Pion observables with the Minkowski Space Pion Model

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Main aspects

- **Non-perturbative hadronic physics**
- **Quark model propagador (with self energy)**
- **Running mass**
- **Fit lattice data**
- **Reproduce experimental pion data**

$\Rightarrow$ **Main references:**

- **E. Rojas, J. de Melo, B. El-Bennich, O. Oliveira and T. Frederico, JHEP 1310, 193 (2013)**
- **Clayton S. Mello, J.de Melo, T. Frederico, Physics Letters B 766 (2017) 86.**
- **M. Parappilly, P. Bowman, U. Heller, D. Leinweber, A. Williams and J. Zhang, Phys. Rev. D 73, 054504 (2006)**

Pion Model with Quark Self-energy

- **Pion is the Goldstone Boson $\Longleftrightarrow$ Dynamical Chiral Symmetry Breaking**
- **Bound state equations need consistent axial Ward vector identity**

$$\Phi(k,p) = S(\eta_1 p + k) S(\eta_2 p - k) \int \frac{dk'}{(2\pi)^4} \, \imath K(k,k',p) \Phi(k,p)$$

- **Källen-Lehman Spectral Representation:**

$$S(p) = \int_0^\infty d\gamma \frac{\rho(\gamma)}{p^2 - \gamma^2 + \imath\epsilon}$$

$$\begin{aligned}\Phi(k,p) \;\; &= \;\; \int_0^\infty d\gamma \frac{\rho(\gamma)}{(\eta_1 p + k)^2 - \gamma^2 + \imath\epsilon} \int_0^\infty d\gamma \frac{\rho(\gamma)}{(\eta_2 p - k)^2 - \gamma^2 + \imath\epsilon} \\ & \int \frac{d^4 k'}{(2\pi)^4} \imath K(k,k',p) \Phi(k',p)\end{aligned}$$

- **In the present work, we use the following for quark propagador**

$$S_F(k) = \imath\, Z(k^2) \left[\not{k} - M(k^2) + \imath\epsilon \right]^{-1}$$

- **Renormalization factor (simplification)** $\Longrightarrow \; Z(k^2) = 1$

The dressed quark propagator

$$S_F(k) = \imath \frac{\not{k} + M(k^2)}{(k^2 - M^2(k^2) + \imath\epsilon)}$$

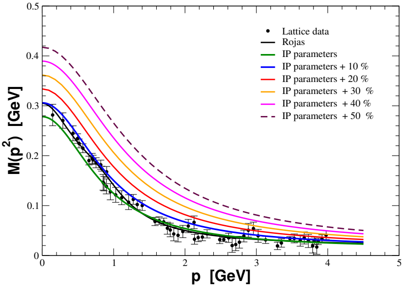
- **Quark Mass Running Parametrization:**

$$M(k^2) = m_0 - m^3 \left[k^2 - \lambda^2 + i\epsilon \right]^{-1}$$

- **Parameters (all in GeV):** $m_0 = 0.014$, $m = 0.574$ *and* $\lambda = 0.846$
- $M(k^2 = 0) = m_0 + \frac{m^3}{\lambda^2} = 0.278 \; GeV$
- **Ref.** **E. Rojas, J. P. B. C. de Melo, B. El-Bennich, O. Oliveira and T. Frederico, JHEP 1310, 193 (2013)**

%	m_0	m_q	m	λ	$M(0)$
0	0.0140	0.220	0.574	0.846	0.2781
10	0.0154	0.242	0.6314	0.9306	0.3061
15	0.0161	0.253	0.6601	0.9729	0.31996
20	0.0168	0.264	0.6888	1.0152	0.3389
25	0.0175	0.275	0.7175	1.0575	0.3478
30	0.0182	0.268	0.7462	1.0998	0.3671
35	0.0189	0.297	0.7749	1.1421	0.3751
40	0.0196	0.308	0.8036	1.1844	0.38952
45	0.0203	0.318	0.8323	1.2267	0.40343
50	0.0210	0.330	0.8610	1.269	0.41736

Tabela-I: **Values of the parameters utilized in the present model.**



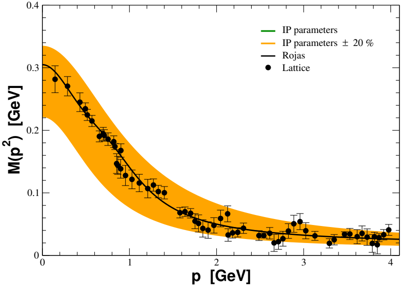


Figure: The running quark mass, as function of the momentum p, with the parameters utilized in the previus work (IP), and with the present new parameters calculations, also, compared with the lattice, and the calculations from Rojas et al.

