Light-front quantum mechanics and quantum field theory

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Dates

- **1949: Dirac introduced 3 simplified "forms of relativistic dynamics" Identified by having the largest kinematic (interaction-free) subgroups.**

$$[K^i, P^j] = i\delta_{ij}(H_0 + V)$$

- **1965: Infinite momentum frame**
- **1973: Light front QFT**
- **1976: Light front RQM**
- **1991: First light-front conference**

Heidelberg 1991 (#1)

$$\vdots$$

Palaiseau 2019 (#38).

What is a light front / light-front dynamics?

- **Light front $:=$ hyperplane tangent to a light cone:**

$$\boxed{x^{+} := x^{0} + \hat{\mathbf{z}} \cdot \mathbf{x} = 0}$$

- **The light front is invariant under a 7 parameter (kinematic) subgroup of the Poincaré (1873 L'Ecole Polytechnique) group.**
- **Kinematic subgroup includes 3 translations tangent to the light-front, a 3 parameter subgroup of light-front preserving boosts, and rotations about the $\hat{\mathbf{z}}$ axis.**
- **Light-front dynamics: Interactions appear in the (3) generators of transformations that do not preserve $x^{+} = 0$.**
- **Light-front dynamics has the largest kinematic subgroup of Dirac's forms of dynamics.**

Special relativity in quantum theories

- **Quantum measurements:**

$$P = |\langle\psi|\phi\rangle|^2 \qquad \langle\psi|A|\psi\rangle \qquad \mathbf{Tr}(\rho A)$$

- **Inertial reference frames are related by Poincaré transformations:** $x^\mu \to x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu + a^\mu$**.**
- **Special relativity (QM) - quantum measurements cannot be used to distinguish inertial reference frames.**

$$P' = P \qquad \langle\psi'|A'|\psi'\rangle = \langle\psi|A|\psi\rangle \qquad \mathbf{Tr}(\rho' A') = \mathbf{Tr}(\rho A)$$

- **Wigner - 1939 - necessary and sufficient conditions for relativistic invariance:**

$$|\psi'\rangle = U(\Lambda, a)|\psi\rangle \qquad A' = U(\Lambda, a) A U^\dagger(\Lambda, a)$$

$$\rho' = U(\Lambda, a)\rho U^\dagger(\Lambda, a)$$

$SL(2,\mathbb{C}) \sim$ Lorentz group

$$X = x^\mu \sigma_\mu = \begin{pmatrix} x^+ & \mathbf{x}_\perp^* \\ \mathbf{x}_\perp & x^- \end{pmatrix} \qquad \det(X) = (x^0)^2 - \mathbf{x}^2$$

$$\boxed{X' = \Lambda X \Lambda^\dagger + A \qquad A := a^\mu \sigma_\mu \qquad \Lambda = e^{\frac{\mathbf{z}}{2}\cdot\boldsymbol{\sigma}}}$$

$$\det(\Lambda) = 1, \quad x^\pm = x^0 \pm x^3, \quad x_\perp = x^1 + ix^2$$

$$\Lambda^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(\sigma_\mu \Lambda \sigma_\nu \Lambda^\dagger) \qquad x^\mu = \frac{1}{2}\mathrm{Tr}(\sigma_\mu X)$$

Kinematic subgroup (preserves $x^+ = 0$):

$$\Lambda_{fb}(p) = \begin{pmatrix} \sqrt{p^+/m} & 0 \\ p_\perp/\sqrt{p^+ m} & \sqrt{m/p^+} \end{pmatrix} \qquad \textbf{light-front boosts}$$

$$\Lambda_{fr}(\phi) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \qquad \textbf{rotations about } \hat{\mathbf{z}}$$

$$A_f = \begin{pmatrix} 0 & a_\perp^* \\ a_\perp & a^- \end{pmatrix} \qquad \textbf{translations tangent to LF}$$

Poincaré group generated by 10 1-parameter subgroups

- **Generators**

$$P^\mu \qquad J^{\mu\nu} = -J^{\nu\mu}$$

- **Generators transform like tensors**

$$U(\Lambda, 0) P^\mu U^\dagger(\Lambda, 0) = \Lambda^{-1\mu}{}_\nu P^\nu$$

$$U(\Lambda, 0) J^{\mu\nu} U^\dagger(\Lambda, 0) = \Lambda^{-1\mu}{}_\alpha \Lambda^{-1\nu}{}_\beta J^{\alpha\beta}$$

