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Purely relativistic states: their content and EM form factors

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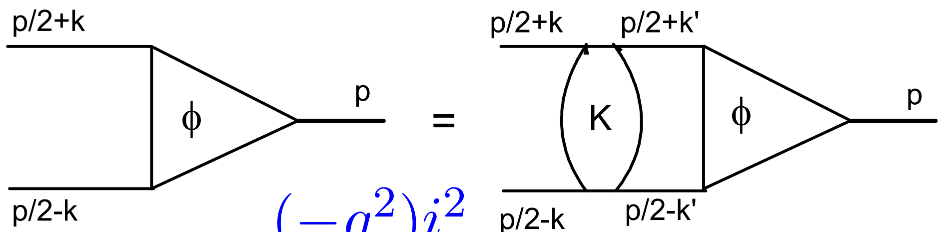
• Bethe-Salpeter bound state equation

Schrödinger equation in the momentum space:

$$\psi(\vec{k}) = \frac{1}{(\vec{k}^2 - mE)} \int V(\vec{k} - \vec{k}')\psi(\vec{k}') \frac{d^3k'}{(2\pi)^3}$$

E.E. Salpeter, H. Bethe, 1951



$$\Phi(k,p) = \frac{(-g^2)i^2}{\left((\frac{p}{2}+k)^2 - m^2 + i\epsilon\right)\left((\frac{p}{2}-k)^2 - m^2 + i\epsilon\right)} \leftarrow \frac{1}{(\vec{k}^2 - mE)}$$

$$\times \int \frac{d^4k'}{(2\pi)^4} \frac{i\Phi(k',p)}{[(k-k')^2 - \mu^2 + i\epsilon]}, \quad \mu = 0 \leftarrow \int V(\vec{k} - \vec{k}')\psi(\vec{k}') \frac{d^3k'}{(2\pi)^3}$$

$$\alpha = \frac{g^2}{16\pi m^2} \rightarrow V(r) = -\frac{\alpha}{r}, \quad c = 1$$

• Non-relativistic limit

Relativity exists since the speed of light c is finite
and the same in any frame.
Non-relativistic limit is $c \to \infty$.
We should restore c in the BS equation and take the limit $c \to \infty$
(analytically and/or numerically).

Restoring c:

$$m \to mc^2,\ M \to Mc^2,\ \alpha = \frac{e^2}{\hbar c} \to \frac{\alpha}{c}.$$

G. Wanders, *Limite non-relativiste d’une équation de Bethe-Salpeter*,
Helvetica Physica Acta, 1957

• Dependence $\alpha(c)$

Ground (normal) state

We repeat the calculations for a set of values of speed of light $1 \leq c \leq 10$ and find the dependence $\alpha(c)$.

