Complex poles, spectral function and reflection positivity violation of Yang-Mills theory

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References:
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arXiv:1812.03116[hep-th]; Y. Hayashi and K.-I. Kondo, PRD99, 074001 (2019).
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§ Introduction

⊙ Confinement is encoded in the (full dressed) gluon and ghost propagators!

We consider the Yang-Mills theory in the Lorenz type gauge.

The Euclidean gluon propagator and ghost propagator in the Landau gauge $\partial_\mu \mathscr{A}_\mu = 0$ are defined by

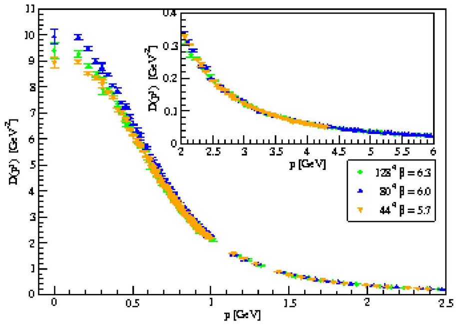
$$\mathscr{D}_{AB}^{\mu\nu}(k_E) = \delta^{AB}\left(\delta^{\mu\nu} - \frac{k_E^\mu k_E^\nu}{k_E^2}\right)\mathscr{D}(k_E^2), \quad \Delta_{AB}^{gh}(k_E) = \delta^{AB}\Delta_{gh}(k_E^2). \tag{1}$$

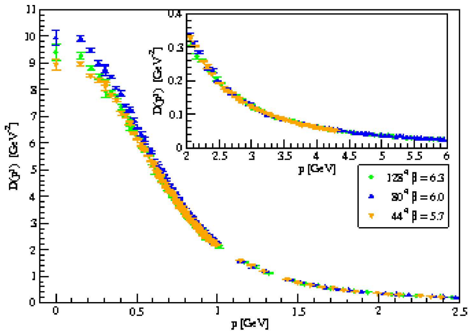
1997– **Scaling solution**: consistent with the Gribov/Zwanziger prediction [realized for $D = 2$]

$$\mathscr{D}(k_E^2) \sim (k_E^2)^{\alpha-1} \downarrow 0, \ \Delta_{gh}(k_E^2) \sim \frac{Z_{gh}}{(k_E^2)^{1-\beta}} \uparrow \infty \quad (1 < \alpha = -2\beta < 2) \quad \text{as} \quad k_E^2 \downarrow 0 \tag{2}$$

2006– **Decoupling solution**: massive gluon and massless ghost [realized for $D = 4, 3$]

$$\mathscr{D}(k_E^2) := \frac{F(k_E^2)}{k_E^2} \to \text{const.}, \ \Delta_{gh}(k_E^2) := \frac{G(k_E^2)}{k_E^2} \sim \frac{Z_{gh}}{k_E^2} \uparrow \infty \quad \text{as} \quad k_E^2 \downarrow 0 \tag{3}$$

$\mathscr{D}(k_E^2)$ 

$G(k_E^2)$ 

[Duarte-Oliveira-Silva, PRD94, 014502 (2016)], arXiv: 1605.00594[hep-lat]

⊙ We consider the massive Yang-Mills model described by the ordinary massless Yang-Mills (YM) Lagrangian in the manifestly Lorentz covariant gauge of the Lorenz type: gauge-fixing (GF) term and the associated Faddeev-Popov (FP) ghost term plus a naive gluon mass term,

$$
\begin{aligned}
\mathscr{L}_{\rm mYM} =& \mathscr{L}_{\rm YM} + \mathscr{L}_{\rm GF} + \mathscr{L}_{\rm FP} + \mathscr{L}_{\rm m}, \\
& \mathscr{L}_{\rm YM} = -\frac{1}{4}\mathscr{F}^{\mu\nu A}\mathscr{F}_{\mu\nu}^{A}, \\
& \mathscr{L}_{\rm GF} = \mathscr{N}^{A}\partial^{\mu}\mathscr{A}_{\mu}^{A} + \frac{\alpha}{2}\mathscr{N}^{A}\mathscr{N}^{A} \to -\frac{1}{2}\alpha^{-1}(\partial^{\mu}\mathscr{A}_{\mu}^{A})^{2} \\
& \mathscr{L}_{\rm FP} = i\bar{\mathscr{C}}^{A}\partial^{\mu}\mathscr{D}_{\mu}[\mathscr{A}]^{AB}\mathscr{C}^{B} = i\bar{\mathscr{C}}^{A}\partial^{\mu}(\partial_{\mu}\mathscr{C}^{A} + gf_{ABC}\mathscr{A}_{\mu}^{B}\mathscr{C}^{C}), \\
& \mathscr{L}_{\rm m} = \frac{1}{2}M^{2}\mathscr{A}^{\mu A}\mathscr{A}_{\mu}^{A},
\end{aligned}
\tag{4}
$$

Here g, M and $\alpha(\to 0)$ are the parameters of the massive Yan-Mills model.

In this talk we regard the massive Yang-Mills model as a low-energy effective theory for describing the $D = 4$ **decoupling solution of the Yang-Mills theory** and examine **gluon and quark confinement**.

• In the Euclidean region, the **massive Yang-Mills model with (at least one-loop) quantum corrections** being included well reproduces propagators and vertices of the decoupling solution in the covariant Landau gauge in the confining phase of the Yang-Mills theory, as demonstrated in the last ten years by [Wschebor, Tissier, Serreau, Reinosa, ...].
This can be done in the so-called **infrared safe renormalization scheme** by taking the renormalization conditions resulting from the **non-renormalization theorem** (in the Landau gauge), initiated by [Tissier and Wschebor, 2010].

⊙ Fitting of the numerical simulations of the Yang-Mills theory to the massive Yang-Mills model

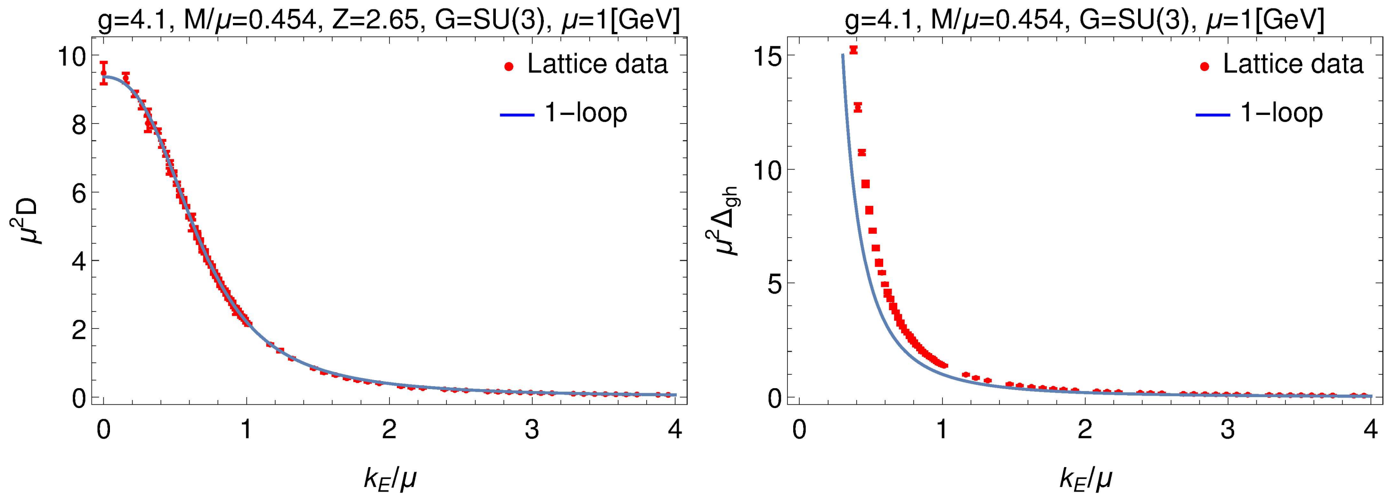


Figure 1: The gluon propagator $\mathscr{D}$ and ghost propagator Δ_{gh} as functions of the Euclidean momentum k_E.

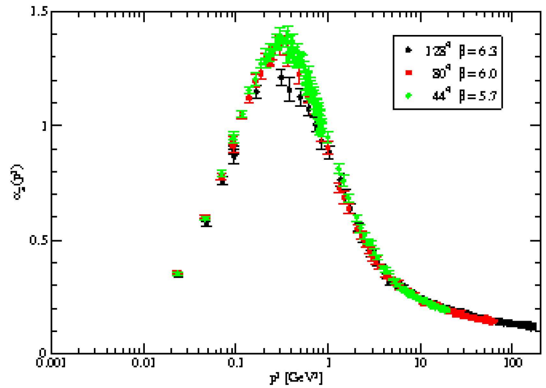
Both gluon propagator and ghost propagator in the decoupling solution of the Yang-Mills theory in the Landau gauge are well reproduced by the massive Yang-Mills model at the specific values of parameters g and M, which we call the **physical point** for the Yang-Mills theory:

$$g = 4.1 \pm 0.1 \ \left(\Leftrightarrow \ \lambda := \frac{g^2 C_2(G)}{16\pi^2} = 0.32 \pm 0.02 \right),$$
$$\frac{M}{\mu} = 0.454 \pm 0.004 \ \left(\Leftrightarrow \ \tilde{m}^2 := \frac{M^2}{\mu^2} = 0.206 \pm 0.004 \right),$$
$$[Z = 2.65 \pm 0.02]. \quad (5)$$

The origin of such a gluon mass term can be discussed separately. ...

⊙ Running coupling constant is always finite and asymptotic free in the infrared as well as the ultraviolet.