- **Invariants:**

$$M^2 = -P^\mu P_\mu \qquad W^2 = W^\mu W_\mu = M^2 S^2$$

$$W_\mu = \frac{1}{2} \epsilon_{\nu\alpha\beta\mu} P^\nu J^{\alpha\beta}$$

Light-front generators

- **Kinematic**

$K^3 = J^{30}, E^1 = J^{10} - J^{31}, E^2 = J^{20} + J^{23}$ **light-front boosts**

$J^3 = J^{12}$ **rotations about $\hat{\mathbf{z}}$**

$P^+ = P^0 + P^3, \mathbf{P}_\perp = (P^1, P^2)$ **translations tangent to LF**

- **Dynamical**

$$F^1 = J^{10} + J^{31}, F^2 = J^{20} - J^{23}, P^- = P^0 - P^3$$

- **Light-front dispersion relation and spectral conditions**

$$\boxed{P^- = \frac{M^2 + \mathbf{P}_\perp^2}{P^+} \geq 0 \qquad P^+ \geq 0}$$

Particles, commuting observables, bases

- **Light-front spin (P operators)**

$$S_f^i = \frac{1}{2}\epsilon_{ijk}\Lambda_{fb}^{-1}(P)^j{}_\mu\Lambda_{fb}^{-1}(P)^k{}_\nu J^{\mu\nu}$$

- **Commuting observables**

$$M^2, S^2, \underbrace{P^+, P^1, P^2}_{\tilde{\mathbf{P}}}, S_f^3$$

- **Single-particle (irreducible) basis vectors**

$$|(m,s)\tilde{p},\mu\rangle$$

- **Finite Poincaré transforms** $(p' = \Lambda p)$

$$U(\Lambda,a)|(m,s)\tilde{p},\mu\rangle = e^{ip'\cdot a}|(m,s)\tilde{p}',\nu\rangle\sqrt{\frac{p^{+\prime}}{p^+}}D^s_{\nu\mu}[\Lambda_{fb}^{-1}(\Lambda p)\Lambda\Lambda_{fb}(p)]$$

Observations

- $U(\Lambda, a)$ **unitary.**
- $(\Lambda, a) \in$ **kinematic subgroup** $\Rightarrow$

$$e^{ip'\cdot a}\sqrt{\frac{p^{+\prime}}{p^{+}}}D^{s}_{\nu\mu}[\Lambda^{-1}_{fb}(\Lambda p)\Lambda\Lambda_{fb}(p)]$$

independent of m!

- **Light-front boosts are a subgroup** $\Rightarrow$

If $\Lambda = \Lambda_{fb}(p'')$

$$R_{wlf}(\Lambda_{fb}, p) = \Lambda^{-1}_{fb}(\Lambda_{fb}(p'')p)\Lambda_{fb}(p'')\Lambda_{fb}(p) = I$$

i.e. the light-front Wigner rotation of a light-front boost is the identity (no rotations in subgroup).

More light-front observations

- **The light-front hyperplane contains light-like separated points - it is not a suitable initial value surface** $\cdots$
- $\cdots$ **but - if** P^- **is self-adjoint then** $e^{-\frac{i}{2}P^- x^+}$ **is a strongly continuous unitary one-parameter group.**
- **If** R **is a rotation (not about the z-axis)**

$$R_{wlf}(R,p) = \Lambda_{fb}^{-1}(Rp) R \Lambda_{fb}(p) \neq R$$

depends on the momentum and mass and is not R**.**

- **The means that general rotations are dynamical and light-front spins cannot be added with** $SU(2)$ **Clebsch-Gordan coefficients.**
- **The equation defining the particle's rest frame** $p^+ = p^- = m$ **is dynamical!**

Equivalence to Dirac's instant form:

- **Polar decomposition of $SL(2,\mathbb{C})$ matrices, canonical boosts and Melosh rotations**

$$\Lambda = \underbrace{(\Lambda\Lambda^{\dagger})^{1/2}}_{\textbf{positive}} \underbrace{(\Lambda\Lambda^{\dagger})^{-1/2}\Lambda}_{\textbf{unitary}} = \Lambda_c(p)R(p)$$

- $\Lambda_c(p) =$ **canonical boost; when** $\Lambda = \Lambda_{fb}(p)$ **the rotation** $R(p) = R_m(p)$ **is called a Melosh rotation:**

$$R_m(p) = \Lambda_c^{-1}(p)\Lambda_{bf}(p)$$

- **Canonical spin defined by:**

$$S_c^i = \frac{1}{2}\epsilon_{ijk}\Lambda_c^{-1}(P)^j{}_\mu \Lambda_c^{-1}(P)^k{}_\nu J^{\mu\nu} = R_m^{ij}(P)S_f^j$$

- **The instant and light-front single particle bases are related by:**

$$|(m,s)\mathbf{p},\mu_c\rangle = \sum |(m,s)\tilde{\mathbf{p}},\mu_f\rangle \sqrt{\frac{p^+}{\epsilon_m(p)}} D^s_{\mu_f\mu_c}[R_m^{-1}(p)]$$

Equivalence to Dirac's instant form:

- **Any unitary representation of the Poincaré group can be decomposed into a direct integral of irreducible representations.**
- **Instant and front-form dynamics are related by**

$$|(m,s)\mathbf{p},\mu_c\rangle = \sum |(m,s)\tilde{\mathbf{p}},\mu_f\rangle \sqrt{\frac{p^+}{\epsilon_m(p)}} D^s_{\mu_f\mu_c}[R_m^{-1}(p)]$$

on each irreducible subspace.

- **The coefficients**

$$\sqrt{\frac{p^+}{\epsilon_m(p)}} D^s_{\mu_f\mu_c}[R_m^{-1}(p)]$$

are dynamical - they require diagonalizing M **and** S

Systems of particles (useful basis choices)

- **Tensor product basis**

$$|(m_1,s_1)\tilde{\mathbf{p}}_1,\mu_1,\cdots,(m_n,s_n)\tilde{\mathbf{p}}_n,\mu_n\rangle = \otimes_i|(m_i,s_i)\tilde{\mathbf{p}}_i,\mu_i\rangle$$

$$\otimes_i\langle(m_i,s_i)\tilde{\mathbf{p}}_i,\mu_i|U(\Lambda_k,a_k)|\psi\rangle = \langle\psi|U_0^\dagger(\Lambda_k,a_k)|\otimes_i\langle(m_i,s_i)\tilde{\mathbf{p}}_i,\mu_i\rangle^*$$

- **Alternative (LF boost invariant) basis**

$$|\tilde{P},\xi_1,\mathbf{k}_{\perp 1},\mu_1,\cdots,\xi_n,\mathbf{k}_{\perp n},\mu_n\rangle$$

$$\tilde{\mathbf{P}} := \sum_i \tilde{\mathbf{p}}_i, \qquad \xi_i := \frac{p_i^+}{P^+}, \qquad \sum_i \xi_i = 1$$

$$\mathbf{k}_{\perp i} := \mathbf{p}_{\perp i} - \xi_i \mathbf{P}_\perp \qquad \sum_i \mathbf{k}_{\perp i} = 0$$

$$\prod d\tilde{\mathbf{p}}_i \to d\tilde{\mathbf{P}} \prod_i (d\xi_i d^2\mathbf{k}_{\perp i})\delta(\sum_j \xi_j - 1)\delta(\sum_k \mathbf{k}_{\perp k})$$

- $\xi_i, \mathbf{k}_{\perp i}$ **and** μ_i **are invariant with respect to light-front boosts! (no Lorentz contractions!)**

Irreducible light-front bases

- **Clebsch-Gordan coefficients for the Poincaré group in light-front irreducible bases (construction summarized below):**
  - **Convert single-particle spins to canonical spins with Melosh rotations.**
  - **Canonical spin - Wigner rotation of a rotation is the rotation $\rightarrow$ they can be added with $SU(2)$ Clebsch-Gordan coefficients**
  - **Add canonical spins and relative orbital angular momenta with $SU(2)$ Clebsch-Gordan coefficients in two-body rest frame $(\Lambda_c(Rp)^{-1}R\Lambda_c(0) = R)$.**
  - **Boost result with light-front boost.**
- **Needed to formulate scattering asymptotic conditions.**