- **Model with the quark propagator poles**

$$
\begin{aligned}
m_i^2 &= M^2(m_i^2) \\
m_i\left(m_i^2-\lambda^2\right) &= \pm\left[m_0\left(m_i^2-\lambda^2\right)-m^3\right]
\end{aligned}
$$

- **Denominator of the quark model propagator as factorized**

$$
S_F(k)=\imath\,\frac{\left(k^2-\lambda^2\right)^2(\not{k}+m_0)-\left(k^2-\lambda^2\right)\,m^3}{\prod_{i=1,3}(k^2-m_i^2+\imath\epsilon)}\,.
$$

The parameters above given (Initial parameters,IP),

$m_1=0.371\ GeV,\quad m_2=0.644\ GeV,$ **and** $m_3=0.954\ GeV$,

- **Feynman quark propagators decomposition**

$$S_F(k) = \imath \left[\frac{(k^2 - \lambda^2)^2 \not{k}}{\prod_{i=1,3}(k^2 - m_i^2 + \imath\epsilon)} \right] + \imath \left[\frac{(\lambda^2 - k^2) m^3}{\prod_{i=1,3}(k^2 - m_i^2 + \imath\epsilon)} + \frac{(k^2 - \lambda^2)^2 m_0}{\prod_{i=1,3}(k^2 - m_i^2 + \imath\epsilon)} \right]$$

- **Källen-Lehmann (KL) Spectral decomposition**

$$S_F(k) = \imath \left[A(k^2)\not{k} + B(k^2)\right]$$

Following expressions,

$$\begin{aligned} A(k^2) &= \int_0^\infty d\mu^2 \frac{\rho_A(\mu^2)}{k^2 - \mu^2 + \imath\varepsilon} \ , \\ B(k^2) &= \int_0^\infty d\mu^2 \frac{\rho_B(\mu^2)}{k^2 - \mu^2 + \imath\varepsilon} \end{aligned}$$

- **Spectral densities**

$$\rho_A(\mu^2) = -\frac{1}{\pi}\,\mathbf{Im}\left[A(\mu^2)\right] \ \ \textbf{and} \ \ \rho_B(\mu^2) = -\frac{1}{\pi}\,\mathbf{Im}\left[B(\mu^2)\right]$$

- **Positivity constraints**

$\mathcal{P}_a = \rho_A(\mu^2) \geq 0$ **and** $\mathcal{P}_b = \mu\,\rho_A(\mu^2) - \rho_B(\mu^2) \geq 0$.

- **From de Eq. for $S(k^2)$, we have**

$$
\begin{aligned}
A(k^2) &= \frac{\left(k^2-\lambda^2\right)^2}{\prod_{i=1,3}(k^2-m_i^2+\imath\epsilon)} \\
B(k^2) &= \frac{(\lambda^2-k^2)m^3}{\prod_{i=1,3}(k^2-m_i^2+\imath\epsilon)} + A(k^2)m_0
\end{aligned}
$$

- Partial fractions

$$
\begin{aligned}
\frac{\left(k^2-\lambda^2\right)^2}{\prod_{i=1,3}(k^2-m_i^2)} &= \frac{D_1}{(k^2-m_1^2)}+\frac{D_2}{(k^2-m_2^2)}+\frac{D_3}{(k^2-m_3^2)} \\
&= \frac{D_1(k^2-m_2^2)(k^2-m_3^2)+D_2(k^2-m_1^2)(k^2-m_3^2)+D_3(k^2-m_1^2)(k^2-m_2^2)}{\prod_{i=1,3}(k^2-m_i^2)} \\
&= \frac{(D_1+D_2+D_3)k^4-\left[(m_3^2+m_2^2)D_1+(m_3^2+m_1^2)D_2+(m_1^2+m_2^2)D_3\right]k^2}{\prod_{i=1,3}(k^2-m_i^2)}+ \\
&+\frac{(D_1 m_2^2 m_3^2+D_2 m_3^2 m_1^2+D_3 m_2^2 m_1^2)}{\prod_{i=1,3}(k^2-m_i^2)}
\end{aligned}
$$

$$
\left\{
\begin{array}{l}
D_1+D_2+D_3 = 1 \\
(m_3^2+m_2^2)D_1+(m_3^2+m_1^2)D_2+(m_1^2+m_2^2)D_3 = 2\lambda^2 \\
D_1 m_2^2 m_3^2+D_2 m_3^2 m_1^2+D_3 m_2^2 m_1^2 = \lambda^4
\end{array}
\right.
$$

$$\frac{k^2 m^3 - \lambda^2 m^3}{\prod_{i=1,3}(k^2 - m_i^2)} = \frac{-E_1}{(k^2 - m_1^2)} + \frac{-E_2}{(k^2 - m_2^2)} + \frac{-E_3}{(k^2 - m_3^2)} + \frac{D_1 m_0}{(k^2 - m_1^2)} + \frac{D_2 m_0}{(k^2 - m_2^2)} + \frac{D_3 m_0}{(k^2 - m_3^2)}$$

$$= \frac{D_1 m_0 - E_1}{(k^2 - m_1^2)} + \frac{D_2 m_0 - E_2}{(k^2 - m_2^2)} + \frac{D_3 m_0 - E_3}{(k^2 - m_3^2)}$$

$$= \frac{(D_1 m_0 - E_1)(k^2 - m_2^2)(k^2 - m_3^2) + (D_2 m_0 - E_2)(k^2 - m_1^2)(k^2 - m_3^2)}{\prod_{i=1,3}(k^2 - m_i^2)} +$$

$$+ \frac{(D_3 m_0 - E_3)(k^2 - m_1^2)(k^2 - m_2^2)}{\prod_{i=1,3}(k^2 - m_i^2)}$$

$$= \frac{[(D_1 m_0 + D_2 m_0 + D_3 m_0) - (E_1 + E_2 + E_3)]k^4}{\prod_{i=1,3}(k^2 - m_i^2)} +$$