- decoupling solution of the Yang-Mills theory on a lattice



running coupling at $\mu = 4$ GeV for a physical volume of (8 fm)^4

- one-loop quantum corrections of the massive Yang-Mills model

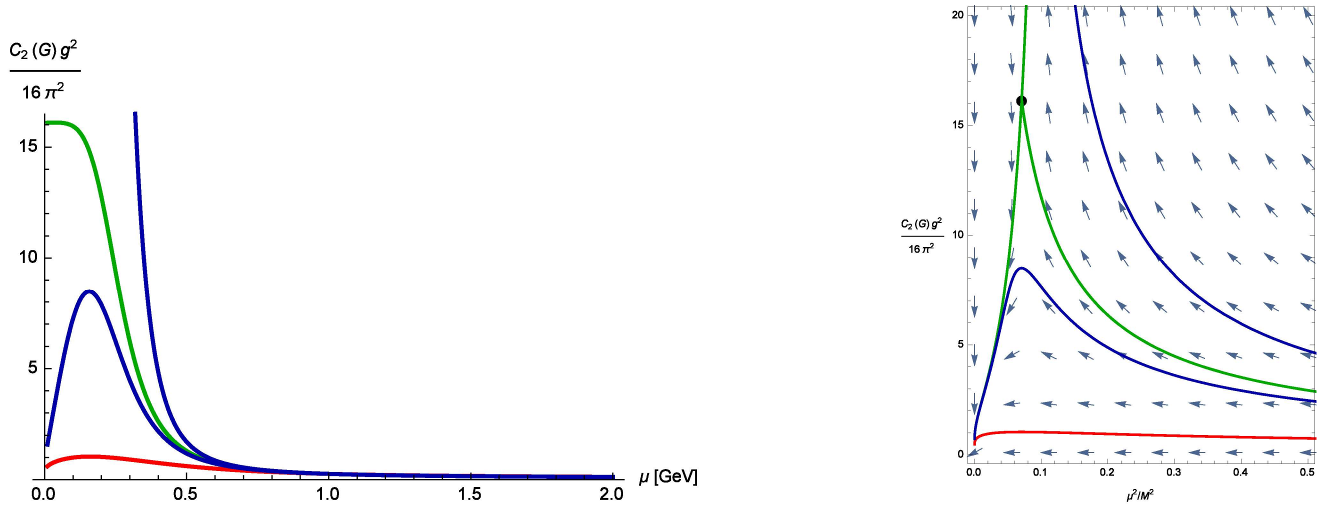


Figure 2: (left) Running coupling constant: Landau pole, scaling solution, decoupling solution, physical point, (right) RG flow trajectories in the plane ($\nu := \mu^2/M^2$, $\lambda := \frac{g^2 C_2(G)}{16\pi^2}$)

§ Reflection positivity violation in the Euclidean region

⊙ Reflection positivity is one of the the **Osterwalder-Schrader (OS) axioms** which are general properties to be satisfied for the **Euclidean quantum field theory** formulated in the Euclidean space:
(OS.3) **Reflection positivity** : Any complex-valued test function $f_0 \in \mathbb{C}_1, f_1 \in \mathscr{S}_+(\mathbb{R}^D), \cdots, f_N \in \mathscr{S}_+(\mathbb{R}^{DN})$,

$$\sum_{n,m=0}^{N} S_{n+m}(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+m}) f_n(\theta x_n, \theta x_{n-1}, \cdots, \theta x_1)^* f_m(x_{n+1}, \cdots, x_{n+m}) \geq 0,$$

where $\mathscr{S}_+(\mathbb{R}^D)$ denotes a complex-valued test (Schwartz) function with support in $\{(\boldsymbol{x}, x_D); x_D > 0\}$ and θ is a **reflection** with respect to a hyperplane $x^0 = 0$: for a function $f_n \in \mathscr{S}(\mathbb{R}^{Dn})$,

$$\theta x = \theta(x^0, \boldsymbol{x}) = (-x^0, \boldsymbol{x}), \ (\theta f_n)(x_1, \cdots, x_n) = f_n(\theta x_1, \cdots, \theta x_n). \tag{1}$$

This is a Euclidean version of the positivity axiom in the **Wightman axioms** for the **relativistic quantum field theory** formulated in the Minkowski spacetime.

(W.3) **Positivity**: For all $f_0 \in \mathbb{C}_1, f_1 \in \mathscr{S}(\mathbb{R}^D), \cdots, f_N \in \mathscr{S}(\mathbb{R}^{DN})$, $(N = 0, 1, 2, \cdots)$

$$\sum_{n,m=0}^{N} W_{n+m}(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+m}) f_n(x_n, x_{n-1}, \cdots, x_1)^* f_m(x_{n+1}, \cdots, x_{n+m}) \geq 0.$$

The violation of reflection positivity in the Euclidean region is regarded as a necessary condition for gluon confinement. To demonstrate the **violation of reflection positivity**, one counterexample suffices.

We focus on a special case ($N = 2$) of a single propagator $S_2 = \mathscr{D}$. Then the reflection positivity reads

$$\int d^D x \int d^D y f^*(\boldsymbol{x}, -x_D)\mathscr{D}(\boldsymbol{x} - \boldsymbol{y}, x_D - y_D) f(\boldsymbol{y}, y_D) \geq 0, \;\; f \in \mathscr{S}_+(\mathbb{R}^D), \tag{2}$$

which is rewritten as

$$\begin{aligned} &\int dx_D \int dy_D \int d^{D-1}\boldsymbol{p} f^*(\boldsymbol{p}, -x_D) f(\boldsymbol{p}, y_D)\Delta(\boldsymbol{p}, x_D - y_D) \\ =& \int_0^\infty dt \int_0^\infty dt' \int d^{D-1}\boldsymbol{p} f^*(\boldsymbol{p}, t) f(\boldsymbol{p}, t')\Delta(\boldsymbol{p}, -(t + t')) \geq 0, \end{aligned} \tag{3}$$

where we defined the **Schwinger function** $\Delta(\boldsymbol{p}, x_D - y_D)$ by

$$\mathscr{D}(x - y) := \int d^{D-1}\boldsymbol{p}\; e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\Delta(\boldsymbol{p}, x_D - y_D). \tag{4}$$

For the inequality (3) to hold for any $f \in \mathscr{S}_+(\mathbb{R}^D)$, the Schwinger function Δ must satisfy the positivity

$$\Delta(\boldsymbol{p}, -(t + t')) = \Delta(\boldsymbol{p}, t + t') \geq 0. \tag{5}$$

We consider a specific **Schwinger function** defined by the Fourier transform of the propagator:

$$\Delta(t) := \Delta(\boldsymbol{p}, t)|_{\boldsymbol{p}=0} := \int d^{D-1}x e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\mathscr{D}(\boldsymbol{x}, t)|_{\boldsymbol{p}=0} = \int_{-\infty}^{+\infty} \frac{dp_E^D}{2\pi} e^{ip_E^D t}\tilde{\mathscr{D}}(\boldsymbol{p} = \boldsymbol{0}, p_E^D). \tag{6}$$

For the free massive theory ($g = 0$), we find $\Delta(t)$ is positive for any t:

$$\tilde{\mathscr{D}}(p) = \frac{1}{p^2 + m^2} \quad (m > 0) \Longrightarrow \Delta(t) = \int_{-\infty}^{+\infty} \frac{dp_D}{2\pi} e^{ip_D t} \frac{1}{p_D^2 + m^2} = \frac{1}{2m} e^{-m|t|} > 0. \tag{7}$$

There is no reflection-positivity violation for the free massive propagator, as expected. For unconfined particles, the reflection positivity should hold.

⊙ The reflection positivity is violated for the massive Yang-Mills model at the physical point of parameters, as shown by the numerical calculations.

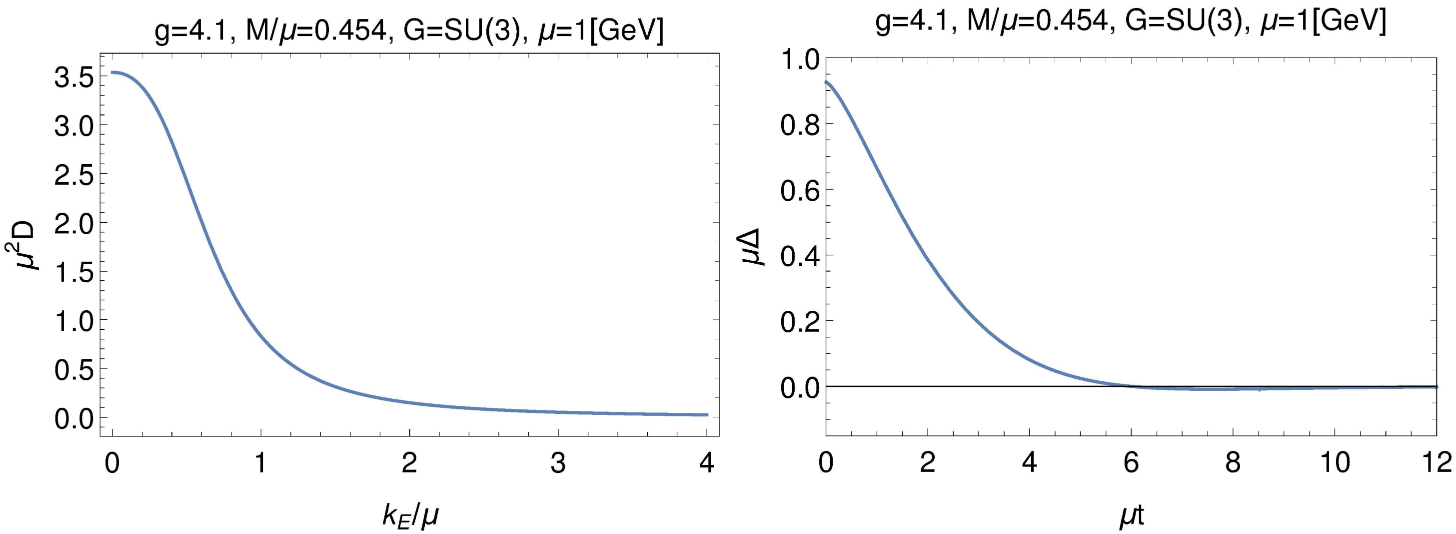


Figure 3: (left) gluon propagator $\mathscr{D}$ as a function of k_E, (right) the Schwinger function Δ as a function of t, at the physical point of the parameters.

