Ward-Takahashi Identity in QED_4 at the One-loop Level in the Light-Front

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Outline

1. Basics
   - Notations and Conventions
   - Tree-level Ward-Takahashi Identities
2. $N = 1$ Photon One-loop Ward-Takahashi Identity
   - Covariant Case
   - Light Front Case

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Basics:
Notations and Conventions

Our definition and notation for the light-front coordinates are:

$$\begin{bmatrix} x^+ \\ x^- \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^3 \end{bmatrix}, \quad (1)$$

and similarly for momentum variables

$$\begin{bmatrix} k^+ \\ k^- \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} k^0 \\ k^3 \end{bmatrix}. \quad (2)$$

The transverse part is the Euclidean 2D space $\mathbf{x}^{\perp} = (x^1, x^2)$.

Basics:
Notations and Conventions

Minkowski metric

- Our Minkowski metric signs are $g^{\mu\nu} = (+1, -1, -1, -1)$.
- Then, the metric tensor for $(+, -, 1, 2)$ coordinates is:

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{3}$$

- The covariant and contravariant indices are related by:

$$\begin{aligned} a_{\pm} &= a^{\mp}, \\ \mathbf{a}_{\perp} &= -\mathbf{a}^{\perp}. \end{aligned} \tag{4}$$

Basics:
Notations and Conventions

Scalar Product

- For two arbitrary four-vectors:

$$\begin{aligned} a_\mu b^\mu &= a_+ b^+ + a_- b^- + \mathbf{a}_\perp \mathbf{b}^\perp, \\ &= a^- b^+ + a^+ b^- - \mathbf{a}^\perp \mathbf{b}^\perp. \end{aligned} \tag{5}$$

Gauge Fixing

- Light-like vector n for the gauge fixing: $n \cdot A = A^+ = 0$

$$\begin{aligned} n_\mu &= (n_+, n_-, n_1, n_2) = (1, 0, 0, 0), \\ n^\mu &= (n^+, n^-, n^1, n^2) = (0, 1, 0, 0). \end{aligned} \tag{6}$$

Basics:
Ward-Takahashi Identities

- They relate various correlation functions in QFT;
- Result from gauge symmetry or other invariances;
- Quantum version of classical Noether's theorem.

• No electrons, N photons

$$(k_j)_{\mu_j} V_N^{\mu_1\mu_2\cdots\mu_N}(k_1, k_2, \cdots, k_N) = 0\,, \quad \forall j. \tag{7}$$

• Example: Photon's Vacuum Polarization Tensor $\Pi^{\mu\nu}(k)$

$$k_\mu \Pi_1^{\mu\nu}(k) \quad = \quad 0. \tag{8}$$

Basics:
Ward-Takahashi Identities

- Next, for $N = 2$ and $p - p' = k_1 + k_2 + \cdots + k_N$, we have

Two electrons, N photons

$$(k_j)_{\mu_j} V_N^{\mu_1\mu_2\cdots\mu_N}(p', p; \{k_j\}) = eS_{N-1}^{\mu_1\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_N}(p', p-k_j) - eS_{N-1}^{\mu_1\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_N}(p'+k_j, p). \quad (9)$$

Example: QED_4 Vertex $\Gamma_1^{\mu}(p', p, k)$

$$k_\mu \Gamma_1^{\mu}(p', p, k) = e\left\{S_0^{-1}(p') - S_0^{-1}(p)\right\}. \quad (10)$$

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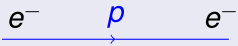
$N = 1$ Photon