$$+ \frac{[-(m_2^2 + m_3^2)D_1 m_0 - (m_1^2 + m_3^2)D_2 m_0 - (m_1^2 + m_2^2)D_3 m_0]k^2}{\prod_{i=1,3}(k^2 - m_i^2)} +$$

$$+ \frac{[(m_2^2 + m_3^2)E_1 + (m_1^2 + m_3^2)E_2 + (m_1^2 + m_2^2)E_3]k^2}{\prod_{i=1,3}(k^2 - m_i^2)}$$

$$+ \frac{(D_1 m_0 - E_1)m_2^2 m_3^2 + (D_2 m_0 - E_2)m_1^2 m_3^2 + (D_3 m_0 - E_3)m_1^2 m_2^2}{\prod_{i=1,3}(k^2 - m_i^2)}$$

$$\begin{cases} \blacktriangleright\ E_1 + E_2 + E_3 = (D_1 + D_2 + D_3)m_0 \\ \blacktriangleright\ (m_3^2 + m_2^2)E_1 + (m_3^2 + m_1^2)E_2 + (m_1^2 + m_2^2)E_3 = \\ \quad = m^3 + (m_2^2 + m_3^2)D_1 m_0 + (m_1^2 + m_3^2)D_2 m_0 + (m_1^2 + m_2^2)D_3 m_0 \\ \blacktriangleright\ m_2^2 m_3^2 E_1 + m_3^2 m_1^2 E_2 + m_2^2 m_1^2 E_3 = \\ \quad = \lambda^2 m^3 + D_1 m_0 m_2^2 m_3^2 + D_2 m_0 m_1^2 m_3^2 + D_3 m_0 m_1^2 m_2^2 \end{cases}$$

$$D_1 = 1.49, \qquad D_2 = -0.594 \quad \textbf{and} \qquad D_3 = -0.094$$

- **We can write spectral density for $A(k^2)$**

$$\int_0^\infty d\mu^2 \frac{\rho_A(\mu^2)}{k^2 - \mu^2} = \sum_{i=1}^{3} \frac{D_i}{k^2 - m_i^2} = \sum_{i=1}^{3} \int_0^\infty d\mu^2 \; \frac{D_i \; \delta(\mu^2 - m_i^2)}{k^2 - \mu^2} \tag{1}$$

$$E_1 = 0.486687, \qquad E_2 = -0.381616 \quad \text{and} \qquad E_3 = -0.0910713$$

- **Calculate the spectral density $\rho_B(\mu^2)$**

$$\int_0^\infty d\mu^2 \frac{\rho_B(\mu^2)}{k^2 - \mu^2} = \sum_{i=1}^{3} \frac{E_i}{k^2 - m_i^2} = \sum_{i=1}^{3} \int_0^\infty d\mu^2 \; \frac{E_i \; \delta(\mu^2 - m_i^2)}{k^2 - \mu^2} \tag{2}$$

- **Pion-quark-antiquark vertex $\Gamma_\pi(k,P)$ / Bethe-Salpeter amplitude (BSA)**

$$\Gamma_\pi(k;P) = \gamma_5[\imath E_\pi(k;P) + \not{P} F_\pi(k;P) + k^\mu P_\mu \not{k} G_\pi(k;P) + \sigma_{\mu\nu} k^\mu P^\nu H_\pi(k;P)]$$

- **Chiral Limit: $m_\pi = 0$, also current quark mass vanishes**

$$f_\pi E_\pi(k,P) = \frac{M(k)}{\sqrt{Z(k^2)}}$$

- f_π **: Electroweak pion decay constant**
- **The model**

$$\Gamma_\pi(k,p) = \imath \mathcal{N} \gamma_5 M(k^2)|_{m_0=0} \ , \ Z(k^2) = 1$$

- $\mathcal{N} \implies$ **Normalization constante:** $F_\pi(0) = 1$

The present model for the pion BSA

$$\Psi_\pi(k;P) = S_F(k+P/2)\,\Gamma_\pi(k;P)\,S_F(k-P/2)$$

- **Other examples for pion vertex:**

$$\Lambda_\pi(k,p) = \mathcal{N}\left[\frac{1}{k^2 - m_R^2 + \imath\epsilon} + \frac{1}{((p-k)^2 - m_R^2 + \imath\epsilon)}\right]$$

- **Ref.** de Melo, Frederico, Pace and Salmé Nucl. Phys. A707, 399 (2002)

ibid., Braz. J. Phys. 33, 301 (2003)

$$\Lambda_\pi(k,p) = \mathcal{N}\left[\frac{1}{(k^2 - m_R^2 + \imath\epsilon)}\frac{1}{((p-k)^2 - m_R^2 + \imath\epsilon)}\right]$$

- **Ref.** C. Fanelli, E. Pace, G. Romanelli, G. Salme and M. Salmistraro, Eur. Phys. J. C 76, no. 5, 253 (2016)