We can prove that the reflection positivity is violated for any choice of the parameters in the massive Yang-Mills model.

This suggests the reflection positivity violation for the decoupling solution of the Yang-Mills theory.

In order to consider the origin, we proceed the complex analysis of the Yang-Mills theory.

§ Complex analysis of the gluon propagator

⊙ Spectral representation of a propagator

In the Minkowski region with time-like momentum $k^2 > 0$, a propagator $\mathscr{D}(k^2)$ has the Källén–Lehmann **spectral representation** under assumptions of the general principles: (i) the spectral condition, (ii) the Poincaré invariance and (iii) the completeness of the state space

$$\mathscr{D}(k^2) = \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 - k^2}, \ k^2 \geq 0, \tag{1}$$

with the weight function $\rho(\sigma^2)$ called the **spectral function**

$$\theta(k_0)\rho(k^2) := (2\pi)^d \sum_n |\langle 0|\phi(0)|P_n\rangle|^2 \delta^D(P_n - k), \tag{2}$$

The spectral function ρ has contributions from a **stable single-particle state** with physical mass m_P (pole mass) and intermediate many-particle states $|p_1, ..., p_n\rangle$ with a continuous spectrum,

$$\begin{aligned} \rho(k^2) =& Z\delta(k^2 - m_P^2) + \tilde{\rho}(k^2), \ k^2 \geq 0, \\ \tilde{\rho}(k^2) =& (2\pi)^d \sum_{n=2}^{\infty} |\langle 0|\phi(0)|p_1, ..., p_n\rangle|^2 \delta^D(p_1 + ... + p_n - k). \end{aligned} \tag{3}$$

Then it is written as the sum of contributions from the real pole $k^2 = m_P^2$ and the others $k^2 \in [\sigma_{\min}^2, \infty)$

$$\mathscr{D}(k^2) = \frac{Z}{m_P^2 - k^2} + \int_{\sigma_{\min}^2}^\infty d\sigma^2 \frac{\tilde{\rho}(\sigma^2)}{\sigma^2 - k^2}, \ k^2 \geq 0. \tag{4}$$

The spectral representation can be extended to the complex $k^2 \in \mathbb{C}$.

⊙ In the absence of complex simple poles, the spectral representation can be extended to the complex momentum $k^2 \in \mathbb{C}$. A propagator $\mathscr{D}(k^2)$ as a complex function of $z = k^2 \in \mathbb{C}$ has

$$\mathscr{D}(k^2) = \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 - k^2}, \ k^2 \in \mathbb{C} - [\sigma_{\min}^2, \infty), \quad \rho(\sigma^2) := \frac{1}{\pi} \mathrm{Im}\, \mathscr{D}(\sigma^2 + i\epsilon). \tag{5}$$

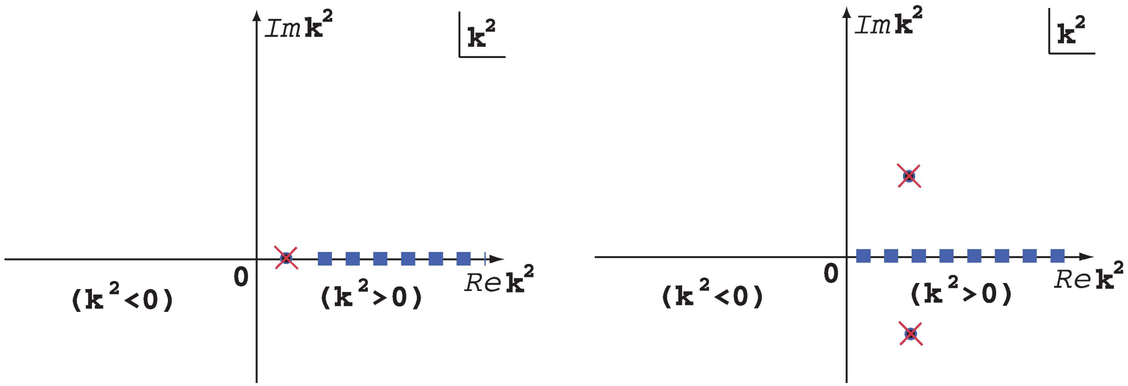


Figure 4: Possible singularities of the propagator on the complex k^2 plane, (Left) a real pole and the branch cut on the positive real axis, (Right) a pair of complex conjugate poles and the branch cut.

⊙ In the presence of complex simple poles, the propagator has the **generalized spectral representation**,

$$\mathscr{D}(k^2) = \mathscr{D}_p(k^2) + \mathscr{D}_c(k^2), \ k^2 \in \mathbb{C} - ([\sigma_{\min}^2, \infty) \cup \{z_\ell\}_\ell),$$

$$\mathscr{D}_p(k^2) := \frac{Z}{(v + iw) - k^2} + \frac{Z^*}{(v - iw) - k^2}, \ Z := \oint_\gamma \frac{dk^2}{2\pi i} \mathscr{D}(k^2),$$

$$\mathscr{D}_c(k^2) := \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 - k^2}, \ \rho(\sigma^2) := \frac{1}{\pi} \mathrm{Im}\, \mathscr{D}(\sigma^2 + i\epsilon). \tag{6}$$

⊙ The gluon propagator $\mathscr{D}(k^2)$ in the massive Yang-Mills model has a pair of complex conjugate poles on the complex momentum on the complex momentum k^2 plane.

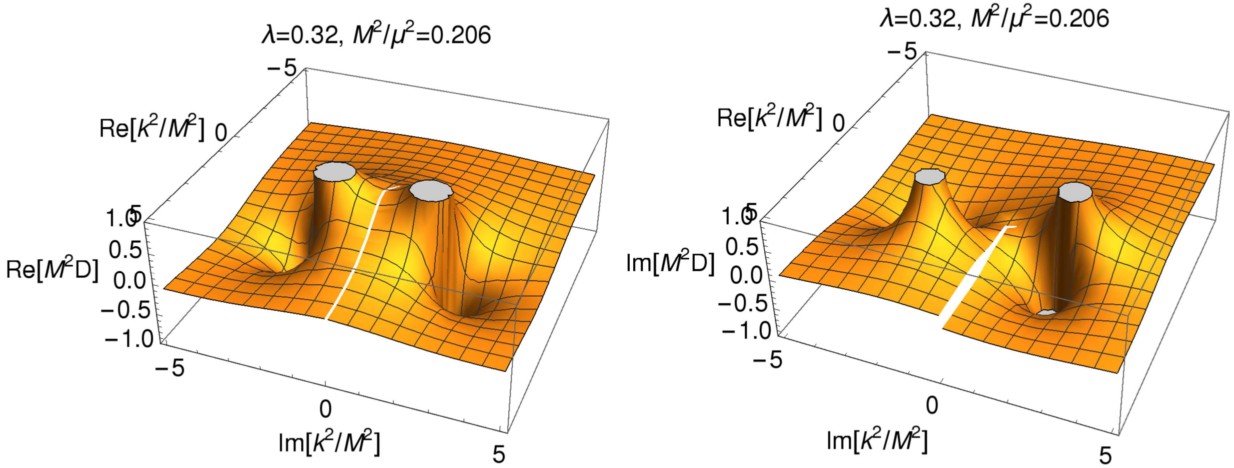


Figure 5: (left) the real part $\mathrm{Re}\,\mathscr{D}(k^2)$, and (right) imaginary part $\mathrm{Im}\,\mathscr{D}(k^2)$ of the gluon propagator $\mathscr{D}(k^2)$ on the complex $k^2 \in \mathbb{C}$. Poles at $k^2 = v \pm iw$, $v/M^2 = 1.123$, $w/M^2 = 2.044$.

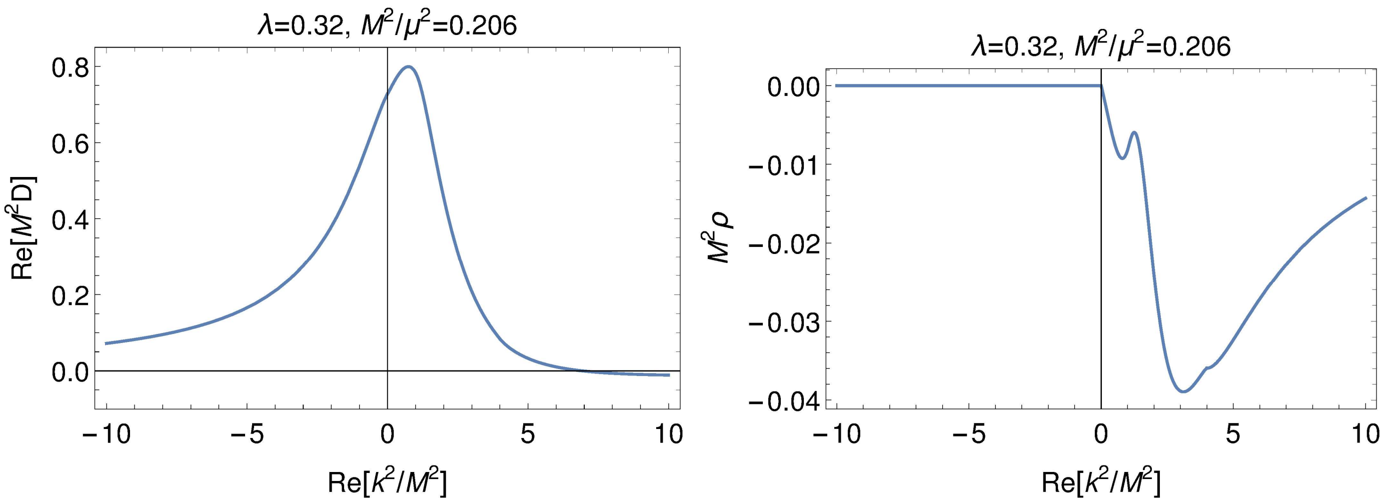


Figure 6: The gluon propagator $\mathscr{D}(k^2)$ as a function of k^2 restricted on the real axis $k^2 \in \mathbb{R}$, (left) the real part $\mathrm{Re}\,\mathscr{D}(k^2)$, (right) the scaled imaginary part $\mathrm{Im}\,\mathscr{D}(k^2 + i\epsilon)/\pi = \rho(k^2)$.