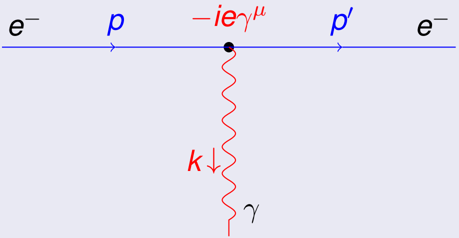
Tree-level Ward-Takahashi Identities

Figure 1: Electron Propagator



$$= S_0(p) = \frac{i}{\not{p} - m}$$

Figure 2: Vertex



$$= V_1^\mu(p', p, k)$$

$$\equiv \frac{i}{\not{p}' - m}(-ie\gamma^\mu)\frac{i}{\not{p} - m}$$

$N = 1$ Photon
Tree-level Ward-Takahashi Identities

The contraction of photon momentum k_μ with V_1^μ yields

$$k_\mu V_1^\mu(p', p, k) = -ie\frac{i}{\not{p}' - m}(k_\mu \gamma^\mu \equiv \not{k})\frac{i}{\not{p} - m}. \quad (11)$$

With momentum conservation, $k_\mu = p_\mu - p'_\mu$,

$$\not{k} = \not{p} - \not{p}' \equiv (\not{p} - m) - (\not{p}' - m). \quad (12)$$

$$k_\mu V_1^\mu(p', p, k) = \frac{-ie}{\not{p} - m} + \frac{ie}{\not{p}' - m} \equiv -eS_0(p, p) + eS_0(p', p').$$

$N = 1$ WTI

$$k_\mu V_1^\mu(p', p, k) = eS_0(p', p - k) - eS_0(p' + k, p). \quad (13)$$

$N = 1$ Photon
Tree-level Ward-Takahashi Identities

Rewriting

$$V_1^{\mu}(p', p, k) = \frac{i}{\not{p}' - m}(-ie\gamma^{\mu})\frac{i}{\not{p} - m} \equiv S_0(p')\Gamma_1^{\mu}(p', p, k)S_0(p)$$

It follows from Equation (13) that

$$S_0(p')k_{\mu}\Gamma_1^{\mu}(p', p, k)S_0(p) = eS_0(p') - eS_0(p). \quad (14)$$

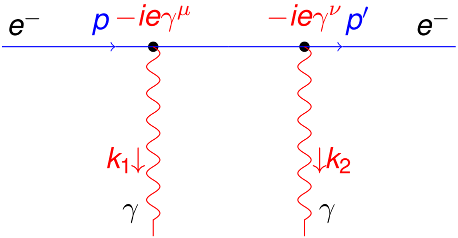
So, we can write alternatively

$N = 1$ WTI Alternative Form

$$k_{\mu}\Gamma_1^{\mu}(p', p, k) = eS_0^{-1}(p) - eS_0^{-1}(p'). \quad (15)$$

$N = 2$ Photons

Tree-level Ward-Takahashi Identities



$$+ \; (\mu \leftrightarrow \nu) \; = \; V_2^{\mu\nu}(p', p; k_1, k_2)$$

$N=2$ Photons

Tree-level Ward-Takahashi Identities

$$
\begin{aligned}
V_2^{\mu\nu}(p',p;k_1,k_2) &= \frac{i}{\not{p}'-m}(-ie\gamma^\mu)\frac{i}{\not{p}'-\not{k}_1-m}(-ie\gamma^\nu)\frac{i}{\not{p}-m} \\
&+ \frac{i}{\not{p}'-m}(-ie\gamma^\nu)\frac{i}{\not{p}+\not{k}_1-m}(-ie\gamma^\mu)\frac{i}{\not{p}-m}
\end{aligned}
$$

$N=2$

$$
k_{1\mu}V_2^{\mu\nu}(p',p;k_1,k_2) = eS_1^\nu(p',p-k_1;k_2) - eS_1^\nu(p'+k_1,p;k_2)
$$

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Covariant Case:
One-loop Radiative Correction

Now comes the $N = 1$ photon, one-loop case:

- From alternative form, Equation (15):

$$k_\mu \Gamma_1^\mu(p', p, k) = eS_0^{-1}(p) - eS_0^{-1}(p')\,, \;\; p' = p - k, \tag{16}$$

