Non-perturbative study of the three-body system using the Bethe-Salpeter approach

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Introduction

- Understanding the structure of non-perturbative few-body systems, from an fundamental point of view is important for applications in hadron physics, e.g. for studies of the nucleon.
- One important aspect is to obtain a reliable solution directly in Minkowski space, so that dynamical observables such as form factor can be calculated.
- In this talk, the solutions of the Bethe-Salpeter equation for a bound-state system of three bosons, bounded through a (two-body) zero range interaction, using three different approaches are discussed:
  - LF projection, i.e. only retaining the valence component, in Minkowski space.
  - Solution of the BS equation in Euclidean space, through Wick rotation
  - Solution of the BS equation in Minkowski space by direct integration.

- Three-body Bethe-Salpeter equation [1]:

$$v(q,p) = 2\mathrm{i}F(M_{12}) \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\epsilon} \frac{\mathrm{i}}{(p-q-k)^2 - m^2 + i\epsilon} v(k,p)$$

- Equal-mass case, bare propagators.
- $v(q,p)$ is one of the Faddeev components of the total vertex function.
- $F(M_{12})$: two-body scattering amplitude characterized by scattering length a and $M_{12}^2 = (p-q)^2$.
- 1) $a < 0$: Borromean system, no two-body bound state, 2) $a > 0$: two-body bound state exists.

[1] T. Frederico, PLB 282 (1992) 409

- LF equation:
  - After the LF projection, i.e. introducing $k_\pm = k_0 \pm k_z$ and integrating over k_-, one obtains the three-body LF equation [1, 2]:

$$\Gamma(k_\perp, x) = \frac{F(M_{12})}{(2\pi)^3} \int_0^{1-x} \frac{dx'}{x'(1-x-x')} \int_0^\infty \frac{d^2k'_\perp}{M_0^2 - M_3^2} \Gamma(k'_\perp, x')$$

with $M_0^2 = (k'^2_\perp + m^2)/x' + (k^2_\perp + m^2)/x + ((k'_\perp + k_\perp)^2 + m^2)/(1-x-x')$

- Euclidean BS equation:
  - Through a change of variables $k = k' + \frac{p}{3}$ and $q = q' + \frac{p}{3}$, and a subsequent Wick rotation [3]:

$$v_{\mathrm{E}}(q'_4, q'_v) = \frac{2F(-M'^2_{12})}{(2\pi)^3} \int_{-\infty}^\infty dk'_4 \int_0^\infty \frac{dk'_v \Pi(q'_4, q'_v, k'_4, k'_v)}{(k'_4 - \frac{\mathrm{i}}{3}M_3)^2 + k'^2_v + m^2} v_{\mathrm{E}}(k'_4, k'_v),$$

with $M'^2_{12} = (\frac{2}{3}\mathrm{i}M_3 + q'_4)^2 + q'^2_v$. The kernel Π is here given by

$$\Pi(q'_4, q'_v, k'_4, k'_v) = \frac{k'_v}{2q'_v} \log \frac{(k'_4 + q'_4 + \frac{i}{3}M_3)^2 + (q'_v + k'_v)^2 + m^2}{(k'_4 + q'_4 + \frac{i}{3}M_3)^2 + (q'_v - k'_v)^2 + m^2}. \qquad (1)$$

- Both the equations can be solved with standard methods, e.g. by using splines.

[1] J. Carbonell and V.A. Karmanov, PRC 67 (2003) 037001

[2] T. Frederico, PLB 282 (1992) 409

[3] E. Ydrefors et al, PLB 770 (2017) 131

Direct method

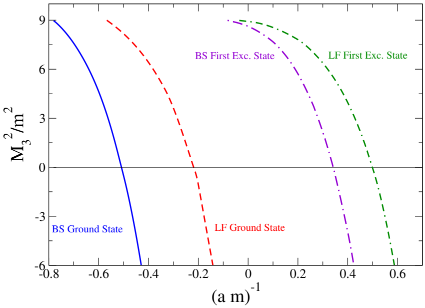
- Direct integration of the BS equation, treating explicitly the singularities.
- A similar approach was introduced by Carbonell and Karmanov [1] to solve the two-body problem (finite-range interaction).
- The equation for the vertex function, $v(q_0, q_v)$ can be written in the "non-singular" form