Light-front Clebsch-Gordan coefficients

$$|(k,j)\tilde{\mathbf{P}},\mu;l,s\rangle =$$

$$\sum |(m_1,s_1)\tilde{\mathbf{p}}_1,\mu_1\rangle|(m_2,s_2)\tilde{\mathbf{p}}_2,\mu_2\rangle\times$$

$$D^{s_1}_{\mu_1\nu_1}[\Lambda^{-1}_{bf}(k_1)\Lambda^{-1}_{c}(k_1)]D^{s_2}_{\mu_2\nu_2}[\Lambda^{-1}_{bf}(k_2)\Lambda^{-1}_{c}(k_2)]Y_{lm}(\hat{\mathbf{k}}_1)\times$$

$$\langle s_s,\nu_1,s_2,\nu_2|s,\nu_s\rangle\langle s,\nu_s,l,m|j,\mu\rangle\sqrt{\frac{p_1^+p_2^+(\epsilon_1(k)+\epsilon_2(k))}{\epsilon_1(k)\epsilon_2(k)(p_1^++p_2^+)}}$$

where the variables are related by

$$\tilde{\mathbf{P}}=\tilde{\mathbf{p}}_1+\tilde{\mathbf{p}}_2 \qquad k_1=k=\Lambda^{-1}_{bf}(P)p_1 \qquad k_2=\Lambda^{-1}_{bf}(P)p_2$$

$$M=\sqrt{\mathbf{k}^2+m_1^2}+\sqrt{\mathbf{k}^2+m_2^2}=\epsilon_1(k)+\epsilon_2(k)$$

Dynamics

$$P^- = \frac{\mathbf{P}_{0\perp}^2 + M_0^2 + V}{P_0^+}$$

$$[V, G_i] = 0 \qquad G_i = \textbf{kinematic generators}$$

- **These conditions preserve the kinematic subgroup, but they are not sufficient to ensure a relativistically (rotationally) invariant dynamics.**
- **A necessary and sufficient condition for rotational covariance is the requirement that the results of any calculations are independent of the orientation of the light front (Karmanov, Fuda).**
- **Fuda operator $F(R)$ changes orientation of light front**

$$F(R) = U(R^{-1}, 0) U_0(R, 0) \qquad R\hat{\mathbf{z}} = \hat{\mathbf{n}}$$

If $U(R)$ exists $F(R)$ is an S-matrix preserving unitary transformation.

Bakamjian-Thomas solution (LF QM)

- **The light-front spin ($\mathbf{s}_f$) and dynamical angular momentum components (J)are related by**

$$\mathbf{J}_\perp = \frac{1}{P^+}[\frac{1}{2}(P^+ - P^-)(\hat{\mathbf{z}}\times\mathbf{E}_\perp) - (\hat{\mathbf{z}}\times\mathbf{P}_\perp)K^3 + \mathbf{P}_\perp s_f^3 + M\mathbf{s}_{f\perp}]$$

- **Require in addition $[V, \mathbf{s}_{f0}] = 0$ (i.e. the spectrum of s is independent of interactions, therefore choose $\mathbf{s}_f = \mathbf{s}_{f0}$).**
- **For $N > 2$ violates $U(\Lambda, a) \to U_A(\Lambda, a) \otimes U_B(\Lambda, a)$ for asymptotically separated subsystems A and B (needed for localized tests of special relativity).**
- **Cluster properties for ($N \geq 3$) can be recovered in a recursive construction (Sokolov) that generates frame-dependent many-body interactions that maintain cluster properties in all inertial frames.**

Sokolov construction ($N = 3$ example)

$$P^- =$$

$$e^{\sum \ln A^\dagger_{ij,k}} \left(\sum (A_{ij,k} P^-_{ij\otimes} k A^\dagger_{ij,k} - 2 \otimes_i P^-_i + V_{123}/P^+ \right) e^{\sum \ln A_{ij,k}}$$