The spectral function $\rho(k^2)$ of the massive Yang-Mills model is always negative.

This is consistent with the negativity of $\rho(k^2)$ at large k^2 shown by [Oehme and Zimmermann,1980].

⊙ The Schwinger function is also separated into the two parts:

$$\Delta(t) = \Delta_p(t) + \Delta_c(t), \quad \Delta_{p,c}(t) := \int_{-\infty}^{+\infty} \frac{dk_E}{2\pi} e^{ik_E t} \mathscr{D}_{p,c}(-k_E^2), \tag{7}$$

• The cut part $\Delta_c(t)$ is directly written as an integral of the spectral function as

$$\Delta_c(t) = \int_{-\infty}^{+\infty} \frac{dk_E}{2\pi} e^{ik_E t} \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 + k_E^2} = \int_0^\infty d\sigma^2 \rho(\sigma^2) \frac{1}{2\sqrt{\sigma^2}} e^{-\sqrt{\sigma^2} t}. \tag{8}$$

To one-loop order in the massive Yang-Mills model, the spectral function $\rho(\sigma^2)$ takes the negative value

$$\rho(\sigma^2) < 0 \text{ for } {}^\forall \sigma^2 > 0 \ \Rightarrow \ \Delta_c(t) < 0 \text{ for } {}^\forall t \geq 0, \tag{9}$$

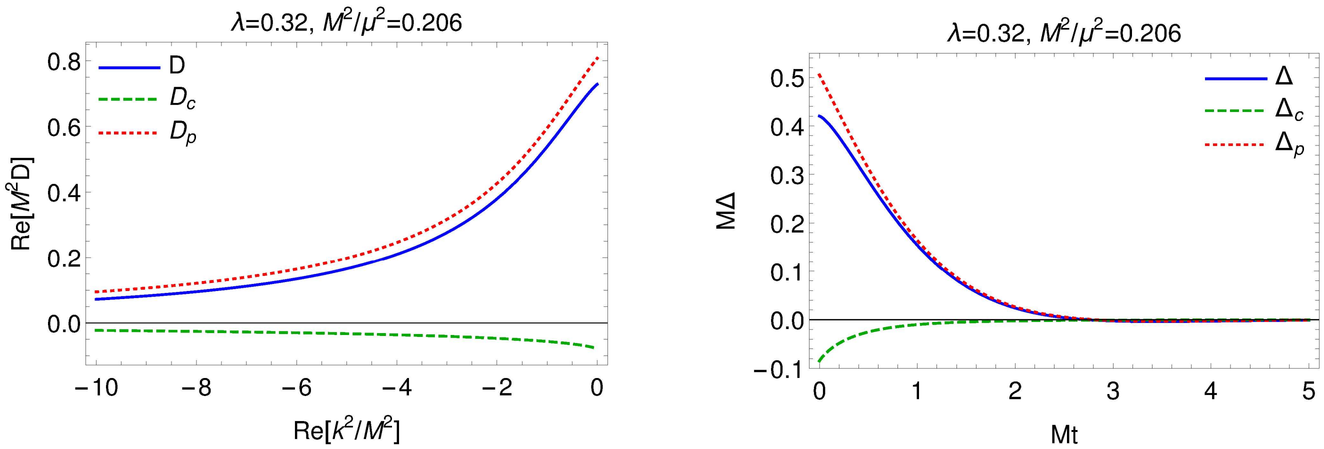


Figure 7: (left) the gluon propagator in the Euclidean region $\mathscr{D}(k_E^2) = \mathscr{D}_p(-k_E^2) + \mathscr{D}_c(-k_E^2)$ where $\mathscr{D}_c(-k_E^2) < 0$ and (right) the gluon Schwinger function $\Delta(t) = \Delta_p(t) + \Delta_c(t)$ for $k^2 = -k_E^2 < 0$.

• The pole part: In the presence of a pair of complex conjugate poles at $k^2 = v \pm iw (v, w > 0)$ with the respective residues $Z, Z^* \in \mathbb{C}$, the pole part of the propagator in the Euclidean region reads

$$\mathscr{D}_p(-k_E^2) = \frac{Z}{k_E^2 + (v + iw)} + \frac{Z^*}{k_E^2 + (v - iw)} = 2\frac{\mathrm{Re}[Z]k_E^2 + (v\,\mathrm{Re}[Z] + w\,\mathrm{Im}[Z])}{k_E^4 + 2vk_E^2 + (v^2 + w^2)}. \quad (10)$$

This pole part of the propagator agrees with the **Gribov-Stingl form** [Stingl, 1986]. This is in good agreement with the lattice results.

The pole part of the Schwinger function is exactly obtained as

$$\Delta_p(t) = \frac{\sqrt{\mathrm{Re}(Z)^2 + \mathrm{Im}(Z)^2}}{(v^2 + w^2)^{1/4}} \exp[-t(v^2 + w^2)^{1/4} \cos\varphi] \cos[t(v^2 + w^2)^{1/4} \sin\varphi + \varphi - \delta],$$

$$\varphi := \frac{1}{2}\arctan\frac{w}{v}, \; \delta := \arctan\frac{\mathrm{Im}(Z)}{\mathrm{Re}(Z)}. \quad (11)$$

Therefore, the pole part has negative value at a certain value of t,

$$\Delta_p(t) < 0 \text{ for } {}^{\exists}\, t \geq 0. \quad (12)$$

Thus, $\Delta(t) = \Delta_c(t) + \Delta_p(t)$ has necessarily negative value at a certain value of t,

$$\Delta(t) = \Delta_c(t) + \Delta_p(t) < 0 \text{ for } {}^{\exists}\, t \geq 0. \quad (13)$$

Thus we complete the proof that the reflection positivity is always violated in the massive Yang-Mills model to one-loop order (irrespective of the choice of the parameters g and M).

§ Relation between the number of poles and the winding number of a propagator

The negativity of the spectral function and the existence of complex conjugate poles are interrelated: It is shown that the negative spectral function yields one pair of complex conjugate poles (or Euclidean real poles of multiplicity 2) [Hayashi and Kondo, 2019]. arXiv:1812.03116[hep-th]

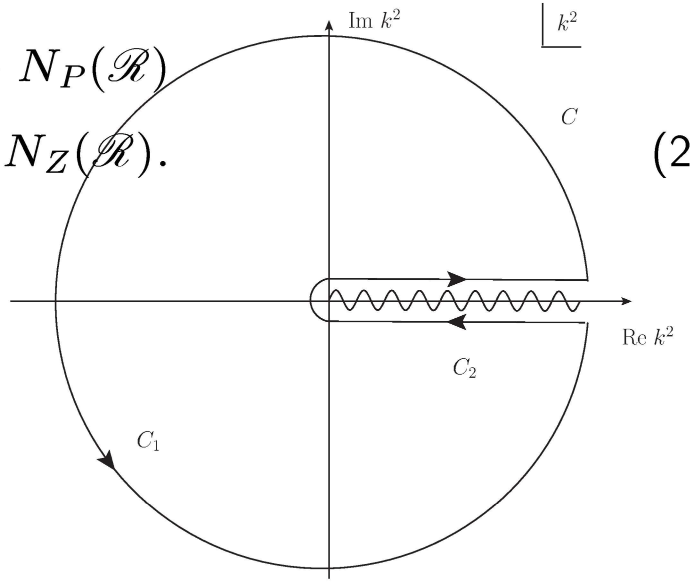
Theorem (principle of argument) For a given region $\mathscr{R} \subset \mathbb{C}$, let $N_W(C)$ be the **winding number** of the phase $\theta(k^2)$ of the propagator $\mathscr{D}(k^2)$ for a closed contour C as a boundary $C = \partial\mathscr{R}$ of the region $\mathscr{R}$ where the phase $\theta(k^2)$ is equal to the argument of $\mathscr{D}(k^2) = |\mathscr{D}(k^2)|e^{i\theta(k^2)}$:

$$N_W(C) := \frac{1}{2\pi i}\oint_C dk^2 \frac{\mathscr{D}'(k^2)}{\mathscr{D}(k^2)} = \oint_C \frac{d\theta(k^2)}{2\pi} \in \mathbb{Z}. \tag{1}$$

Then $N_W(C)$ is equal to the difference between the number of zeros $N_Z(\mathscr{R})$ and the number of poles $N_P(\mathscr{R})$ in $\mathscr{R}$:

$$\begin{aligned} N_W(C = \partial\mathscr{R}) &= N_Z(\mathscr{R}) - N_P(\mathscr{R}) \\ \Longleftrightarrow N_P(\mathscr{R}) &= -N_W(C = \partial\mathscr{R}) + N_Z(\mathscr{R}). \end{aligned} \tag{2}$$

For the contour C divided into two paths $C = C_1 + C_2$, $N_W(C) = N_W(C_1) + N_W(C_2)$.



(Case I) Positive spectral function

Suppose that a propagator exhibits the following behaviors. $z := k^2 \in \mathbb{C}$.

(i) The complex propagator has the leading asymptotic behavior: $\mathscr{D}(z) \sim -\frac{1}{z}\tilde{D}(z)$ as $|z| \to \infty$, where $\tilde{D}(z)$ is a real and positive function $\tilde{D}(z) > 0$ for large $|z|$.

(ii) The spectral function is always positive: $\rho(\sigma^2) > 0$, *i.e.*, $\operatorname{Im} \mathscr{D}(\sigma^2 + i\epsilon) > 0$ for $\sigma^2 > 0$.

(iii) The Euclidean propagator is positive in the n.b.d. of the origin: $\mathscr{D}(-\epsilon) > 0$ for sufficiently small $\epsilon > 0$.