- where

$$\begin{aligned} \Gamma_1^\mu(p', p; k) &= i\left\{e\gamma^\mu + \Lambda_{1\ell}^\mu(p', p; k)\right\}, \\ S_0^{-1}(p) &= i\left\{\not{p} - m - \Sigma_{1\ell}(p)\right\}. \end{aligned} \tag{17}$$

Covariant Case:
One-loop Radiative Correction

- Then, left-hand-side of Equation (16) is

$$k_\mu \Gamma_1^\mu(p', p, k) = ie\not{k} + ik_\mu \Lambda_{1\ell}^\mu(p', p; k), \tag{18}$$

- while the right-hand-side of Equation (16) is given by

$$eS_0^{-1}(p) - eS_0^{-1}(p') = ie\not{k} + ie\left\{\Sigma_{1\ell}(p') - \Sigma_{1\ell}(p)\right\}. \tag{19}$$

Covariant Case:
One-loop Radiative Correction

1ℓ WTI

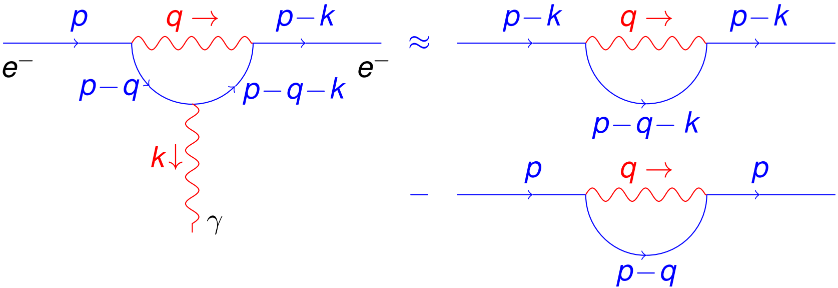
$$k_\mu \Lambda^\mu_{1\ell}(p', p; k) = e\left\{\Sigma_{1\ell}(p') - \Sigma_{1\ell}(p)\right\}. \tag{20}$$

where:

$$\begin{aligned}
\Lambda^\mu_{(1\ell)}(p,k) &= (-ie)^3 \int \frac{d^D q}{(2\pi)^D} T^\mu_\Lambda(p,k,q), \\
\Sigma_{(1\ell)}(p) &= (-ie)^2 \int \frac{d^D q}{(2\pi)^D} T_\Sigma(p,q), \\
T^\mu_\Lambda(p,k,q) &\equiv \gamma^\alpha \left[iS_0(p-k-q)\right] \gamma^\mu \left[iS_0(p-q)\right] \gamma^\beta \left[iD_{\alpha\beta}(q)\right], \\
T_\Sigma(p,q) &\equiv \gamma^\alpha \left[iS_0(p-q)\right] \gamma^\beta \left[iD_{\alpha\beta}(q)\right].
\end{aligned} \tag{21}$$

Covariant Case:
One-loop Radiative Correction

Diagrammatically...



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Light Front Case:
One-loop Radiative Correction

1ℓ WTI in LF

$$k_\mu \Lambda^\mu_{(1\ell)}(p;k) = e\left\{\Sigma_{(1\ell)}(p-k) - \Sigma_{(1\ell)}(p)\right\}, \quad p' = (p-k). \quad (22)$$

where now $\Lambda^\mu_{(1\ell)}(p;k)$ and $\Sigma_{(1\ell)}(p)$ are to be calculated using

LF Photon and Electron Propagators:

$$\begin{aligned} iD^{\rm LF}_{\alpha\beta}(q) &= \frac{-i}{(q^2+i\varepsilon)}\left\{g_{\alpha\beta} - \frac{q_\alpha n_\beta + q_\beta n_\alpha}{q^+} + \frac{q^2 n_\alpha n_\beta}{(q^+)^2}\right\}, \\ iS^{\rm LF}_0(p) &= \frac{i(\not p + m)}{(p^2 - m^2 + i\varepsilon)}. \end{aligned} \quad (23)$$

Light Front Case:
One-loop Radiative Correction

We can separate the contributions in two pieces:

Separation COV + LF

$$\begin{aligned} \Lambda^{\mu}_{(1\ell)}(p,k) &= \Lambda^{\mu[\mathrm{COV}]}_{(1\ell)}(p,k) + \Lambda^{\mu[\mathrm{LF}]}_{(1\ell)}(p,k), \\ \Sigma_{(1\ell)}(p) &= \Sigma^{\mathrm{COV}}_{(1\ell)}(p) + \Sigma^{\mathrm{LF}}_{(1\ell)}(p). \end{aligned} \tag{24}$$

Results for the COV part are standard:

$$\Lambda^{\mu[\mathrm{COV}]}_{(1\ell)}(p,k) = \gamma^{\mu} F_1^{(1\ell)}(q^2) + \frac{i}{2m}\sigma^{\mu\nu} q_{\nu} F_2^{(1\ell)}(q^2). \tag{25}$$

Light Front Case:
One-loop Radiative Correction - Vertex Part

For notational convenience, we write:

$$k_\mu \Lambda^{\mu[\mathrm{LF}]}_{(1\ell)}(p,k) \equiv L_a(p,p') + L_b(p,p'), \;\; p' = (p-k), \tag{26}$$

where

$$\begin{aligned} L_a(p,p') &\equiv -e^3 \{ 2\not{k} \left[Q_{[0]}(p) + Q_{[0]}(p') \right] \\ &\quad -2\not{k} \left[p^+ Q_{[+]}(p) + p'^+ Q_{[+]}(p') \right] \\ &\quad -\not{k}\gamma^+\gamma_\alpha Q^\alpha_{[+]}(p) - \gamma_\alpha\gamma^+\not{k} Q^\alpha_{[+]}(p') \} , \\ L_b(p,p') &\equiv 2e^3 \left\{ 2\not{k}\Theta_{[++]}(p,p') + \not{k}\gamma^+\gamma_\alpha \Theta'^\alpha_{[++]}(p,p') \right. \\ &\quad \left. +\gamma_\alpha\gamma^+\not{k}\Theta^\alpha_{[++]}(p,p') + k^+\gamma_\alpha\gamma^+\gamma_\beta \Theta^{\alpha\beta}_{[++]}(p,p') \right\} . \end{aligned}$$

Light Front Case:
One-loop Radiative Correction - Vertex Part

Q-type LF integrals:

$$
\begin{aligned}
Q_{[0]}(p) &= \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + i\varepsilon)\left[(p-q)^2 - m^2 + i\varepsilon\right]}, \\
Q_{[+]}(p) &= \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + i\varepsilon)\left[(p-q)^2 - m^2 + i\varepsilon\right]} \frac{1}{q^+}, \\
Q^{\alpha}_{[+]}(p) &= \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + i\varepsilon)\left[(p-q)^2 - m^2 + i\varepsilon\right]} \frac{q^\alpha}{q^+},
\end{aligned} \tag{27}
$$

Light Front Case:
One-loop Radiative Correction - Vertex Part

Θ-type LF integrals:

$$\begin{aligned}
\Theta_{[++]}(p,p') &= \int \frac{d^dq}{(2\pi)^d}\frac{1}{D(p')D(p)}\frac{(p'^+ - q^+)(p^+ - q^+)}{(q^+)^2},\\
\Theta'^{\alpha}_{[++]}(p,p') &= \int \frac{d^dq}{(2\pi)^d}\frac{1}{D(p')D(p)}\frac{(p'^+ - q^+)(q^\alpha)}{(q^+)^2},\\
\Theta^{\alpha}_{[++]}(p,p') &= \int \frac{d^dq}{(2\pi)^d}\frac{1}{D(p')D(p)}\frac{(p^+ - q^+)(q^\alpha)}{(q^+)^2},\\
\Theta^{\alpha\beta}_{[++]}(p,p') &= \int \frac{d^dq}{(2\pi)^d}\frac{1}{D(p')D(p)}\frac{(q^\alpha)(q^\beta)}{(q^+)^2}
\end{aligned} \tag{28}$$