- **Useful identity**

$$\frac{1}{((k+\frac{p}{2})^2-\mu'^2+\imath\epsilon)(k^2-\lambda^2+\imath\epsilon)(k-\frac{p}{2})^2-\mu^2+\imath\epsilon)} = \\ = \int_{-\infty}^{+\infty} d\gamma \int_{-1}^{1} dz \frac{g(\gamma,z;\mu',\mu,p)}{(k^2+z\,k\cdot p+\gamma+\imath\epsilon)^3}$$

with,

$$g(\gamma,z;\mu',\mu,p) = \\ = \frac{\theta(\alpha)\theta(1-\alpha)}{\frac{1}{2}-\alpha}\left[\theta(1-2\alpha-z)\theta(z)-\theta(z-1+2\alpha)\theta(-z)\right]$$

and,

$$\alpha = \frac{\frac{p^2}{4}+\lambda^2-\mu^2-z^{-1}(\lambda^2+\gamma)}{\mu^2-\mu'^2+2z^{-1}(\lambda^2+\gamma)}$$

- **Pion BS Amplitude**

$$\psi_\pi(k;P) \;=\; -\left[A(k_q^2)\,\not{k}_q + B(k_q^2)\right] \frac{\mathcal{N}\,\gamma_5\,m^3}{k^2-\lambda^2+\imath\epsilon} \left[A(k_{\overline{q}}^2)\,\not{k}_{\overline{q}} + B(k_{\overline{q}}^2)\right]$$

- **with** $k_q = k + P/2$ **and** $k_{\overline{q}} = k - P/2$

Integral representation for BSA

$$\psi_\pi(k;P) = \gamma_5\,\chi_1(k,P) + \not{k}_q\,\gamma_5\,\chi_2(k,P) + \gamma_5\,\not{k}_{\overline{q}}\,\chi_3(k,P) + \not{k}_q\,\gamma_5\,\not{k}_{\overline{q}}\,\chi_4(k,P)$$

here,

$$\chi_i(k,P) = \int_{-\infty}^{+\infty} d\gamma \int_{-1}^{1} dz \frac{g_i(\gamma,z;p)}{(k^2 + z\,k\cdot p + \gamma + \imath\epsilon)^3}$$

The Nakanishi weight functions are,

$$g_i(\gamma,z;p) = -\mathcal{N} \int_0^\infty d\mu^2 \int_0^\infty d\mu'^2 \rho_{C_i'}(\mu'^2)\rho_{C_i}(\mu^2)\,g(\gamma,z;\mu',\mu,p)$$

• $(C_1', C_1) = (B,B)$, $(C_2', C_2) = (A,B)$, $(C_3', C_3) = (B,A)$ **and** $(C_4', C_4) = (A,A)$

Electroweak decay constant

Partially conserved axial-vector current

$$\langle 0|A_i^\mu|\pi_j\rangle = \imath P^\mu f_\pi \delta_{ij}$$

• **we take** $A_i^\mu = \bar{q}\gamma^\mu\gamma^5\frac{\tau_i}{2}q$ **and our model for the pion BSA**
After the color and isospin algebra, we have the decay constant:

$$P^\mu f_\pi = N_c \int \frac{d^4k}{(2\pi)^4}\mathbf{Tr}\left[\gamma^\mu\gamma^5\psi_\pi(k;P)\right]$$

• N_c **number of colors.**
• **Integration on** k^-**and plus component of the axial current** $(A_i^+ = A_i^0 + A_i^3)$

Pion electromagnetic form factor: Space like region

- **The pion-photon vertex is:**

$$-\imath\,\Gamma^{\mu}_{\pi}(P,P';q) \equiv \left\langle \pi(P') \right| J^{\mu} \left| \pi(P) \right\rangle = (P+P')^{\mu} F_{\pi}(Q^2)$$

$\Rightarrow$ **Where $Q^2 = -q^2$, $|\pi(P)\rangle$ is the pion state and J^{μ} the electromagnetic current operator**

The building blocks of the impulse approximation are:
(i) the nonperturbative quark-photon vertex $\Gamma^{\mu}_{q}(k',k;q)$ with $q = k' - k$, namely the dressed quark current
(ii) the dressed quark and antiquark propagators;
(iii) the pion-quark vertex.

• **Impulse approximation, meson-photon vertex is the sum of two terms**

$$\Gamma^{\mu}_{\pi^+}(P,P';q) = \hat{Q}_u \Gamma^{\mu}_{\pi^+,u}(P,P';q) + \hat{Q}_{\bar{d}} \Gamma^{\mu}_{\pi^+,\bar{d}}(P,P';q)$$

• **Isospin symmetric limit ($m_u = m_{\bar{d}}$)**

$$\Gamma^{\mu}_{\pi,u}(P,P';q) = \Gamma^{\mu}_{\pi,\bar{d}}(P,P';q)$$

• **In the present model there is only one independent quark electromagnetic vertex**

- **Dressed quark current operator**
- **To satisfy**

$$q_\mu \Gamma^\mu_{\pi^+,u}(P,P';q) = 0$$

- **The quark-photon vertex satisfies the Ward-Takahashi identity (WTI)**

$$q_\mu \Gamma^\mu_q(p',p;q) = S_F^{-1}(p') - S_F^{-1}(p)$$

- **The quark-photon vertex**

$$-\imath\, \Gamma^\mu_q(p',p;q) = \gamma^\mu - \frac{m^3(p'+p)^\mu}{\mathcal{D}(p'^2)\mathcal{D}(p^2)}$$

- **Denominator is defined by the function**

$$\mathcal{D}(p^2) = \left(p^2 - \lambda^2 + i\epsilon\right)$$

Observables: Results

%	m_q	$< r^2 >$ $[fm^2]$	f_π [MeV]
0	0.220	0.6720	89.840
10	0.242	0.6115	90.015
15	0.253	0.5853	90.089
20	0.264	0.5610	90.157
25	0.275	0.5389	90.217
30	0.268	0.5184	90.273
35	0.297	0.4992	90.324
40	0.308	0.4815	90.366
45	0.318	0.4650	90.407
50	0.330	0.4490	90.441

Tabela-II: Pion observables, electromagnetic radius and weak decay.