(i) First of all, $\tilde{D}(z)$ often depends only on $|z|$: $\tilde{D}(z) = \tilde{D}(|z|)$. For example, for the "physical propagator" that has the spectral representation,

$$\mathscr{D}_{phys}(k^2) := \frac{Z}{M^2 - k^2} + \int_{\sigma_0^2}^{\infty} d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 - k^2}, \ Z > 0, \ \ \rho(\sigma^2) > 0, \tag{3}$$

$\tilde{D}(z)$ is a constant. Indeed, for sufficiently large $|z|$,

$$\mathscr{D}_{phys}(z) \sim -\frac{1}{z}\left[Z + \int_{\sigma_0^2}^{\infty} d\sigma^2 \rho(\sigma^2)\right] = -\frac{1}{z}\tilde{D}(z), \tag{4}$$

(i) yields the contribution from the large circle C_1

$$\frac{\mathscr{D}'(k^2)}{\mathscr{D}(k^2)} = (\log \mathscr{D}(k^2))' = (-\log k^2 + \ldots)' = \frac{-1}{k^2} + \ldots . \Longrightarrow N_W(C_1) = \frac{1}{2\pi i}\oint_{C_1} dk^2 \frac{-1}{k^2} = -1. \tag{5}$$

As shown in the next slide, (i),(ii),(iii) lead to

$$N_W(C_2) = +1. \tag{6}$$

Hence, the net winding number is zero

$$N_W(C) = N_W(C_1) + N_W(C_2) = 0. \tag{7}$$

Therefore we obtain

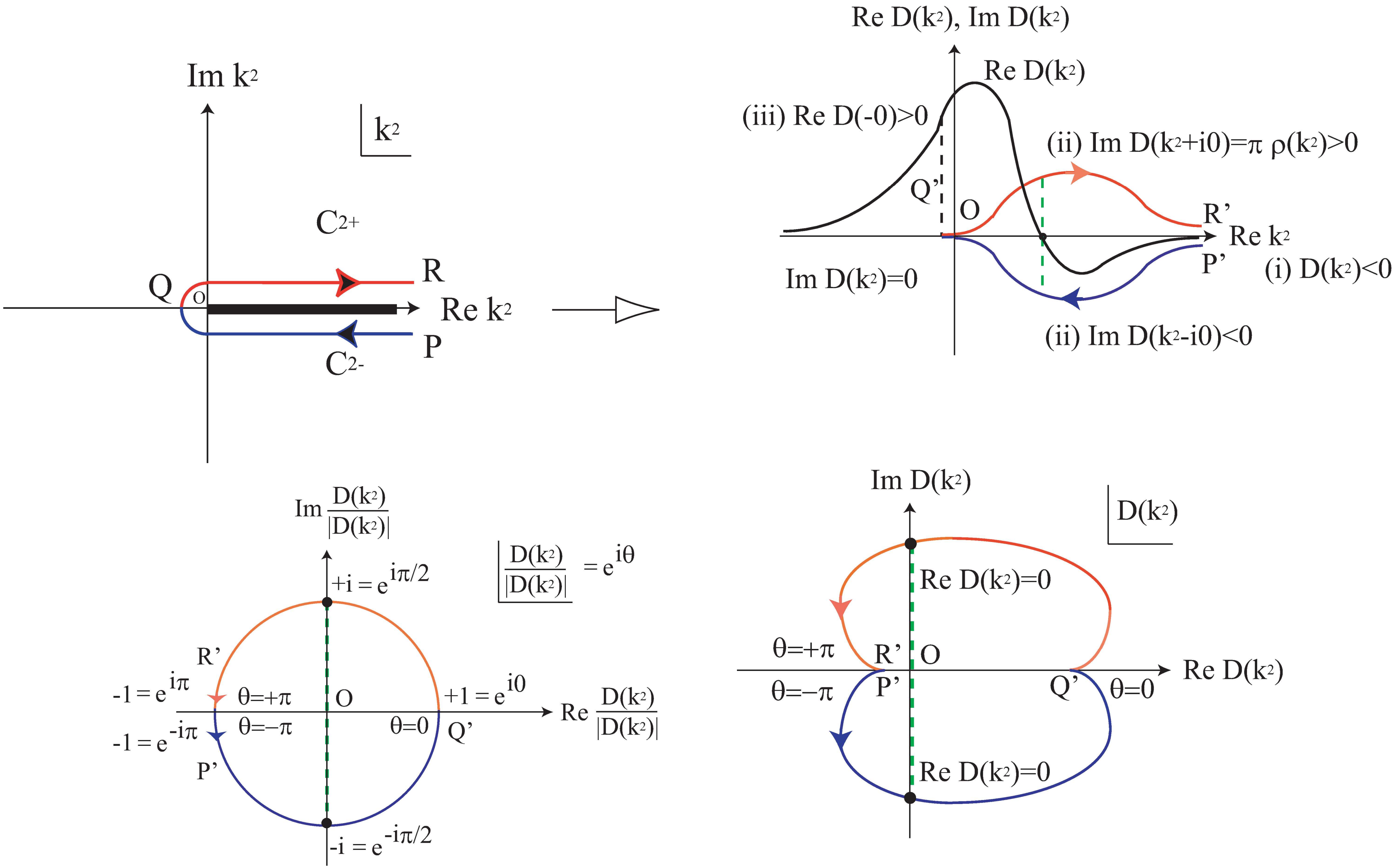
$$N_P = N_Z. \tag{8}$$

If the complex "physical propagator"(3) has no zeros $N_Z = 0$. it has no unphysical poles

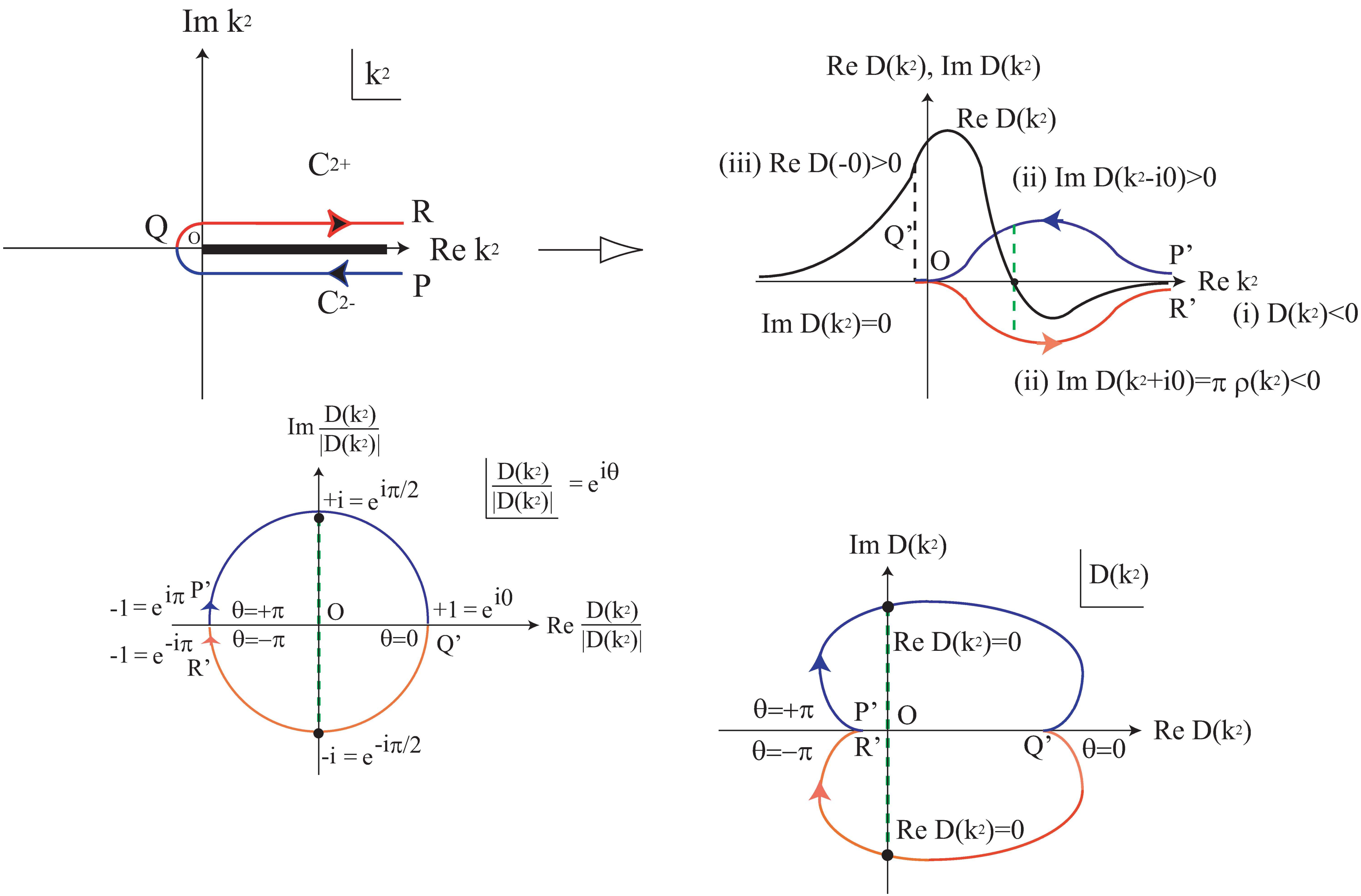
$$N_P = 0 \tag{9}$$

We call poles which are not located on the positive real axis **unphysical poles**, and particularly poles located on the negative real axis **Euclidean poles** or **tachyonic poles**.