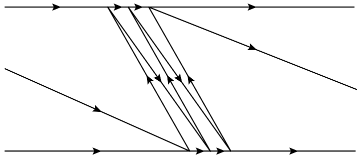
$$v(q_0,q_v) = \frac{\mathcal{F}(M_{12})}{(2\pi)^4}\int_0^\infty k_v^2 dk_v \Bigg\{ i\frac{[\Pi(q_0,q_v;\varepsilon_k,k_v)v(\varepsilon_k,k_v)+\Pi(q_0,q_v;-\varepsilon_k,k_v)v(-\varepsilon_k,k_v)]}{2\varepsilon_k}$$
$$-2\int_{-\infty}^0 dk_0\left[\frac{\Pi(q_0,q_v;k_0,k_v)v(k_0,k_v)-\Pi(q_0,q_v;-\varepsilon_k,k_v)v(-\varepsilon_k,k_v)}{k_0^2-\varepsilon_k^2}\right] \tag{2}$$
$$-2\int_0^\infty dk_0\left[\frac{\Pi(q_0,q_v;k_0,k_v)v(k_0,k_v)-\Pi(q_0,q;\varepsilon_k,k_v)v(\varepsilon_k,k_v)}{k_0^2-\varepsilon_k^2}\right]\Bigg\},$$

  using, e.g, $[k_0^2-k_v^2-m^2+i\epsilon]^{-1} = PV[k_0^2-\varepsilon_k^2]^{-1} - i\pi/(2\varepsilon_k)[\delta(k_0-\varepsilon_k)+\delta(k_0+\varepsilon_k)]$. Above, where $\varepsilon_k = \sqrt{k_v^2+m^2}$, $k_v = |\vec{k}|$ and the kernel Π only has weak, logaritmic, singularities. For $a<0$ (considered here) $F(M_{12})$ has no pole.
- The singularities at $k_0 = \pm\varepsilon_k$ were subtracted.
- We have solved the above equation by using a spline expansion for v, i.e. $v(q_0,q_v) = \sum_{ij} C_{ij} S_i(q_0) S_j(q_v)$.

[1] J. Carbonell and V.A. Karmanov, PRD 90 (2014) 056002

Binding energy versus inverse scattering length (EBS vs LF)





- The (complete) BS equation gives a stronger bound system compared to the LF one for all a.
- Exists a region with $a < 0$ (i.e. a Borromean system) the solution with the smallest M_3^2, i.e. the formal ground state, is physical.
- However, for $a > 0$, i.e. a two-body bound state exists, the lowest state is unphysical.
- $M_3^2 > -\infty$: No Thomas collapse in the non-relativistic sense, i.e. an effective short-range repulsion.
- The higher-Fock state contributions beyond the valence to the kernel can be interpreted as an effective three-body force of relativistic origin.

Transverse amplitudes

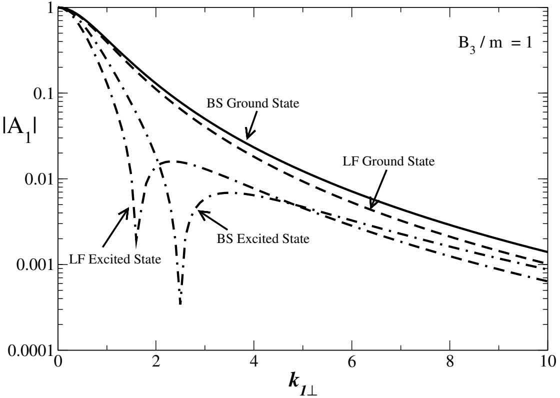
- The LF and (Euclidean) BS vertex functions cannot be directly compared with each other.
- However, we can define the transverse amplitudes

$$A^{\mathrm{LF}}(\vec{k}_{1\perp},\vec{k}_{2\perp}) = A_1^{\mathrm{LF}} + A_2^{\mathrm{LF}} + A_3^{\mathrm{LF}} = \frac{-\sqrt{2\pi}}{4}$$
$$\times \int_0^1 dx_1 \int_0^{1-x_1} \frac{dx_2}{x_1 x_2 (1-x_1-x_2)} \frac{\Gamma(\vec{k}_{1\perp},x_1)+\Gamma(\vec{k}_{2\perp},x_2)+\Gamma(\vec{k}_{3\perp},x_3)}{M_0^2 - M_3^2}$$

and

$$A^{\mathrm{EBS}}(\vec{k}_{1\perp},\vec{k}_{2\perp}) = A_1^{\mathrm{EBS}} + A_2^{\mathrm{EBS}} + A_3^{\mathrm{EBS}} =$$
$$-i\int dk_{14}dk_{1z}dk_{24}dk_{2z}[v_E(k_{14},k_{1v})+v_E(k_{24},k_{2v})+v_E(k_{34},k_{3v})]\Pi_1\Pi_2\Pi_3$$

where $\Pi_j^{-1} = (k_{j4}^2 - i\frac{1}{3}M_3)^2 + k_{jz}^2 + k_{j\perp}^2 + m^2$



- In both frameworks the first excited state has one node, and the ground state has no node. This confirms these assignments.
- The extra contributions included in the "full" BS solution has a significant impact on the transverse amplitude, especially for the first excited state.