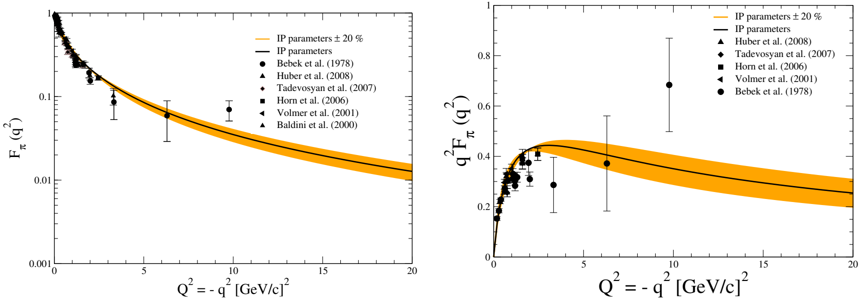


Figure: The pion electromagnetic form factor calculated in the present model and compared with the experimental data. In the present work, we evaluated the effects in changing the parameters by ±20 %.

Preliminary calculation with a complete basis

- BSA

$$\psi_\pi(k;P) = \gamma_5\,\chi_1(k,P) + \not{k}_q\,\gamma_5\,\chi_2(k,P) + \gamma_5\,\not{k}_{\overline{q}}\,\chi_3(k,P) + \not{k}_q\,\gamma_5\,\not{k}_{\overline{q}}\,\chi_4(k,P)$$

where

$$\chi_i(k,P) = \int_0^{+\infty} d\gamma \int_{-1}^{1} dz \frac{g_i(\gamma,z;p)}{\left(k^2 + z\,k\cdot p + \gamma + \imath\epsilon\right)^3}$$

- **Nakanishi weight functions**

$$g_i(\gamma,z;M) = -\mathcal{N} \int_0^{\infty} d\mu^2 \int_0^{\infty} d\mu'^2 \rho_{C_i'}(\mu'^2)\rho_{C_i}(\mu^2)\,g(\gamma,z;\mu',\mu,M)$$

- **The integral representation identity are useful**

$$\frac{1}{((k+\frac{p}{2})^2-\mu'^2+\imath\epsilon)(k^2-\lambda^2+\imath\epsilon)(k-\frac{p}{2})^2-\mu^2+\imath\epsilon)} = \\ = \int_0^{+\infty} d\gamma \int_{-1}^{1} dz \frac{g(\gamma,z;\mu',\mu,p)}{(k^2+z\,k\cdot p+\gamma+\imath\epsilon)^3}$$

where

$$g(\gamma,z;\mu',\mu,p) = \\ = \frac{\theta(\alpha)\theta(1-\alpha)}{\frac{1}{2}-\alpha}\left[\theta(1-2\alpha-z)\theta(z)-\theta(z-1+2\alpha)\theta(-z)\right]$$

and

$$\alpha = \frac{\frac{p^2}{4}+\lambda^2-\mu^2-z^{-1}(\lambda^2+\gamma)}{\mu^2-\mu'^2+2z^{-1}(\lambda^2+\gamma)}$$

- **Comparing, term by term, both members of the above equation**

$$\begin{cases} \gamma_5\,\chi_1(k,P) = -\mathcal{N}\gamma_5\; B(k_q^2)\;\frac{1}{k^2-\lambda^2+\imath\epsilon}\;B(k_{\overline{q}}^2); \\ \not{k}_q\;\gamma_5\,\chi_2(k,P) = -\mathcal{N}\;\not{k}_q\gamma_5\;A(k_q^2)\frac{1}{k^2-\lambda^2+\imath\epsilon}\;B(k_{\overline{q}}^2) \\ \gamma_5\;\not{k}_{\overline{q}}\;\chi_3(k,P) = -\mathcal{N}\;\gamma_5\;\not{k}_{\overline{q}}\;B(k_q^2)\;\frac{1}{k^2-\lambda^2+\imath\epsilon}\;A(k_{\overline{q}}^2) \\ \not{k}_q\,\gamma_5\,\not{k}_{\overline{q}}\,\chi_4(k,P) = -\mathcal{N}\;\not{k}_q\;\gamma_5\;\not{k}_{\overline{q}}\;A(k_q^2)\;\frac{1}{k^2-\lambda^2+\imath\epsilon}\;A(k_{\overline{q}}^2) \end{cases}$$

