in a silicon photonic circuit *Science* **333** 729
[13] Ramezani H *et al.* 2012 $\mathcal{PT}$-symmetric Talbot effects *Phys. Rev. Lett.* **109** 033902
[14] Regensberger A *et al.* 2012 Parity-time synthetic photonic lattices *Nature* **488** 167
[15] Razzari L and Morandotti R 2012 Gain and loss mixed in the same cauldron *Nature* **488** 163
[16] Liang G Q and Chong Y D 2013 Optical resonator analog of a two-dimensional topological insulator *Phys. Rev. Lett.* **110** 203904
[17] Luo X *et al.* 2013 Pseudo-parity-time symmetry in optical systems *Phys. Rev. Lett.* **110** 243902

[18] Regensberger A *et al.* 2013 Observation of defect states in $\mathcal{PT}$-symmetric optical lattices *Phys. Rev. Lett.* **110** 223902
[19] Hang C, Huang G and Konotop V V 2013 $\mathcal{PT}$ symmetry with a system of three-level atoms *Phys. Rev. Lett.* **110**, 083604
[20] Rubinstein J, Sternberg P and Ma Q 2007 Bifurcation diagram and pattern formation of phase slip centers in superconducting wires driven by electric currents *Phys. Rev. Lett.* **99** 167003
[21] Chtchelkatchev N *et al.* 2012 Stimulation of the fluctuation superconductivity by $\mathcal{PT}$ symmetry *Phys. Rev. Lett.* **109** 150405
[22] Dietz B *et al.* 2011 Exceptional points in a microwave billiard with time-reversal invariance violation *Phys. Rev. Lett.* **106** 150403
[23] Bittner S *et al.* 2012 $\mathcal{PT}$ symmetry and spontaneous symmetry breaking in microwave billiards *Phys. Rev. Lett.* **108** 024101
[24] Chong Y D, Ge L and Stone A D 2011 $\mathcal{PT}$-symmetry breaking and laser-absorber modes in optical scattering systems *Phys. Rev. Lett.* **106** 093902
[25] Liertzer M *et al.* 2012 Pump induced exceptional points in lasers *Phys. Rev. Lett.* **108** 173901
[26] Bender N *et al.* 2013 Observation of asymmetric transport in structures with active nonlinearities *Phys. Rev. Lett.* **110** 234101
[27] Schomerus H 2010 Quantum noise and self-sustained radiation of $\mathcal{PT}$-symmetric systems *Phys. Rev. Lett.* **104** 233601
[28] West C T, Kottos T and Prosen T 2010 $\mathcal{PT}$-symmetric wave chaos *Phys. Rev. Lett.* **104** 054102
[29] Fagotti M *et al.* 2011 Armchair graphene nanoribbons: $\mathcal{PT}$-symmetry breaking and exceptional points without dissipation *Phys. Rev.* B **83** 241406(R)
[30] Szameit A *et al.* 2012 $\mathcal{PT}$-symmetry in honeycomb photonic lattices *Phys. Rev.* A **84** 021806
[31] Peng B *et al.* 2014 Parity-time-symmetric whispering-gallery microcavities *Nature Physics* **10** 394
[32] Peng B *et al.* 2014 Loss-induced suppression and revival of lasing *Science* **346**, 328
[33] Kartashov Y V, Konotop V V and Abdullaev F Kh 2013 Gap solitons in a spin-orbit-coupled Bose-Einstein condensate *Phys. Rev. Lett.* **111** 060402
[34] Castaldi G *et al.* 2013 $\mathcal{PT}$ metamaterials via complex coordinate transformation optics *Phys. Rev. Lett.* **110** 173901
[35] Lazarides N and Tsironis G P 2013 Gain-driven discrete breathers in $\mathcal{PT}$-symmetric nonlinear metamaterials *Phys. Rev. Lett.* **110** 053901
[36] Yin X and Zhang X 2013 Unidirectional light propagation at exceptional points *Nature Materials* **12** 175
[37] Feng L *et al.* 2013 Experimental demonstration of a unidirectional reflectionless parity-time metamaterial at optical frequencies *Nature Materials* **12** 108
[38] Assawawarrait S, Yu X and Fan S 2017 Robust wireless power transfer using a nonlinear parity-time-symmetric circuit *Nature* **546** 387
[39] Bender C M 2016 $\mathcal{PT}$ symmetry in quantum physics *Europhysics News* **47/2** 17
[40] Bender C M *et al.* 2019 *PT Symmetry: in Quantum and Classical Physics* (World Scientific, Singapore)
[41] Christodoulides D N and Yang J, eds. 2018 *Parity-time symmetry and its applications* (Springer Tracts in Modern Physics **280**)
[42] Bender C M *et al.* 2018 PT-symmetric quantum field theory in D dimensions *Physical Review D* **98**, 125003
[43] Mezard M, Parisi G and Virasoro M 1987 *Spin Glass Theory and Beyond* (World Scientific, Singapore)