QCD AT FINITE TEMPERATURE AND DENSITY
FROM THE CURCI-FERRARI MODEL

Urko Reinosa*

(based on various collaborations with J. A. Gracey, J. Maelger,
M. Peláez, M. Tissier, J. Serreau and N. Wschebor)

*Centre de Physique Théorique, Ecole Polytechnique,
CNRS & Institut Polytechnique de Paris

Light Cone 2019, 16-20 September 2019, Ecole Polytechnique.
This talk and Julien Serreau’s this afternoon are an invitation to look at the low energy properties of QCD from a new perspective.

Two main ideas to be conveyed:

− This talk: (some) low energy features of the pure gauge sector of the theory could be accessible via perturbative methods;
− Julien Serreau’s talk: and (some) low energy features of QCD from a systematic expansion, controlled by small parameters.

The aim is not to compete with other methods but to bring a complementary view on certain aspects of the problem.
OUTLINE

1. Yang-Mills theory from perturbative methods.

2. Heavy-quark QCD from perturbative methods.

Julien Serreau’s talk this afternoon:

3. QCD with small parameters.
YANG-MILLS THEORY
FROM PERTURBATIVE METHODS
Perturbation theory is not applicable to YM theories at low energies because the coupling grows large in this regime:

This statement is vague for at least two reasons.
1. The very notions of a renormalized coupling and a perturbative expansion depend on a choice of renormalization scheme but also on a choice of gauge.

2. Our ability to say anything about the fate of the running coupling at low energies relies on our ability to fix the gauge consistently.
THE NEED FOR GAUGE FIXING

The YM action is invariant under a gauge symmetry group:

\[ S_{YM}[A^U] = S_{YM}[A] \]

with

\[(A^U)^a_\mu(x)t^a \equiv U(x)A^a_\mu(x)t^aU^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x)\]

Prevents the definition of Feynman diagrams (no gluon propagator).
FADDEEV-POPOV GAUGE FIXING

Constrained formulation:

\[ S_{YM}[A] \quad \text{with} \quad \partial_\mu A_\mu^a = 0 \quad [\text{Landau gauge}] \]

Unconstrained formulation:

\[ S_{FP} = S_{YM} + \int_X \left\{ i h^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu D_\mu c^a \right\} \quad [\text{Faddeev-Popov}] \]

The equivalence between the two formulations is known to rely on a mathematically incorrect assumption ("Gribov copy problem").

A bit more subtle in fact:

- At high energies, the equivalence is seen to hold.
- At low energies, we have tangible evidence that it does not.
SCALING VS DECOUPLING SOLUTIONS

Kugo and Ojima: unconstrained formulation + confinement

⇒ “scaling” solution
\[
\begin{align*}
J(q^2) &\equiv q^2\langle c(-q)\bar{c}(q)\rangle \to \infty \text{ as } q \to 0 \\
D(q^2) &\equiv P_{\mu\nu}(q)\langle A_\mu(-q)A_\nu(q)\rangle \to 0
\end{align*}
\]

At odds with the “decoupling” solution found on the lattice (constrained formulation):

[I. L. Bogolubsky, E. M. Ilgenfritz, M. Müller-Preussker, A. Sternbeck, PLB 676, 69 (2009)]
RUNNING COUPLING FROM THE LATTICE

\[ \alpha_s(q^2) \]

\[ q^2 \text{ [GeV}^2\text{]} \]

\[ \beta = 5.7 \]

BEYOND THE FADDEEV-POPOV APPROACH

From the above considerations the following scenario emerges:

- The FP action is a model that needs to be extended.
- This could open a window into a new weakly coupled regime.

How to find the appropriate extension of the FP action?

- semi-constructive approaches: Gribov-Zwanziger, ...;
- phenomenological approach: add new operators and try to constrain their couplings, or even discard them, using experiments/lattice simulations.
THE CURCI-FERRARI MODEL

The Curci-Ferrari (CF) model is one example of such an extension:

\[
S_{FP} \to S_{CF} = S_{FP} + \int_x \frac{m_0^2}{2} A^a_\mu A^a_\mu
\]

It is renormalizable: only one additional parameter \( m_0 \) to be dealt with.