(Case I) Positive spectral function $N_W(C_2) = +1$, $(N_W(C_1) = -1,\ N_P = N_Z)$

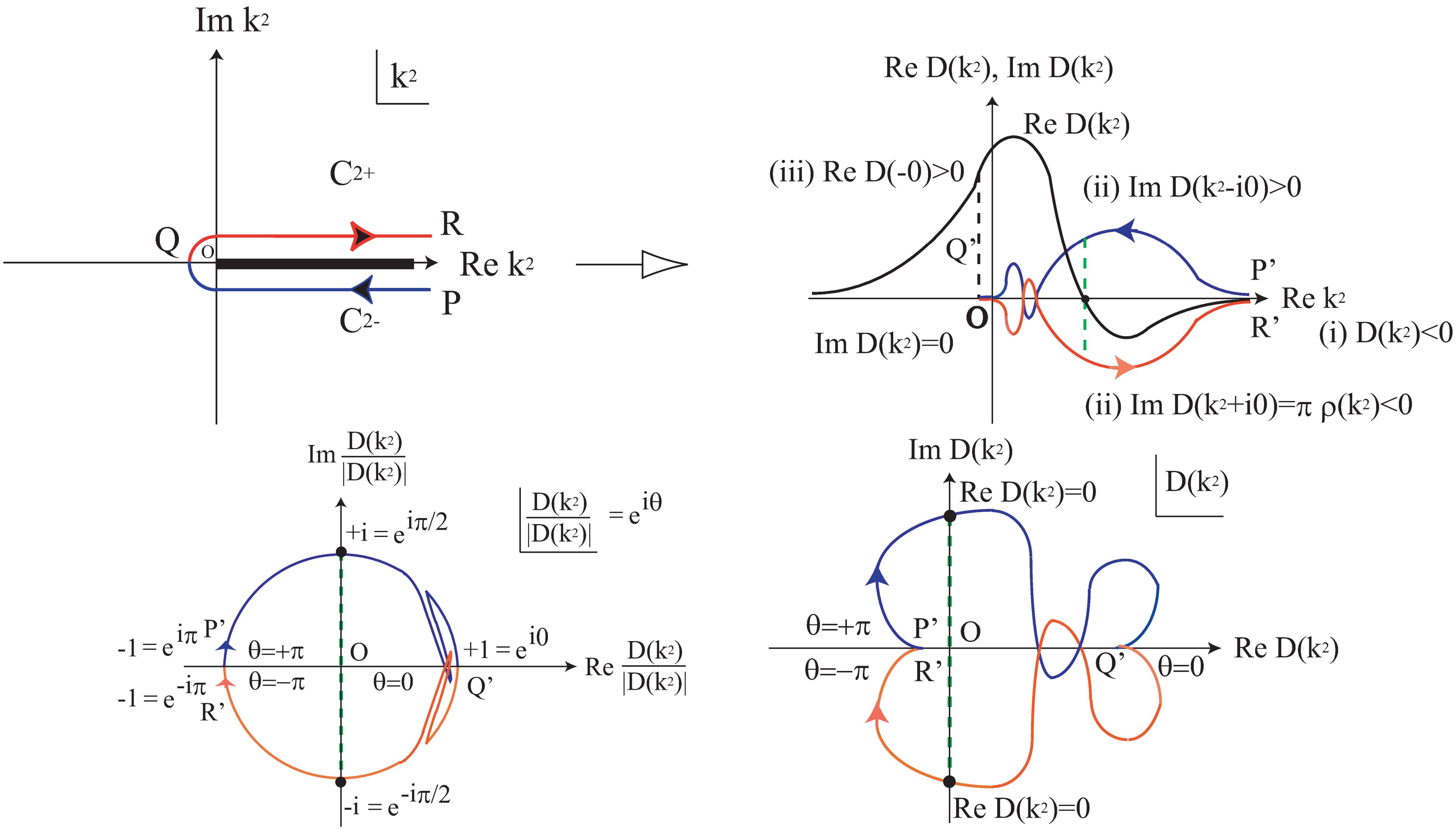


(Case II) Negative spectral function $N_W(C_2) = -1$, $(N_W(C_1) = -1,\ N_P = N_Z + 2)$



(Case II)' Quasi-negative spectral function $N_W(C_2) = -1$, $(N_W(C_1) = -1,\ N_P = N_Z + 2)$

We call a spectral function **quasi-positive** (resp. **quasi-negative**) if and only if $\rho(k_0^2) > 0$ (resp. $\rho(k_0^2) < 0$) at all real and positive zeros k_0^2 of $\mathrm{Re}\ D(k^2)$ *i.e.*, $\mathrm{Re}\ D(k_0^2) = 0\ (k_0^2 > 0)$.



§ Conclusion and discussion

⊙ We have considered the analytic continuation of the gluon propagator from the Euclidean region to the complex momentum plane towards the Minkowski region. Then we have derived general relationships between the number of complex poles of a propagator and the sign of the spectral function originating from the branch cut in the Minkowski region under some assumptions on the asymptotic behaviors of the propagator.

⊙ We have applied this relation to the massive Yang-Mills model with one-loop quantum corrections, which is to be identified with a low-energy effective theory of the Yang-Mills theory in the sense that the confining decoupling solution for the Euclidean gluon and ghost propagators of the Yang-Mills theory in the Landau gauge obtained by numerical simulations on the lattice are reproduced with good accuracy from the massive Yang-Mills model under the infrared safe renormalization scheme.

⊙ We have shown that the gluon propagator in the massive Yang-Mills model has a pair of complex conjugate poles (or Euclidean poles of multiplicity two), in accordance with the fact that the spectral function is always negative, while the ghost propagator has at most one "unphysical" pole.

The complex structure of the propagator enables us

to give an analytical proof that the reflection positivity is violated for any choice of the parameters in the massive Yang-Mills model, including the physical point of the Yang-Mills theory,

and to explain why the Euclidean gluon propagator is well described by the Gribov-Stingl form, which comes from the dominant pole part rather than the relatively small cut part.

⊙ It is well-known that the gluon spectral function becomes negative in the ultraviolet region in the Landau-gauge Yang-Mills theory due to [Oehme and Zimmermann (1980)]. In fact, the negativity of the spectral function in a weak sense, namely, the quasi-negativity of the spectral function is enough to deduce the existence of complex poles.

Backup Slides

⊙ The massive Yang-Mills model well reproduces gluon and ghost propagators of the decoupling solution in the Euclidean region of the pure Yang-Mills theory in the Landau gauge by a suitable choice of parameters g and M.

⊙ The reflection positivity is violated in the Euclidean region of the massive Yang-Mills model at the physical point of the parameters, as shown by observing the negativity of the Schwinger function obtained as the Fourier transform of the gluon propagator in a numerical way.

The violation of reflection positivity in the Euclidean region is regarded as a necessary condition for gluon confinement.

⊙ The violation of reflection positivity in the Euclidean region is understood from the complex structure of the gluon propagator obtained by performing the analytic continuation of the Euclidean propagator to the entire complex squared momentum plane: The violation of reflection positivity follows from
(i) the negativity of the spectral function obtained from the discontinuity of the gluon propagator across the branch cut on the positive real axis of the complex squared momentum plane.
(ii) the existence of a pair of complex conjugate poles in the gluon propagator.

⊙ At the physical point, the contribution from the cut part to the gluon propagator in the Euclidean region is relatively small compared with that from the pole part.
Therefore, the gluon propagator in the Euclidean region is well approximated by the contribution from a pair of complex conjugate poles, and is well described by the Gribov-Stingl form, in agreement with the lattice result.

⊙ In the massive Yang-Mills model we have confirmed at least to one-loop order that the gluon propagator has two unphysical poles, namely, one pair of complex conjugate poles (or Euclidean poles with multiplicity two) with no time-like poles, while the ghost propagator has no complex poles [Hayashi and Kondo, 2019]. This result stems from the negativity of the spectral function $\rho(\sigma^2)$.
It is well-known due to [Oehme and Zimmermann (1980)] that the gluon spectral function becomes negative in the ultraviolet region in the Landau-gauge Yang-Mills theory. In fact, the negativity of the spectral function in a weak sense, namely, the quasi-negativity is enough to deduce the existence of complex poles.

⊙ The presence of complex poles invalidates the ordinary Källén-Lehmann spectral representation and therefore indicates the gluon confinement in the sense that the one gluon particle state must be excluded from the physical spectrum.

Furthermore, the absence of a time-like pole in the gluon propagator also suggests that one gluon asymptotic state does not exist in the asymptotic state space in the Yang-Mills theory. Therefore, our results at the physical point of the massive Yang-Mills model support strongly gluon confinement in the Yang-Mills theory.

⊙ The massive Yang-Mills model in the Landau gauge has the gauge-invariant extension, namely, the complementary gauge-scalar model with a radially fixed fundamental scalar field subject to an appropriate reduction condition. This is performed through the gauge-independent description of the BEH mechanism. [Kondo (2018)]
In other words, the gauge-scalar model with a radially fixed fundamental scalar field subject to the reduction condition can be gauge-fixed to obtain the massive Yang-Mills theory in the covariant Landau gauge.

[a non-gauge theory = a gauge-fixed version of the gauge-invariant theory]

⊙ The result that the massive Yang-Mills model exhibits violation of reflection positivity for any choice of parameters g and M can be understood a consequence of the fact that the complementary gauge-scalar model has both Confinement-like region and Higgs-like region in a single confinement phase which is regarded as the continuum realization of the Fradkin-Shenker continuity.

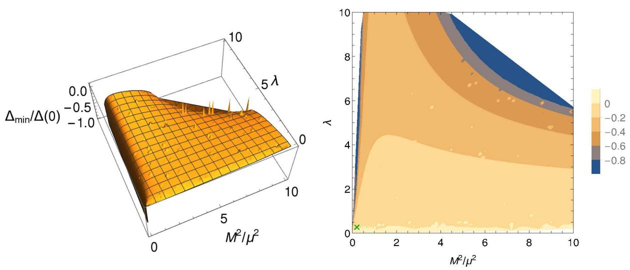


Figure 8: The magnitude of the violation of reflection positivity obtained from the ratio $\min_{0<t<\infty} \Delta(t)/\Delta(t=0)$ of the Schwinger functions in the larger range of parameters, (left) 3D plot, (right) contour plot.

⊙ The massive Yang-Mills theory in the covariant gauge has the gauge-invariant extension.

• [mYM] massive Yang-Mills theory in the covariant Landau gauge has no longer gauge symmetry, (although it has the modified BRST symmetry).