- **After some algebra**

$$
\begin{aligned}
\chi_1(k,P) &= \int_0^{\infty} d\gamma \int_{-1}^{1} dz \frac{1}{\left[k^2 + z\, k\cdot P + \gamma + \imath\epsilon\right]^3} \times (-\mathcal{N}) \\
\times & \Big[(E_1 + m_0 D_1)^2 F_{11} + (E_1 + m_0 D_1)(E_2 + m_0 D_2) F_{12} + (E_1 + m_0 D_1)(E_3 + m_0 D_3) F_{13} \\
+ & (E_2 + m_0 D_2)(E_1 + m_0 D_1) F_{21} + (E_2 + m_0 D_2)^2 F_{21} + (E_2 + m_0 D_2)(E_3 + m_0 D_3) F_{23} \\
+ & (E_3 + m_0 D_3)(E_1 + m_0 D_1) F_{31} + (E_3 + m_0 D_3)(E_2 + m_0 D_2) F_{32} + (E_3 + m_0 D_3)^2 F_{33}\Big]
\end{aligned}
$$

- We can identify the functions $F(\gamma, z\,; \mu'^2, \mu, M)$

$$F_{11} = \frac{\theta(\alpha_{11})\theta(1-\alpha_{11})}{\left(\frac{1}{2}-\alpha_{11}\right)}\left[\theta(1-2\alpha_{11}-z)\theta(z)-\theta(z-1+2\alpha_{11})\theta(-z)\right];$$

$$F_{12} = \frac{\theta(\alpha_{12})\theta(1-\alpha_{12})}{\left(\frac{1}{2}-\alpha_{12}\right)}\left[\theta(1-2\alpha_{12}-z)\theta(z)-\theta(z-1+2\alpha_{12})\theta(-z)\right];$$

$$F_{13} = \frac{\theta(\alpha_{13})\theta(1-\alpha_{13})}{\left(\frac{1}{2}-\alpha_{13}\right)}\left[\theta(1-2\alpha_{13}-z)\theta(z)-\theta(z-1+2\alpha_{13})\theta(-z)\right];$$

$$F_{21} = \frac{\theta(\alpha_{21})\theta(1-\alpha_{21})}{\left(\frac{1}{2}-\alpha_{21}\right)}\left[\theta(1-2\alpha_{21}-z)\theta(z)-\theta(z-1+2\alpha_{21})\theta(-z)\right];$$

$$F_{22} = \frac{\theta(\alpha_{22})\theta(1-\alpha_{22})}{\left(\frac{1}{2}-\alpha_{22}\right)}\left[\theta(1-2\alpha_{22}-z)\theta(z)-\theta(z-1+2\alpha_{22})\theta(-z)\right];$$

$$F_{23} = \frac{\theta(\alpha_{23})\theta(1-\alpha_{23})}{\left(\frac{1}{2}-\alpha_{23}\right)}\left[\theta(1-2\alpha_{23}-z)\theta(z)-\theta(z-1+2\alpha_{23})\theta(-z)\right];$$

$$F_{32} = \frac{\theta(\alpha_{32})\theta(1-\alpha_{32})}{\left(\frac{1}{2}-\alpha_{32}\right)}\left[\theta(1-2\alpha_{32}-z)\theta(z)-\theta(z-1+2\alpha_{32})\theta(-z)\right];$$

$$F_{33} = \frac{\theta(\alpha_{33})\theta(1-\alpha_{33})}{\left(\frac{1}{2}-\alpha_{33}\right)}\left[\theta(1-2\alpha_{33}-z)\theta(z)-\theta(z-1+2\alpha_{33})\theta(-z)\right],$$

and the respective $\alpha' s$

$$\alpha_{11} = \frac{M^2/4 - \lambda^2 - m_1^2 - z^{-1}(\lambda^2+\gamma)}{m_1^2 - m_1^2 + 2z^{-1}(\lambda^2+\gamma)} \qquad \alpha_{12} = \frac{M^2/4 - \lambda^2 - m_2^2 - z^{-1}(\lambda^2+\gamma)}{m_2^2 - m_1^2 + 2z^{-1}(\lambda^2+\gamma)}$$

$$\alpha_{13} = \frac{M^2/4 - \lambda^2 - m_3^2 - z^{-1}(\lambda^2+\gamma)}{m_3^2 - m_1^2 + 2z^{-1}(\lambda^2+\gamma)} \qquad \alpha_{21} = \frac{M^2/4 - \lambda^2 - m_1^2 - z^{-1}(\lambda^2+\gamma)}{m_1^2 - m_2^2 + 2z^{-1}(\lambda^2+\gamma)}$$

$$\alpha_{22} = \frac{M^2/4 - \lambda^2 - m_2^2 - z^{-1}(\lambda^2+\gamma)}{m_2^2 - m_2^2 + 2z^{-1}(\lambda^2+\gamma)} \qquad \alpha_{23} = \frac{M^2/4 - \lambda^2 - m_3^2 - z^{-1}(\lambda^2+\gamma)}{m_3^2 - m_2^2 + 2z^{-1}(\lambda^2+\gamma)}$$

$$\alpha_{31} = \frac{M^2/4 - \lambda^2 - m_1^2 - z^{-1}(\lambda^2+\gamma)}{m_1^2 - m_3^2 + 2z^{-1}(\lambda^2+\gamma)} \qquad \alpha_{32} = \frac{M^2/4 - \lambda^2 - m_2^2 - z^{-1}(\lambda^2+\gamma)}{m_2^2 - m_3^2 + 2z^{-1}(\lambda^2+\gamma)}$$

$$\alpha_{33} = \frac{M^2/4 - \lambda^2 - m_3^2 - z^{-1}(\lambda^2+\gamma)}{m_3^2 - m_3^2 + 2z^{-1}(\lambda^2+\gamma)},$$