[NB: the mass is added to an already (partially) gauge-fixed action, not to the YM action. The CF model IS NOT massive YM (Proca theory)]
ONE-LOOP CURCI-FERRARI CORRELATORS

\[ G(p) \equiv D(p) \]

\[ F(p) \equiv J(p) \]

Tissier and Wschebor, Phys. Rev. D84 (2011);
[For 3-point functions, see also: Peláez, Tissier, Wschebor, Phys. Rev. D88 (2013)]

\[ m_0 \simeq 500 \text{ MeV} \]
Two-loop corrections

\[ \frac{p^2 G(p)}{p (\text{GeV})} \]

IS one-loop results

IS two-loop results

\[ \frac{F(p)}{p (\text{GeV})} \]

IS one-loop results

IS two-loop results

[J.A. Gracey, M. Peláez, U. Reinosa, M. Tissier, PRD 100, 034023 (2019)]
\[
\frac{g^2 N_c}{16 \pi^2} = \frac{\alpha_s N_c}{4 \pi} < 0.3
\]
The confinement/deconfinement transition is conveniently described in terms of the Polyakov loop

$$\ell \equiv \left\langle \text{tr} P \exp \left\{ i g \int_0^\beta d\tau A_0^a(\tau, \vec{x}) t^a \right\} \right\rangle, \quad \tau \in [0, \beta = 1/T]$$

It is directly related to the energy cost $\Delta F_q$ for bringing a heavy test quark into the medium

$$\ell \propto e^{-\beta \Delta F_q} \rightarrow \begin{cases} \ell = 0, \Delta F_q = \infty : \text{confined phase;} \\ \ell \neq 0, \Delta F_q < \infty : \text{deconfined phase.} \end{cases}$$

⇒ Need to evaluate the Polyakov loop potential $V(\ell)$. 
ONE-LOOP POLYAKOV LOOP POTENTIAL

\[ \ell \simeq \text{tr} \, e^{i g \beta \langle A^0 \rangle}, \quad g \beta \langle A^0 \rangle = r_3 \frac{\lambda_3}{2} + r_8 \frac{\lambda_8}{2} \] [gluonic background]
[UR, J. Serreau, M. Tissier, N. Wschebor, PLB 742, 61 (2015)]
CONFINEMENT MECHANISM

Take SU(2) for instance:

\[
W_{SU(2)}(r_3) = \frac{3 T}{\pi^2} \int_0^\infty dq \, q^2 \, \text{Re} \ln \left( 1 - e^{-\beta \sqrt{q^2 + m^2 + ir_3}} \right) \quad \text{massive gluons}
\]

\[
- \frac{T}{\pi^2} \int_0^\infty dq \, q^2 \, \text{Re} \ln(1 - e^{-\beta q + ir_3}) \quad \text{massless ghosts}
\]

**\( T \gg m : \)**

\[
\frac{2 T}{\pi^2} \int_0^\infty dq \, q^2 \, \text{Re} \ln(1 - e^{-\beta q + ir_3})
\]

**\( T \ll m : \)**

\[
- \frac{T}{\pi^2} \int_0^\infty dq \, q^2 \, \text{Re} \ln(1 - e^{-\beta q + ir_3})
\]

This mechanism is common to most continuum approaches (fRG, ...) and is rooted in the properties of the decoupling solution.
## SUMMARY OF RESULTS (ONE-LOOP CF)

<table>
<thead>
<tr>
<th>order</th>
<th>lattice</th>
<th>fRG</th>
<th>CF model at 1-loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU(2)</td>
<td>2nd</td>
<td>2nd</td>
<td>2nd</td>
</tr>
<tr>
<td>SU(3)</td>
<td>1st</td>
<td>1st</td>
<td>1st</td>
</tr>
<tr>
<td>SU(4)</td>
<td>1st</td>
<td>1st</td>
<td>1st</td>
</tr>
<tr>
<td>Sp(2)</td>
<td>1st</td>
<td>1st</td>
<td>1st</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$T_c$ (MeV)</th>
<th>lattice</th>
<th>fRG(*)</th>
<th>CF at 1-loop(**)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU(2)</td>
<td>295</td>
<td>230</td>
<td>238</td>
</tr>
<tr>
<td>SU(3)</td>
<td>270</td>
<td>275</td>
<td>185</td>
</tr>
</tbody>
</table>