- **Method preserves Poincaré invariance, cluster properties and spectral condition (for suitable interactions).**
- **Preserves kinematic subgroup (for suitable interactions).**
- $A_{ij,k}$ **generates frame-dependent many interactions.**
- **Resulting spin is dynamical.**

Scattering Theory - light-front dynamics

$$S = \Omega_+^\dagger \Omega_-$$

$$\Omega_\pm := \lim_{t \to \pm\infty} e^{iHt} \Phi e^{-iH_0 t} = \lim_{t \to \pm\infty} e^{i(P_0^+ + P^-)\frac{t}{2}} \Phi e^{-i(P_0^+ + P_0^-)\frac{t}{2}} =$$

$$\lim_{t \to \pm\infty} e^{iP^- \frac{t}{2}} \Phi e^{-iP_0^- \frac{t}{2}} = \lim_{t \to \pm\infty} e^{i(M^2 + \mathbf{P}_{\perp 0}^2)\frac{t}{2P_0^+}} \Phi e^{-i(M_0^2 + \mathbf{P}_{\perp 0}^2)\frac{t}{2P_0^+}} =$$

$$\lim_{t' \to \pm\infty} e^{iM^2 t'} \Phi e^{-iM_0^2 t'} = \lim_{t' \to \pm\infty} e^{iMt'} \Phi e^{-iM_0 t'}$$

Electron scattering - current matrix elements

- **For space-like momentum transfers all matrix elements of a conserved covariant current can be constructed from matrix elements of $I^+(0)$ in a frame with momentum transfer along the $\hat{\mathbf{z}}$-axis (Drell-Yan-West frame).**

- **These matrix elements are light-front boost invariant**
$(p'' = \Lambda_{fb} p, \quad p''' = \Lambda_{fb} p')$

$$\langle (m,s)\tilde{\mathbf{p}}', \mu' | I^+(0) | (m,s)\tilde{\mathbf{p}}, \mu \rangle =$$

$$\langle (m,s)\tilde{\mathbf{p}}''', \mu' | I^+(0) | (m,s)\tilde{\mathbf{p}}, \mu \rangle =$$

$$\langle (m,s)\mu' \| I^+(0) \| (m,s)\mu \rangle$$

Remarks - observations on currents

- More generally, all components of the operator $I^{\mu}(0)$ can be expressed in terms of $I^{+}(0)$ and the Poincaré generators.

- For one-body currents the momentum transferred to the target is the same as the momentum transferred to a constituent particle in all frames related by light-front boosts. (only form where this is true!)

- Dynamical constraints imply impulse current operators violate Lorentz covariance.

Light-Front quantum field theory

Free fields

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_m(\mathbf{p})}} \left(e^{ip\cdot x} a(\mathbf{p}) + e^{-ip\cdot x} a^\dagger(\mathbf{p}) \right)$$

Change variables $\mathbf{p} \to \tilde{\mathbf{p}}$:

$$|\frac{\partial(\tilde{\mathbf{p}})}{\partial(\mathbf{p})}| = \frac{p^+}{\epsilon_m(\mathbf{p})} \qquad a(\tilde{\mathbf{p}}) := a(\mathbf{p}) \sqrt{\frac{\epsilon_m(\mathbf{p})}{p^+}}$$

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}'] = \delta(\mathbf{p} - \mathbf{p}') \quad \Leftrightarrow \quad [a(\tilde{\mathbf{p}}), a^\dagger(\tilde{\mathbf{p}}'] = \delta(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}')$$

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{dp^+ \theta(p^+)}{\sqrt{2p^+}} d\mathbf{p}_\perp \left(e^{ip\cdot x} a(\tilde{\mathbf{p}}) + e^{-ip\cdot x} a^\dagger(\tilde{\mathbf{p}}) \right)$$

Fields restricted to the light-front are irreducible!