Binding energies: Euclidean solution vs direct method [1]

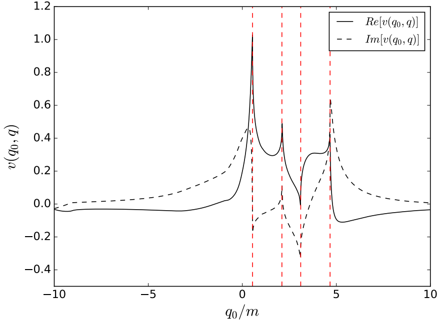
- The three-body binding energy (for fixed a) is calculable both in Minkowski and Euclidean spaces.
- In the table are shown for three cases ($a\,m = -1.28, -1.5, -1.705$), the obtained eigenvalue using the B_3 from the Euclidean calculation.

$a\,m$	B_3/m	λ
-1.28	0.006	$0.999 - 0.0544i$
-1.5	0.395	$1.000 + 0.0023i$
-1.705	1.001	$0.997 + 0.106i$

- Results good for the case $a\,m = -1.5$, but the error in the imaginary is getting quite large for more strongly bound system.
- One reason for the non-zero imaginary part could be the use of finite cutoffs, i.e. $k_{max}/m = 6.0$ and $k_{0max}/m = 13.0$ (first two cases) and $k_{0max}/m = 15.0$ (third case).
- Euclidean solution obtained without cutoffs, i.e. using a mapping.

[1] E. Ydrefors et al, PLB 791 (2019) 276

Three-body vertex function in Minkowski space



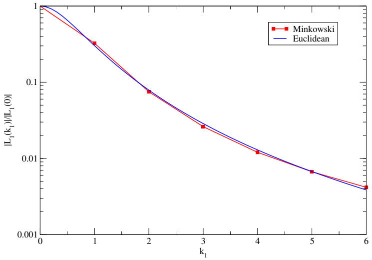
- The figure shows the real and imaginary parts of $v(q_0, q_v)$ at fixed $q_v/m = 0.5$, for the case $B_3/m = 0.395$.
- It is seen that there are four peaks (either singularities or branch cuts). It turns out that they have the positions $q_0 = M_3 \pm \sqrt{q_v^2 + 4m^2}$ and $q_0 = M_3 \pm q_v$, shown by red dashed lines. These are thus moving peaks depending on q_v.
- The non-smooth behavior of v makes the solution of this problem numerically very challenging.

Transverse amplitude in Minkowski space

$$
\begin{aligned}
L_1(\vec{k}_{1\perp},\vec{k}_{2\perp}) = \int_{-\infty}^{\infty} dk_{1z} \Bigg\{ & \frac{i\pi\left[v(\tilde{k}_{10},k_{1v})\chi(\tilde{k}_{10},k_{1z},\vec{k}_{1\perp};\vec{k}_{2\perp}) + v(-\tilde{k}_{10},k_{1v})\chi(-\tilde{k}_{10},k_{1z},\vec{k}_{1\perp};\vec{k}_{2\perp})\right]}{2\tilde{k}_{10}} \\
& - \int_0^{\infty} dk_{10} \frac{v(k_{10},k_{1v})\chi(k_{10},k_{1z},\vec{k}_{1\perp};\vec{k}_{2\perp}) - v(\tilde{k}_{10},k_{1v})\chi(\tilde{k}_{10},k_{1z},\vec{k}_{1\perp};\vec{k}_{2\perp})}{k_{10}^2-\tilde{k}_{10}^2} \\
& - \int_0^{\infty} dk_{10} \frac{v(-k_{10},k_{1v})\chi(-k_{10},k_{1z},\vec{k}_{1\perp};\vec{k}_{2\perp}) - v(-\tilde{k}_{10},k_{1v})\chi(-\tilde{k}_{10},k_{1z},\vec{k}_{1\perp};\vec{k}_{2\perp})}{k_{10}^2-\tilde{k}_{10}^2} \Bigg\},
\end{aligned}
\tag{3}
$$

with $\tilde{k}_{10} = \sqrt{k_{1z}^2 + \vec{k}_{1\perp}^2 + m^2}$ and χ is a known function having only weak, square root, singularities.