SUMMARY OF RESULTS (TWO-LOOP CF)

<table>
<thead>
<tr>
<th>order</th>
<th>lattice</th>
<th>fRG</th>
<th>CF model at 1-loop</th>
<th>CF model at 2-loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU(2)</td>
<td>2nd</td>
<td>2nd</td>
<td>2nd</td>
<td>2nd</td>
</tr>
<tr>
<td>SU(3)</td>
<td>1st</td>
<td>1st</td>
<td>1st</td>
<td>1st</td>
</tr>
<tr>
<td>SU(4)</td>
<td>1st</td>
<td>1st</td>
<td>1st</td>
<td>1st</td>
</tr>
<tr>
<td>Sp(2)</td>
<td>1st</td>
<td>1st</td>
<td>1st</td>
<td>1st</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$T_c$ (MeV)</th>
<th>lattice</th>
<th>fRG(*)</th>
<th>CF model at 1-loop</th>
<th>CF model at 2-loop(****)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU(2)</td>
<td>295</td>
<td>230</td>
<td>238</td>
<td>284</td>
</tr>
<tr>
<td>SU(3)</td>
<td>270</td>
<td>275</td>
<td>185</td>
<td>254</td>
</tr>
</tbody>
</table>

HEAYV-QUARK QCD
FROM PERTURBATIVE METHODS
\[ \cdots + \sum_{f=1}^{N_f} \int_x \left\{ \overline{\psi}_f (\partial - ig A^a t^a + M_f) \psi_f \right\} , \text{with } M_f \gg T \]
COLUMBIA PLOT FROM THE CF MODEL

\( M > M_c : \) first order

\( M = M_c : \) second order

\( M < M_c : \) crossover
COLUMBIA PLOT FROM THE CF MODEL

\[ 1 - e^{-\frac{M_s}{m}} \]

\[ 1 - e^{-\frac{M_u}{m}} \]

Crossover

1st order
\[ R_{N_f} \equiv M_c / T_c \quad \begin{array}{l|ccc}
 N_f = 1 & N_f = 2 & N_f = 3 \\
 \hline
 \text{Lattice}^{(i)} & 7.23 & 7.92 & 8.33 \\
 \text{CF}^{(ii)} & 6.74 & 7.59 & 8.07 \\
 \text{GZ}^{(iii)} & 7.09 & 7.92 & 8.40 \\
 \text{Matrix}^{(iv)} & 8.04 & 8.85 & 9.33 \\
 \text{DSE}^{(v)} & 1.42 & 1.83 & 2.04 \\
 \end{array} \]

At two-loop order, the comparison to lattice data is tricky since the quark mass $M$ is scheme dependent. To reduce scheme dependences we can compare ratios of $R_{N_f} = M_c(N_f)/T_c(N_f)$ at various values of $N_f$.

<table>
<thead>
<tr>
<th>$R_{N_f} \equiv M_c / T_c$</th>
<th>$N_f = 1$</th>
<th>$N_f = 2$</th>
<th>$N_f = 3$</th>
<th>$R_2/R_1$</th>
<th>$R_3/R_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lattice$^{(i)}$</td>
<td>7.23</td>
<td>7.92</td>
<td>8.33</td>
<td>1.10</td>
<td>1.15</td>
</tr>
<tr>
<td>CF 1-loop$^{(ii)}$</td>
<td>6.74</td>
<td>7.59</td>
<td>8.07</td>
<td>1.12</td>
<td>1.20</td>
</tr>
<tr>
<td>CF 2-loop$^{(iib)}$</td>
<td>7.53</td>
<td>8.40</td>
<td>8.90</td>
<td>1.11</td>
<td>1.18</td>
</tr>
<tr>
<td>GZ$^{(iii)}$</td>
<td>7.09</td>
<td>7.92</td>
<td>8.40</td>
<td>1.12</td>
<td>1.19</td>
</tr>
<tr>
<td>Matrix$^{(iv)}$</td>
<td>8.04</td>
<td>8.85</td>
<td>9.33</td>
<td>1.10</td>
<td>1.16</td>
</tr>
<tr>
<td>DS$^{(v)}$</td>
<td>1.42</td>
<td>1.83</td>
<td>2.04</td>
<td>1.29</td>
<td>1.43</td>
</tr>
</tbody>
</table>