- **Fourier transform of field restricted to $x^+ = 0$:**

$$\phi(x^+ = 0, p^+, \mathbf{p}_\perp) =$$

$$\frac{1}{(2\pi)^{3/2}} \int dx^- d\mathbf{x}_\perp e^{i(\frac{x^- p^+}{2} - \mathbf{x}_\perp \cdot \mathbf{p}_\perp)} \phi(0, x^-, \mathbf{x}_\perp) =$$

$$\theta(p^+)\sqrt{\frac{2}{p^+}} a(p^+, \mathbf{p}_\perp) + \theta(-p^+)\sqrt{\frac{2}{-p^+}} a^\dagger(-p^+, \mathbf{p}_\perp)$$

- **Possible to determine both $a(\tilde{\mathbf{p}})$, and $a^\dagger(\tilde{\mathbf{p}})$ without constructing conjugate momentum or going off of the light front.**
- **Canonical case requires $\pi(\mathbf{x}, t = 0) = -i[H, \phi(\mathbf{x}, t = 0)]$ which involves the dynamics.**

Creation and annihilation operators in terms of fields on the light front:

$$a(\tilde{\mathbf{p}}) = \sqrt{\frac{p^+}{2}}\theta(p^+)\phi(x^+ = 0, p^+, \mathbf{p}_\perp)$$

$$a^\dagger(\tilde{\mathbf{p}}) = \sqrt{\frac{p^+}{2}}\theta(p^+)\phi(x^+ = 0, -p^+, \mathbf{p}_\perp)$$

- **This means that if A satisfies**

$$\boxed{[\phi(x^+ = 0, \tilde{\mathbf{x}}), A] = 0 \rightarrow A = \textbf{constant}}$$

Field dynamics - naive expectation

- Noether's theorem on a light front gives expressions for all ten Poincaré generators in terms of fields restricted to the light front.

- Irreducibility of fields on the light front suggests that all of the generators can be expressed in terms of the irreducible Fock algebra restricted to a light front.

- Using Noether's theorem to compute the generators for a scalar field gives

$$P^+ = 4\int_{x^+=0} d\tilde{\mathbf{x}} : \frac{\partial\phi(x)}{\partial x^-}\frac{\partial\phi(x)}{\partial x^-} :$$

$$P^i = 2\int_{x^+=0} d\tilde{\mathbf{x}} : \frac{\partial\phi(x)}{\partial x^-}\frac{\partial\phi(x)}{\partial x^i} :$$

$$J^3 = 2\int_{x^+=0} d\tilde{\mathbf{x}} : \frac{\partial\phi(x)}{\partial x^-}(x^j\frac{\partial\phi(x)}{\partial x^i} - x^i\frac{\partial\phi(x)}{\partial x^j}) :$$

$$E^i = 2\int_{x^+=0} d\tilde{\mathbf{x}} : \frac{\partial\phi(x)}{\partial x^-}(2x^i\frac{\partial\phi(x)}{\partial x^-} - x^+\frac{\partial\phi(x)}{\partial x^i}) :$$

$$K^3 = 4\int_{x^+=0} d\tilde{\mathbf{x}} : \frac{\partial\phi(x)}{\partial x^-}(\frac{\partial\phi(x)}{\partial x^-}x^- - \frac{\partial\phi(x)}{\partial x^+}x^+) :$$

$$P^- = 4\int_{x^+=0} d\tilde{\mathbf{x}} : \frac{\partial\phi(x)}{\partial x^-}\frac{\partial\phi(x)}{\partial x^+} :$$

$$F^i = 2\int_{x^+=0} d\tilde{\mathbf{x}} : \frac{\partial\phi(x)}{\partial x^-}(2x^i\frac{\partial\phi(x)}{\partial x^+} - x^-\frac{\partial\phi(x)}{\partial x^i}) :$$

Observations

- **The expressions for K^3 and the dynamical generators (P^-, F^1, F^2) have $\partial/\partial x^+$ derivatives that are normal to the light front.**

- **These expressions contain no dynamical information! They are independent of both the mass (m^2) and the interaction ($V(\phi(x))$).**

- **These observations are related to the characteristic surface problem, where the operators that generate transformations normal to the light front are not derivable from fields restricted to the light front.**

- **Additional information is required to ensure that the resulting theory is equivalent to a canonical field theory with a given mass and interaction!**