One can apply Nakanishi integral representation (NIR) to the Bethe-Salpeter amplitude, $\chi_1(k,P)$, which is a scalar function, below,

$$\chi_1(k,P) = \int_{-\infty}^{+\infty} d\gamma \int_{-1}^{1} dz \frac{\mathcal{G}_1(\gamma, z; M)}{\left[k^2 + z\, k \cdot P + \gamma + \imath\epsilon\right]^3}$$

with this, we can define:

$$\boxed{\mathcal{G}_1(\gamma, z; P) = (-\mathcal{N}) \sum_{i=1}^{3} \sum_{j=1}^{3} (E_i + m_0 D_i)(E_j + m_0 D_j) F_{ij}}$$

Working with the second equation of the system

$$\chi_2(k,P) = A(k_q^2)\frac{(-\mathcal{N})}{k^2-\lambda^2+\imath\epsilon}\ B(k_{\overline{q}}^2)$$

$$\chi_2(k,P) = \int_{-\infty}^{+\infty} d\gamma \int_{-1}^{1} dz \frac{\mathcal{G}_2(\gamma,z;M)}{\left[k^2 + z\,k\cdot P + \gamma + \imath\epsilon\right]^3}$$

- where we defined,

$$\begin{aligned}
\mathcal{G}_2(\gamma,z;M) &= (-\mathcal{N}) \\
&\times [D_1(E_1+m_0D_1)F_{11} + D_1(E_2+m_0D_2)F_{12} + D_1(E_3+m_0D_3)F_{13} \\
&+ D_2(E_1+m_0D_1)F_{21} + D2(E_2+m_0D_2)F_{22} + D_2(E_3+m_0D_3)F_{23} \\
&+ D_3(E_1+m_0D_1)F_{31} + D3(E_2+m_0D_2)F_{32} + D_3(E_3+m_0D_3)F_{33}]
\end{aligned}$$

$$\boxed{\mathcal{G}_2(\gamma,z;M) = (-\mathcal{N})\sum_{i=1}^{3}\ \sum_{j=1}^{3} D_i\ (E_j+m_0D_j)F_{ij}}$$

- For third amplitude

$$\chi_3(k,P) = \int_{-\infty}^{+\infty} d\gamma \int_{-1}^{1} dz \frac{\mathcal{G}_3(\gamma,z;M)}{\left[k^2 + z\,k\cdot P + \gamma + \imath\epsilon\right]^3}$$

- So Nakanish's third weight function

$$\boxed{\mathcal{G}_3(\gamma,z;M) = (-\mathcal{N}) \sum_{i=1}^{3}\sum_{j=1}^{3}\ (E_i + m_0 D_i) D_j\ Fij}$$

- **With the same procedure, we have the final for $\chi_4(k,P)$**

$$\chi_4(k,P) = -\mathcal{N}\; A(k_q^2)\; \frac{1}{k^2-\lambda^2+\imath\epsilon}\; A(k_{\bar{q}}^2)$$

- **Inserting the Eq. to $A(k)$, and use the identity**

$$\begin{aligned}
\chi_4(k,P) &= \int_0^\infty d\mu'^2 \frac{\rho_A(\mu'^2)}{[(k+\frac{P}{2})^2-\mu'^2+\imath\epsilon]} \frac{(-\mathcal{N})}{[k^2-\lambda^2+\imath\epsilon]} \int_0^\infty d\mu^2 \frac{\rho_A(\mu^2)}{[(k-\frac{P}{2})^2-\mu^2+\imath\epsilon]} \\
&= \int_0^\infty d\mu'^2 \frac{\sum_{i=1}^{3} D_i\;\delta(\mu'^2-m_i^2)]}{[(k+\frac{P}{2})^2-\mu'^2+\imath\varepsilon]} \frac{(-\mathcal{N})}{[k^2-\lambda^2+\imath\epsilon]} \int_0^\infty d\mu^2 \frac{\sum_{i=1}^{3} D_i\;\delta(\mu^2-m_i^2)]}{[(k+\frac{P}{2})^2-\mu^2+\imath\varepsilon]} \\
&= \frac{\sum_{i=1}^{3} D_i\;\theta(m_i^2)}{[(k+\frac{P}{2})^2-m_i^2+\imath\varepsilon]} \frac{(-\mathcal{N})}{[k^2-\lambda^2+\imath\epsilon]} \frac{\sum_{i=1}^{3} D_i\;\theta(m_i^2)}{[(k-\frac{P}{2})^2-m_i^2+\imath\varepsilon]},
\end{aligned}$$

- **Using Feynman's tricks**

$$\chi_4(k,P) = \int_{-\infty}^{+\infty} d\gamma \int_{-1}^{1} dz \frac{\mathcal{G}_4(\gamma,z;M)}{[k^2+z\,k\cdot P+\gamma+\imath\epsilon]^3}$$

- **The fourth weight function of Nakanish**

$$\mathcal{G}_4(\gamma, z; M) = (-\mathcal{N}) \sum_{i=1}^{3} \sum_{j=1}^{3} \; D_i D_j \; F_{ij}$$