$$D(p) = (p-q)^2 - m^2 + i\varepsilon$$

Light Front Case:
One-loop Radiative Correction - Vertex Part

In order to calculate the integrals, need

The Mandelstam-Leibbrandt Prescription:

$$\begin{aligned} \left(\frac{1}{q^+}\right) &\rightarrow \lim_{\epsilon\rightarrow 0}\frac{1}{q^+ + i\epsilon q^-}, \quad \text{or} \\ \left(\frac{1}{q^+}\right) &\rightarrow \lim_{\epsilon\rightarrow 0}\frac{q^-}{q^+q^- + i\epsilon}. \end{aligned} \tag{29}$$

Light Front Case:
One-loop Radiative Correction - Vertex Part

Important Dirac algebra relations in LF:

LF Dirac Gamma matrices:

$$
\begin{aligned}
(\gamma^+)^2 &= (\gamma^-)^2 = 0, \\
\gamma^-\gamma^+\gamma^- &= 2\gamma^-. \\
\gamma^+\gamma^-\gamma^+ &= 2\gamma^+. \\
\gamma^\mu\gamma^+\gamma^- + \gamma^-\gamma^+\gamma^\mu &= 2\gamma^\mu + 2g^{\mu+}\gamma^- - 2g^{\mu-}\gamma^+.
\end{aligned} \tag{30}
$$

Light Front Case:
One-loop Radiative Correction - Vertex Part

Final result for LHS vertex part is

The Vertex Part Result:

$$k_\mu \Lambda^{\mu[\mathrm{LF}]}_{(1\ell)}(p,k)\Big|_{div} = -2e^3 \left\{ \not{k} - k^+\gamma^- - k^-\gamma^+ \right\} I_{\mathrm{div}}. \tag{31}$$

.

where

$$I_{\mathrm{div}} = Q_{[0]}(p) = \frac{i\pi}{(2-d/2)} + f_{[0]}(p) + \mathcal{O}(2-d/2). \tag{32}$$

Light Front Case:
One-loop Radiative Correction - Electron Self-energy Part

For the RHS of WTI we need to calculate the electron self-energy part

$$\Sigma^{\mathrm{LF}}_{(1\ell)}(p) = -2e^2 \left\{ (\not{p} - m) Q_{[0]}(p) - \left(p^+ \gamma_\alpha + p_\alpha \gamma^+\right) Q^\alpha_{[+]}(p) + p^+ T_{[++]}(p) \right\}, \qquad (33)$$

where

T-type LF integral

$$T_{[++]}(p) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{\left[(p-q)^2 - m^2 + i\varepsilon\right] (q^+)^2} \qquad (34)$$

Light Front Case:
One-loop Radiative Correction - Electron Self-energy Part

The electron self-energy part results in

$$\Sigma^{\rm LF}_{(1\ell)}(p)\Big|_{\rm div} - = -2e^2 \left\{(\not{p} - m) - p^+\gamma^- - p^-\gamma^+\right\} I_{\rm div}. \qquad (35)$$

The RHS of WTI Electron Self-energy Part Result:

$$e\left\{\Sigma^{\rm LF}_{(1\ell)}(p')\Big|_{\rm div} - \Sigma^{\rm LF}_{(1\ell)}(p)\Big|_{\rm div}\right\} = -2e^3 \left\{\not{k} - k^+\gamma^- - k^-\gamma^+\right\} I_{\rm div}. \qquad (36)$$

which is the same result as we have in Equation (31)!

Summary

- The WTI for QED_4 in the LF is verified to one-loop order.
- We used the 3-term photon propagator for the LF gauge.
- Mandelstam-Leibbrandt prescription is OK for handling one-loop LF Feynman integrals.

- Outlook
  - Can WTI for QED_4 in the LF be verified for higher loops?
  - Is Mandelstam-Leibbrandt prescription OK for higher loops?

  Issues: $\dfrac{1}{q^+(p^+ - q^+)\cdots} \to ?$