However, [mYM] has a gauge-invariant extension [GIE]. [GIE] is the gauge-scalar model

$$\mathscr{L}_{\rm RF} = -\frac{1}{2}{\rm tr}[\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}] + (D_\mu[\mathscr{A}]\Phi)^\dagger \cdot (D^\mu[\mathscr{A}]\Phi), \tag{1}$$

with a single scalar field in the fundamental rep. of the gauge group G subject to the radially-fixed constraint,

$$f(\Phi(x)) := \Phi(x)^\dagger \cdot \Phi(x) - \frac{1}{2}v^2 = 0, \ (v > 0) \text{ for } G = SU(2), \tag{2}$$

if an appropriate constraint which we call the reduction condition is imposed (off shell),

$$\chi(x) := \mathscr{D}_\mu[\mathscr{A}]\mathscr{W}^\mu(x) = 0, \quad \mathscr{W}^\mu = \mathscr{W}^\mu[\mathscr{A}, \Phi]. \tag{3}$$

Here $\mathscr{W}^\mu(x) = \mathscr{W}^\mu[\mathscr{A}(x), \Phi(x)]$ is the massive vector field mode defined shortly in terms of $\mathscr{A}$ and Φ, which follows from the gauge-independent Brout-Englert-Higgs (BEH) mechanism $M = gv/2$.
In other words, if we take the covariant Landau gauge and eliminate the scalar field, [GIE] reduces to [mYM],

$$\mathscr{L}_{\rm RF} + \mathscr{L}_{\rm GF} + \mathscr{L}_{\rm FP} \to \mathscr{L}_{m{\rm YM}} = \mathscr{L}_{YM} + \mathscr{L}_m + \mathscr{L}_{GF} + \mathscr{L}_{FP} \tag{4}$$

• Based on this correspondence, [mYM] can describe also the Higgs phase by choosing parameters g, M or g, v.
This is regarded as a continuum realization of the Fradkin-Shenker continuity shown on the lattice.

⊙ In the presence of complex poles, the usual superconvergence relation

$$\int_0^\infty d\sigma^2 \rho(\sigma^2) = 0, \tag{5}$$

due to Oehme and Zimmermann does not hold.
In the presence of one pair of complex conjugate poles, instead, we find that the modified superconvergence relation holds

$$2\,\mathrm{Re}\,Z + \int_0^\infty d\sigma^2 \rho(\sigma^2) = 0,\ 2\,\mathrm{Im}\,Z = \frac{1}{\pi}\int_{-\infty}^\infty dk^2\ \mathrm{Re}\,\mathscr{D}(k^2 + i\epsilon). \tag{6}$$

provided that the propagator has the asymptotic behavior $\lim_{|k^2|\to\infty} k^2 \mathscr{D}(k^2) = 0$. For the massive Yang-Mills model to one-loop order at the physical point are

$$\begin{aligned} &\mathrm{Re}\,Z = 0.386322,\ \mathrm{Im}\,Z = 0.861514, \\ &\int_0^\infty d\sigma^2 \rho(\sigma^2) = -0.694533 < 0,\ \frac{1}{\pi}\int_{-\infty}^\infty dk^2\ \mathrm{Re}\,\mathscr{D}(k^2 + i\epsilon) = 1.74006 > 0, \end{aligned} \tag{7}$$

$$2\,\mathrm{Re}\,Z + \int_0^\infty d\sigma^2 \rho(\sigma^2) = 0.0781108,\ 2\,\mathrm{Im}\,Z - \frac{1}{\pi}\int_{-\infty}^\infty dk^2\ \mathrm{Re}\,\mathscr{D}(k^2 + i\epsilon) = -0.0170369. \tag{8}$$

$\odot$ For ghosts, we impose the renormalization condition

$$\Gamma^{(2)}_{gh}(k_E = \mu) = \mu^2 \Longleftrightarrow \hat{\Pi}^{\text{fin}}_{gh}(s = \nu) = 0, \quad \nu := \frac{\mu^2}{M^2}. \tag{9}$$

For gluons, we can take a naive zero-momentum renormalization condition such that

$$\text{[TW1]} \begin{cases} \Gamma^{(2)}_{\mathscr{A}}(k_E = 0) = M^2 \\ \Gamma^{(2)}_{\mathscr{A}}(k_E = \mu) = \mu^2 + M^2 \end{cases} \Longleftrightarrow \begin{cases} \hat{\Pi}^{\text{fin}}_T(s = 0) = 0 \\ \hat{\Pi}^{\text{fin}}_T(s = \nu) = 0 \end{cases} \quad \text{(at } \mu = 1 \text{ GeV)}, \tag{10}$$

However, $\Gamma^{(2)}_{\mathscr{A}}(k_E = 0) = M^2$ or $\hat{\Pi}^{\text{fin}}(s = 0) = 0$ yields the **IR Landau pole**, namely, the coupling constant diverges at a certain momentum in the IR region.
To avoid the IR Landau pole, we replace the zero-momentum renormalization condition by

$$\text{[TW2]} \begin{cases} Z_{M^2} Z_{\mathscr{A}} Z_{\mathscr{C}} = 1 \\ \Gamma^{(2)}_{\mathscr{A}}(k_E = \mu) = \mu^2 + M^2 \end{cases} \Longleftrightarrow \begin{cases} Z_{M^2} Z_{\mathscr{A}} Z_{\mathscr{C}} = 1 \\ \hat{\Pi}^{\text{fin}}_T(s = \nu) = 0 \end{cases} \quad \text{(at } \mu = 1 \text{ GeV)}. \tag{11}$$

There is a well-known **non-renormalization in the Landau gauge** for the coupling [Taylor, 1971] which also holds in the massive Yang-Mills model in the Landau gauge:

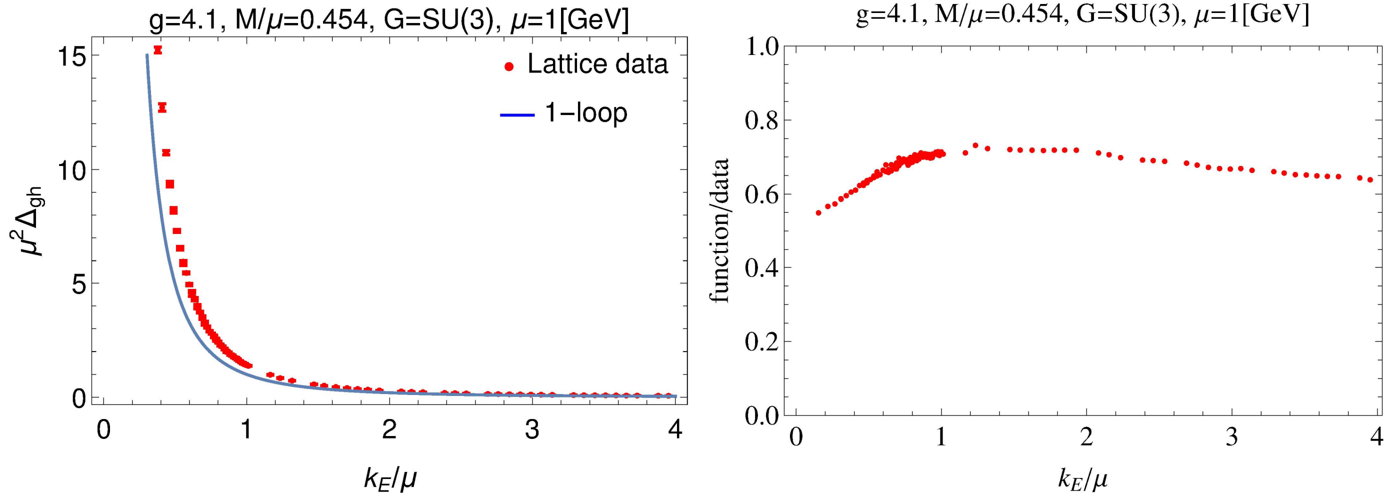
$$Z_g Z_{\mathscr{A}}^{1/2} Z_{\mathscr{C}} = \tilde{Z}_\alpha^2 = 1 \text{ for } \alpha = 0, \tag{12}$$

For the massive Yang-Mills model in the Landau gauge $\alpha = 0$, another identity holds

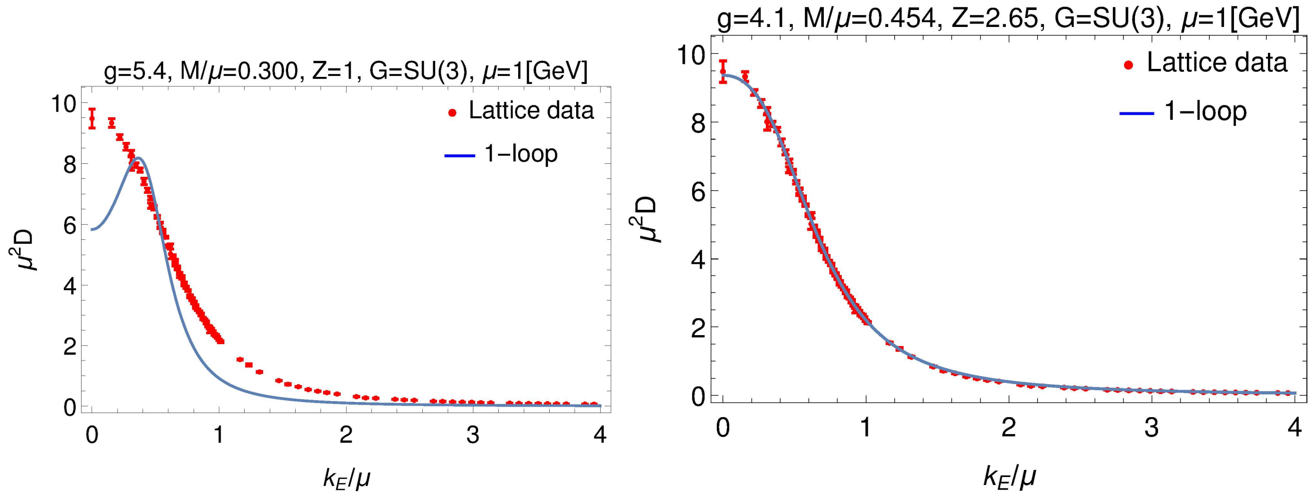
$$Z_{M^2} Z_{\mathscr{A}} Z_{\mathscr{C}} = \tilde{Z}_\alpha^2 = 1 \text{ for } \alpha = 0, , \tag{13}$$