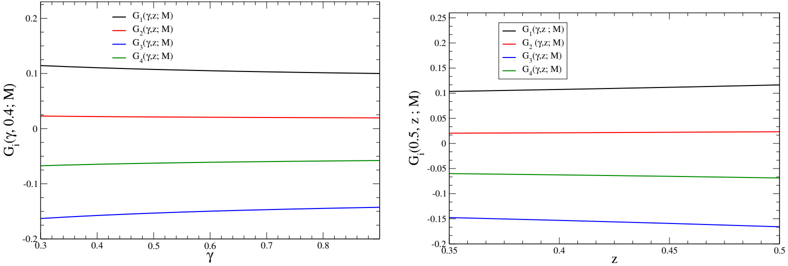


Figure: Nakanishi weight functions $\mathcal{G}_i(\gamma, z; M)$.

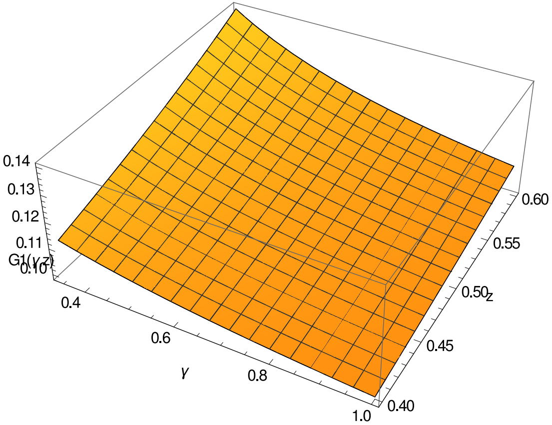


Figure: $G_1(\gamma, z, M)$

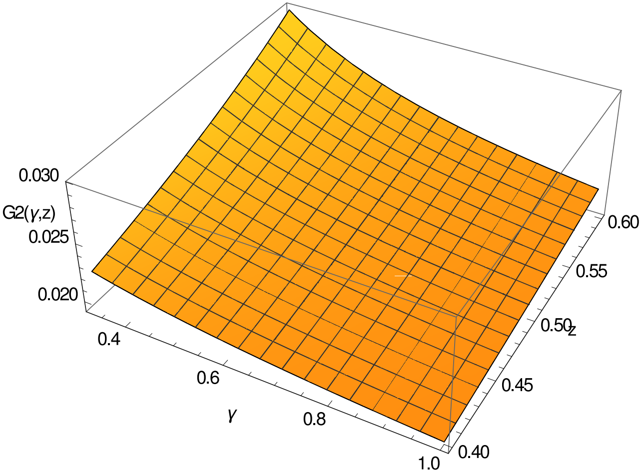


Figure: $G_2(\gamma, z, M)$

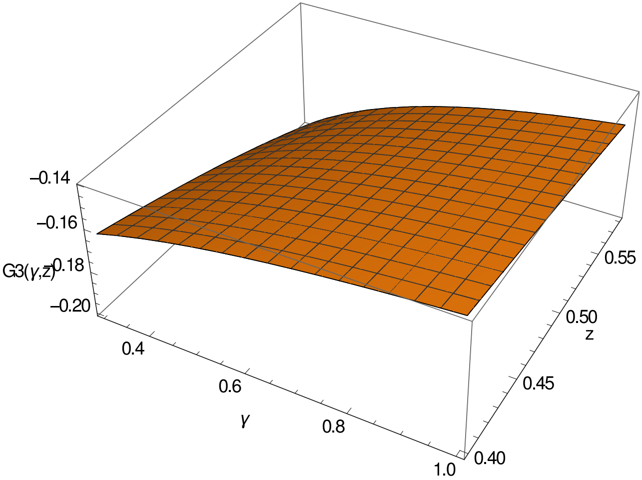


Figure: $G_3(\gamma, z, M)$

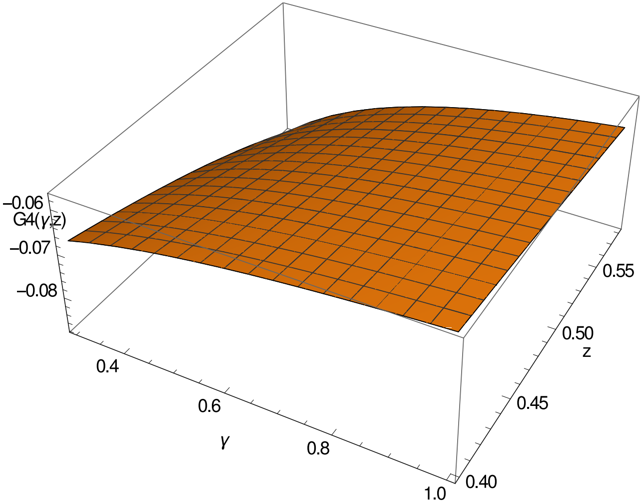


Figure: $G_4(\gamma, z, M)$

Final remarks

- **Light-front approach to Beth-Salpeter Amplitudes**
- **Self-energies, vertex corrections, Landau gauge**
- **Some ingredients from Lattice QCD**
- **A method for solving the fermionic BSE with Nakanishi Integral Representation (NIR)**
- **Connection with data**

Next

- **PDF's and DA's**
- **Kaon**
- **Vector mesons: ρ-meson**

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