IMAGINARY CHEMICAL POTENTIAL

\[ \cdots + \sum_{f=1}^{N_f} \int_x \left\{ \bar{\psi}_f (\partial - igA^a t^a + M_f - i \mu_i \gamma_0) \psi_f \right\} , \text{with } M_f \gg T, \mu \]

Not really physical, but much praised by the lattice community since the sign problem is absent. For us, immense source of data to which we can compare to.

For \( \mu_i = (\pi/3)T \), the action admits a new symmetry (Roberge-Weiss), combining center transformations, abelian gauge transformations and charge conjugation.

The Roberge-Weiss (RW) symmetry is known to be broken for large enough temperatures.
RW TRANSITION FROM THE CF MODEL

\begin{figure}
\centering
\includegraphics[width=\textwidth]{rw_transition_plot}
\caption{Phase transition from the CF model.}
\end{figure}

\begin{equation}
\text{Arg } \ell
\end{equation}

\begin{align*}
T/m &= 0.34 \\
\mu_i/T &= 0.36
\end{align*}
RW TRANSITION FROM THE CF MODEL

$M > M_c(0)$:

$M \in [M_c(i\pi/3), M_c(0)]$:

$M = M_c(i\pi/3)$:

Agrees with lattice [P. de Forcrand, O. Philipsen, Phys.Rev.Lett. 105 (2010)]
\[ \frac{M_c}{T_c} = \frac{M_{\text{tric.}}}{T_{\text{tric.}}} + K \left[ \left( \frac{\pi}{3} \right)^2 + \left( \frac{\mu}{T} \right)^2 \right]^{2/5} \]

\begin{table}
\begin{tabular}{|c|c|c|c|}
\hline
 & our model(*) & lattice(**) & SD(***)
\hline
K & 1.85 & 1.55 & 0.98 \\
\frac{M_{\text{tric.}}}{T_{\text{tric.}}} & 6.15 & 6.66 & 0.41 \\
\hline
\end{tabular}
\end{table}

(**) Fromm et.al., JHEP 1201 (2012) 042.
REAL CHEMICAL POTENTIAL

The critical line moves towards larger quark masses as $\mu$ is increased:

\[ 1 - e^{-\frac{M_s}{m}} \]

\[ 1 - e^{-\frac{M_u}{m}} \]

in full agreement with lattice simulations.
CONCLUSIONS

In the Landau gauge, the Faddeev-Popov action is not a faithful description of Yang-Mills theory at low energies.

A simple model beyond the Faddeev-Popov recipe seems to render perturbation theory viable at all scales (in YM theory/heavy QCD):

- Lattice correlators are reproduced quite satisfactorily at LO.
- The phase structure of pure Yang-Mills is reproduced.
- This applies also to QCD in the heavy-quark limit.
- Two-loop corrections improve the results.

Can one generate the CF model (or something similar) from first principles?
→ In progress.

How to extend our approach to QCD?
→ see Julien Serreau’s talk this afternoon.
THANK YOU ALL
FOR YOUR ATTENTION
AND “BON APPETIT”!
BACKUP SLIDES
REAL CHEMICAL POTENTIAL

A proper discussion of the real chemical potential case requires one to allow for imaginary components of the background gluon field:

\[
\bar{A}_\mu^a = \langle A_\mu^a \rangle \bar{\Lambda} = (\bar{A}_\mu^3 \in \mathbb{R}) \frac{\lambda_3}{2} + (\bar{A}_\mu^8 \in i\mathbb{R}) \frac{\lambda_8}{2}
\]

Only then are \( \ell \) and \( \bar{\ell} \) real and positive, as it should to ensure \( \Delta F, \Delta \bar{F} \in \mathbb{R} \). Moreover, we find \( r_8 \neq 0 \), which implies \( \Delta F \neq \Delta \bar{F} \), as expected since \( \mathcal{C} \):

\[
\Delta F = -T \ln \ell \\
\Delta \bar{F} = -T \ln \bar{\ell}
\]