Recover the desired theory

- **Integrate x^- by parts in the dynamical generators:**

$$K^3 = -4\int_{x^+=0} d\tilde{\mathbf{x}} : \phi(x)\Big(\frac{\partial^2\phi(x)}{\partial x^{-2}}x^- + \frac{\partial\phi(x)}{\partial x^-} - \frac{\partial^2\phi(x)}{\partial x^-\partial x^+}x^+\Big) :$$

$$P^- = -4\int_{x^+=0} d\tilde{\mathbf{x}} : \phi(x)\frac{\partial^2\phi(x)}{\partial x^-\partial x^+} :$$

$$F^i = -2\int_{x^+=0} d\tilde{\mathbf{x}} : \phi(x)\Big(2x^i\frac{\partial^2\phi(x)}{\partial x^-\partial x^+} - x^-\frac{\partial^2\phi(x)}{\partial x^-\partial x^i} - \frac{\partial\phi(x)}{\partial x^i}\Big) :$$

- **Use the field equation to express $\frac{\partial^2\phi(x)}{\partial x^-\partial x^+}$ in terms of quantities in the irreducible light-front algebra.**

$$\frac{\partial^2\phi(x)}{\partial x^-\partial x^+} = -\nabla_\perp^2\phi(x) + m^2\phi(x) + V(\phi(x))$$

- **This selects the mass and interaction.**
- **The dynamical generators are now expressed in terms of the irreducible algebra of fields on the light front.**

Consequences of the spectral condition ($P^+ \geq 0$)

- P^+ **kinematic** $\Rightarrow$

$$P^+ = \sum P_i^+ = 0 \qquad P_i^+ = 0$$

- **Interactions preserve kinematic subgroup** $\Rightarrow$

$$[V, P^+] = 0$$

- **Translational invariance of vacuum** $\Rightarrow$

$$P^+|0\rangle = 0 \qquad P_i^+|0\rangle = 0$$

$$P^+ V|0\rangle = VP^+|0\rangle = 0$$

- **Insert a complete set of eigenstates of** P^+

$$V|0\rangle = |0\rangle\langle 0|V|0\rangle = c|0\rangle$$

- **The light-front Fock vacuum is a normalizable translationally invariant eigenstate of both the free and interacting theory.**

Free field comments

- Solving the mass m field equations using (1) irreducible light-front algebra and (2) the free Fock vacuum gives the correct free field Wightman functions.

- The Wightman functions are moments of the vacuum generating functional.

- This means that solving the field equations using the kinematic vacuum generates the physical (mass m) vacuum.

Is the vacuum trivial?

- **The pure creation terms in the interactions (after normal ordering) are responsible for changing the vacuum.**
- **For a $\phi^4(x)$ interaction the pure creation part of the light-front interaction has the structure:**

$$\int \frac{\theta(p^+)\delta(p^+)dp^+}{(p^+)^2\prod \xi_i} \prod d\mathbf{p}_{i\perp} d\xi_i \delta(\sum \mathbf{p}_{i\perp})\delta(\sum \xi_i - 1)\times$$

$$a^\dagger(\xi_1 p^+, \mathbf{p}_{\perp 1})a^\dagger(\xi_2 p^+, \mathbf{p}_{\perp 2})a^\dagger(\xi_3 p^+, \mathbf{p}_{\perp 3})a^\dagger(\xi_4 p^+, \mathbf{p}_{\perp 4})$$

- **The problem is that while $\frac{\theta(p^+)\delta(p^+)dp^+}{p^{+2}\prod \xi_i}$ vanishes for $p^+ \neq 0$, it is both singular and not well defined for $p^+ = 0$.**

Zero modes

- **Noether's theorem on the light front gives Poincaré generators with no dynamical information.**
- **Dynamics must be put in by hand to be consistent with canonical field theory.**
- **Defining non-trivial theories require both field equations and a renormalization prescription to define local operator products in generators.**
- $p^+ = 0$ **gets mapped into** $p^i = -\infty$ **on changing light front orientation.**
- **Rotational covariance relates** $|\mathbf{p}| \to \infty$ **divergences with** $p^+ \to 0$ **divergences. (see talk by Beuf)**
- **For light-front representations this may require** $p^+ = 0$**-modes to renormalize the theory and maintain equivalence with the renormalized canonical theory.**

Additional consequences of $P^+ > 0$: Let Q be a charge by integrating a not-necessarily conserved current over the light front.