Results for the transverse amplitudes



- The figure compares (as an example) the modulus of the transverse amplitudes for the case $B_3/m = 0.395$.
- The agreement between the two approaches is good.
- Even though the Minkowski space amplitude, $v(q_0, q_v)$, has a non-smooth behavior, a smooth transverse amplitude is obtained.

Alternative: Nakanishi integral representation

- One alternative in order to avoid the numerical difficulties with the direct method, could be to use the Nakanishi integral representation, which has been succesfully been adopted for the two-body system. We then write three-body vertex function in the form:

$$v(q;p) = \int_{-\frac{4}{3}}^{\frac{2}{3}} dz \int_0^\infty \frac{g(\gamma,z)}{\gamma - k^2 - (p\cdot q)z - i\varepsilon} \tag{4}$$

- The two-body scattering amplitude is represented in the spectral form

$$F(M_{12}^2) = \int_{4m^2}^\infty d\gamma \frac{\rho(\gamma)}{M_{12}^2 - \gamma + i\epsilon} \tag{5}$$

with

$$\rho(M_{12}^2) = -\frac{\theta(M_{12}^2 - 4m^2)}{16\pi^2} \frac{y''}{(\frac{y''}{16\pi^2}\log\frac{1+y''}{1-y''} - \frac{1}{16\pi ma})^2 + (\frac{y''}{16\pi})^2}, \tag{6}$$

$$y'' = \frac{\sqrt{M_{12}^2 - 4m^2}}{M_{12}^2}. \tag{7}$$

Equation for the Nakanishi weight function

By assuming the uniqueness of the Nakanishi weight function one can derive the following equation for $g(\gamma, z)$:

$$g(\gamma,z) = -\frac{2}{(4\pi)^2(z+\frac{4}{3})^4}\int_{-\frac{4}{3}}^{\frac{2}{3}} dz'\left(\frac{2}{3}-z'\right)^2\theta\left(\left(\frac{2}{3}-z'\right)-\left(z+\frac{4}{3}\right)\right)$$
$$\int_{4m^2}\rho(\gamma'')d\gamma''\int_{\frac{z+\frac{4}{3}}{\frac{2}{3}-z'}}^{1}\frac{d\alpha_1}{\alpha_1}\int_0^{\frac{z+\frac{4}{3}}{\frac{2}{3}-z'}} d\alpha_3\left[\left(z+\frac{4}{3}\right)-\alpha_3\left(\frac{2}{3}-z'\right)\right]\frac{\partial g(\gamma_0,z')}{\partial\gamma_0}\theta(\gamma_0) \tag{8}$$

where

$$\gamma_0 = \frac{p^2}{9}+m^2-z'\frac{p^2}{3}-\frac{1}{4}\frac{\alpha_3}{\alpha_1}\left(\frac{2}{3}-z'\right)^2p^2-\frac{\alpha_1}{\alpha_3}m^2+\frac{(\frac{2}{3}-z')}{(z+\frac{4}{3})^2}$$
$$\times\left[\left(z+\frac{4}{3}\right)-\alpha_3\left(\frac{2}{3}-z'\right)\right]\left[\gamma-(1-\alpha_1)\gamma''-\frac{2}{3}p^2\left(\frac{2}{3}+z\right)\right]. \tag{9}$$

The numerical solution of this equation will be attempted in the near future.

Conclusions

- We have in this work studied a system of three bosons interacting through a zero-range potential using three approaches. Namely, 1) using the valence LF equation in Minkowski space, 2) Solving the 4-dimensional Euclidean BS equation, 3) Solving the 4-dimensional BS equation by direct integration in Minkowski space.
- The contributions beyond the valence have large impact both on binding energies and transverse amplitudes. These contributions can be interpreted as an effective three-body force of relativistic origin.
- The direct method is of great interest since it can give a BS amplitude defined in Minkowski space, needed to compute dynamical observables.
- This is work is in progress. However, we have shown that the binding energy (at least for modest B_3) is in fair agreement with the Euclidean. The transverse amplitudes are also in fair agreement.
- Unfortunately, the method is numerically very challenging due the treatment of the many singularities.
- One way to solve this could be to use a Nakanishi integral representation (similarly to the two-body case) for the BS amplitude, and it will be done in the near future.