As $Z_{M^2} Z_{\mathscr{A}} = 1 + \delta_{M^2}$ and $Z_{\mathscr{C}} = 1 + \delta_C$ this relation yields

$$\delta_{M^2} = Z_{M^2} Z_{\mathscr{A}} - 1 = Z_{\mathscr{C}}^{-1} - 1 = (1 + \delta_C)^{-1} - 1, \tag{14}$$



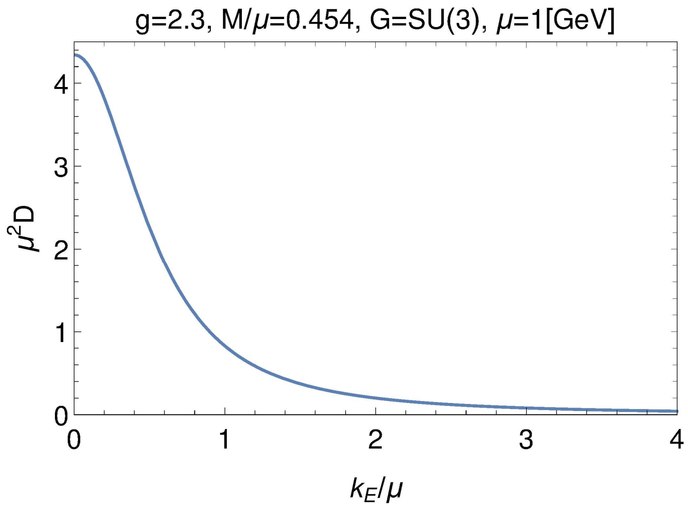
ghost propagator Δ_{gh} as functions of the Euclidean momentum k_E.

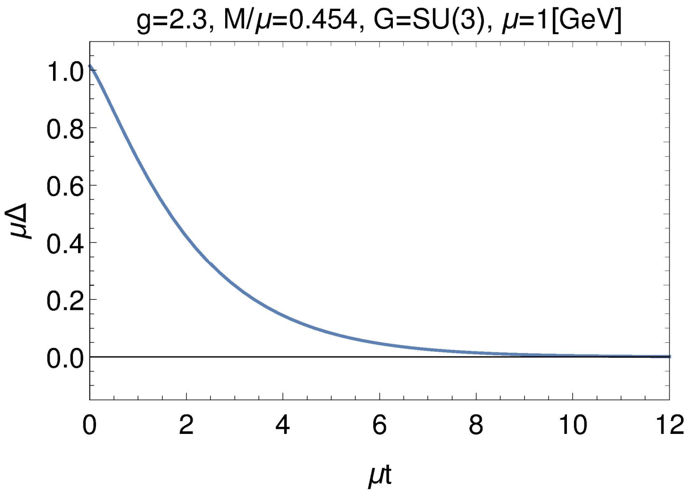


The gluon propagator $\mathscr{D}$ as functions of the Euclidean momentum k_E.

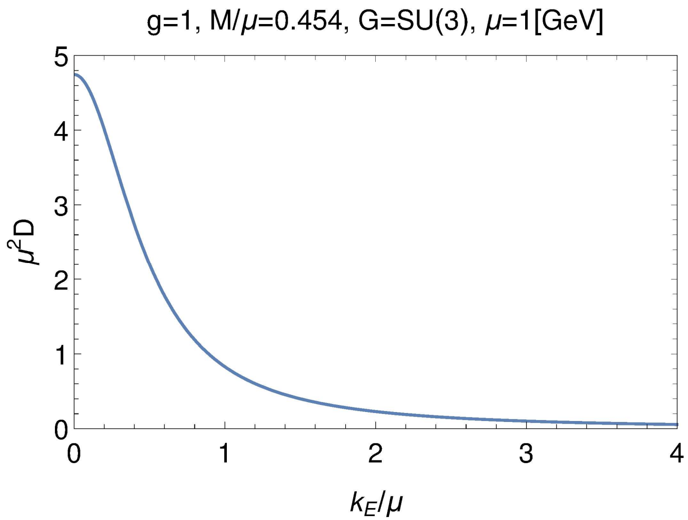
⊙ Positivity violation in the complementary gauge-scalar model

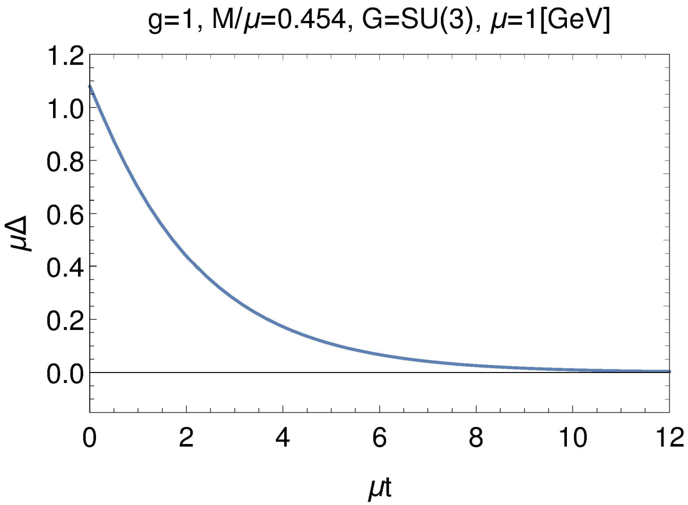
For a smaller coupling constant $g = 2.3$ and $M/\mu = 0.454$



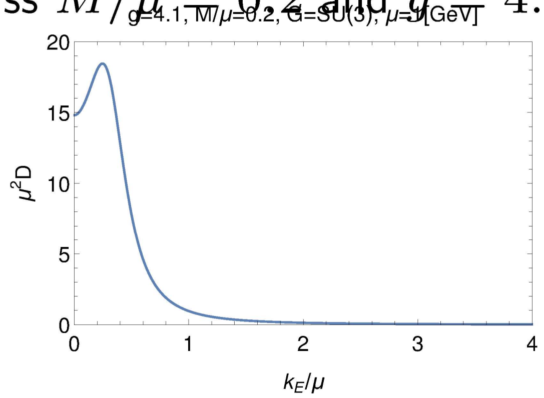


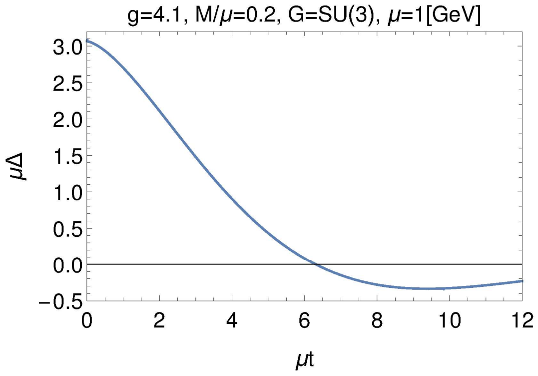
For a more smaller coupling constant $g = 1$ and $M/\mu = 0.454$



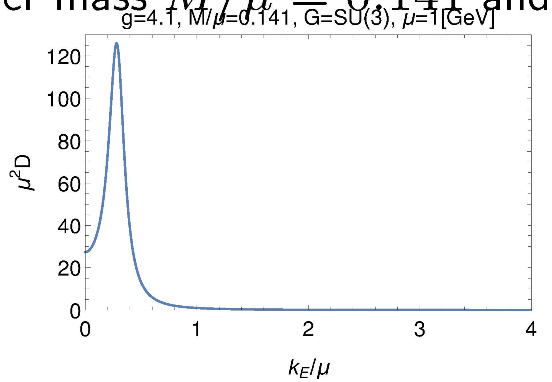


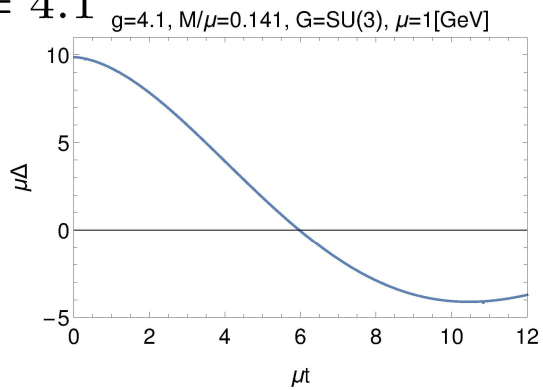
For a smaller mass $M/\mu = 0.2$ and $g = 4.1$



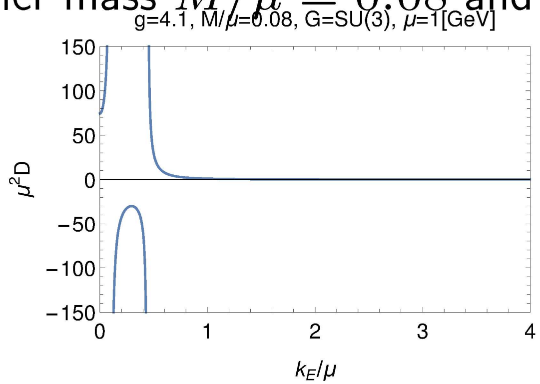


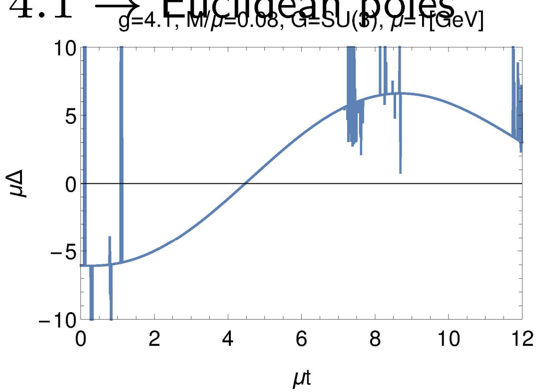
For a more smaller mass $M/\mu = 0.141$ and $g = 4.1$





For a much smaller mass $M/\mu = 0.08$ and $g = 4.1 \to$ Euclidean poles





The reflection positivity is violated for any choice of the parameters g and M.