- **P^+ kinematic, interactions preserve kinematic subgroup**

$$P^+ = \sum P_i^+ = 0 \qquad P_i^+ = 0 \qquad [Q, P^+] = 0$$

- **Translational invariance of vacuum**

$$P^+|0\rangle = 0 \qquad P^+Q|0\rangle = QP^+|0\rangle = 0$$

- **Insert a complete set of eigenstates of P^+**

$$Q|0\rangle = |0\rangle\langle 0|Q|0\rangle = c|0\rangle$$

- **The light-front Fock vacuum is necessarily an eigenstate of Q.**
- **Vacuum contains no information without adding dynamical information - Spontaneous symmetry breaking can be recovered by solving the dynamics with an explicit symmetry breaking term (Beane) and letting the symmetry breaking term $\to 0$. This generates the physical vacuum.**

Summary of properties

- **Largest kinematic subgroup.**
- **3-Parameter subgroups of kinematic boosts and translations.**
- **Light-front boosts have no Wigner rotations.**
- **Angular momentum dynamical.**
- **Orbital angular momentum dynamical.**
- $[J_\perp, M_0] \neq 0$
- **Cluster properties of** $U(\Lambda, a) \to$ **spin dynamical.**
- **Adding angular only makes sense asymptotically with cluster properties.**

- **Frame-independent impulse approximations.**
- **Rest frame defined dynamically.**
- **Irreducibility of fields on the light front.**
- **Dynamical vacuum = Fock vacuum (on irreducible light-front algebra).**
- **Equivalent to other forms of dynamics.**
- **Characteristic surfaces - self adjoint P^- defines dynamics - Noether's theorem - requires additional dynamical information.**
- **Trivial vacuum plus dynamical information generates the physical vacuum (?).**

Two-body scattering (LF QM)

$$M := \sqrt{m_1^2 + \mathbf{k}^2 + 2m_{red}V_{nn}} + \sqrt{m_2^2 + \mathbf{k}^2 + 2m_{red}V_{nn}}$$

$$\langle (k',j')\tilde{\mathbf{P}}', \mu', l', s' | V_{nn} | (k,j)\tilde{\mathbf{P}}, \mu, l, s\rangle =$$

$$\delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}})\langle k', l', s' \| V_{nn}^j \| k, l, s\rangle$$

The invariance principle gives, since $M = M(H_{nr})$:

$$\langle (k',j)l', s' | S_{\mathbf{exp}} | (k,j)l, s\rangle =$$

$$\langle (k',j)l', s' | \Omega_+(H_{nr}, H_{0nr})\Omega_-^\dagger(H_{nr}, H_{0nr}) | (k,j)l, s\rangle =$$

$$\langle (k',j)l', s' | \Omega_+(M, M_0)\Omega_-^\dagger(M, M_0) | (k,j)l, s\rangle$$

Non-relativistic interactions fit to experiment can be reinterpreted as light-front interactions fit to same data.

- For the free field solving gives the field $\phi(x)$ with $x^+ \neq 0$.
- The role of the field equations can be seen for the case of a free scalar field of mass m. The light-front limit of the two-point Wightman function is $(z = x - y)$

$$\langle 0|\phi(x)\phi(y)|0\rangle \to -i\frac{\epsilon(z^-)\delta(\mathbf{z}_\perp^2)}{4\pi} - \frac{m}{4\pi^2\sqrt{\mathbf{z}_\perp^2}}K_1(m\sqrt{\mathbf{z}_\perp^2}).$$

  while a direct calculation using fields restricted to the light front gives

$$\langle 0|\phi(x)\phi(y)|0\rangle =$$

$$\frac{1}{2(2\pi)^3}\int \frac{\theta(q^+)dq^+d\mathbf{q}_\perp}{q^+}e^{-i\frac{q^+}{2}z^- + i\mathbf{q}_\perp\cdot\mathbf{z}_\perp}$$

  which knows nothing about the mass or dynamics (recall that the dynamics had to be put in "by hand".)