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(*   m=1, mu=0.15, B=0.1  *)
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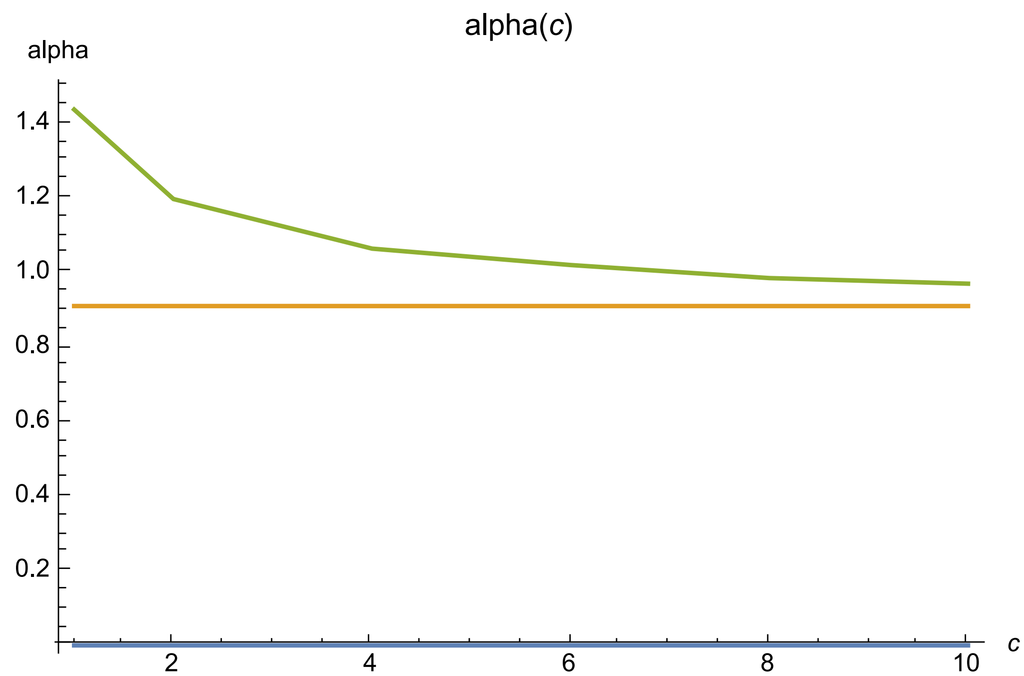
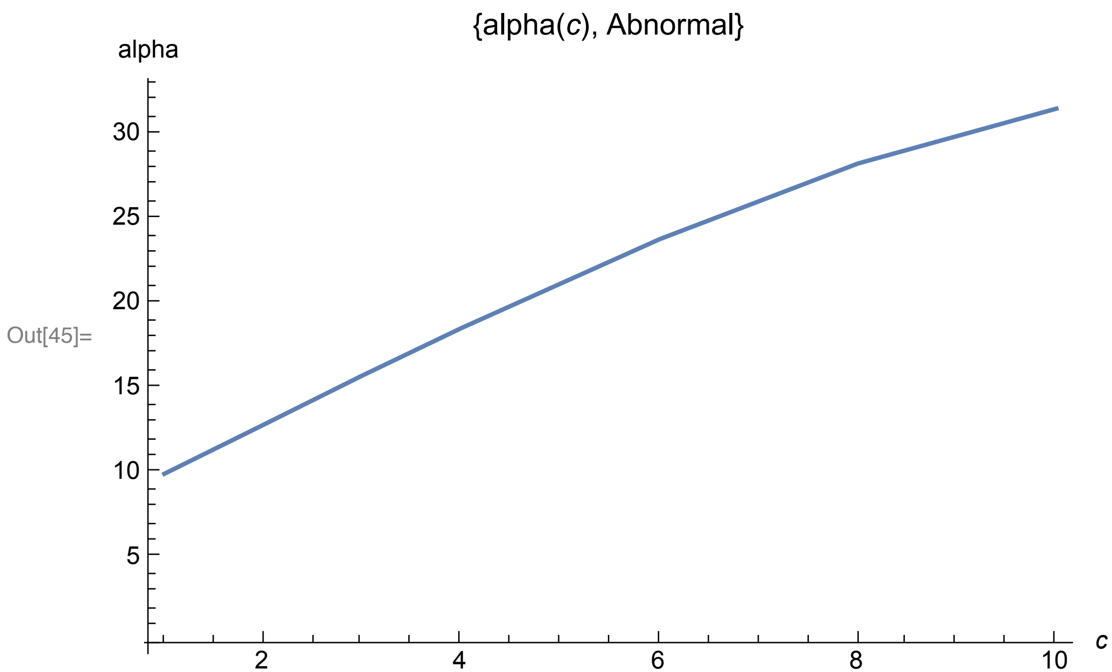


Dependence of the coupling constant α (for the ground state) on speed of light c. Horizontal line is the non-relativistic limit.

• Solutions of the second type



Dependence of the coupling constant α
(for $\mu = 0.15$, $B = 0.1$) on speed of light c

For normal state: $\alpha(c \to \infty) \to$ finite (nonrelativistic) limit.
For abnormal state: $\alpha(c \to \infty)$ increases without any limit.

• Abnormal solutions for $\mu = 0$

In 1954, G.C. Wick and R.E. Cutkosky,
still for massless exchange $\mu = 0$,
solved BS equation and reproduced Balmer series.
In addition, they found another series,
which is absent in the Schrödinger equation.

These new solutions, which disappear in the
non-relativistic limit, were called
"abnormal" solutions.

• Spectrum

In general: $E = E_{nk}, \quad n = 1, 2, 3\ldots, \quad k = 0, 1, 2, 3, \ldots$

If $k = 0$, the normal Balmer series is reproduced (with a relativistic correction):

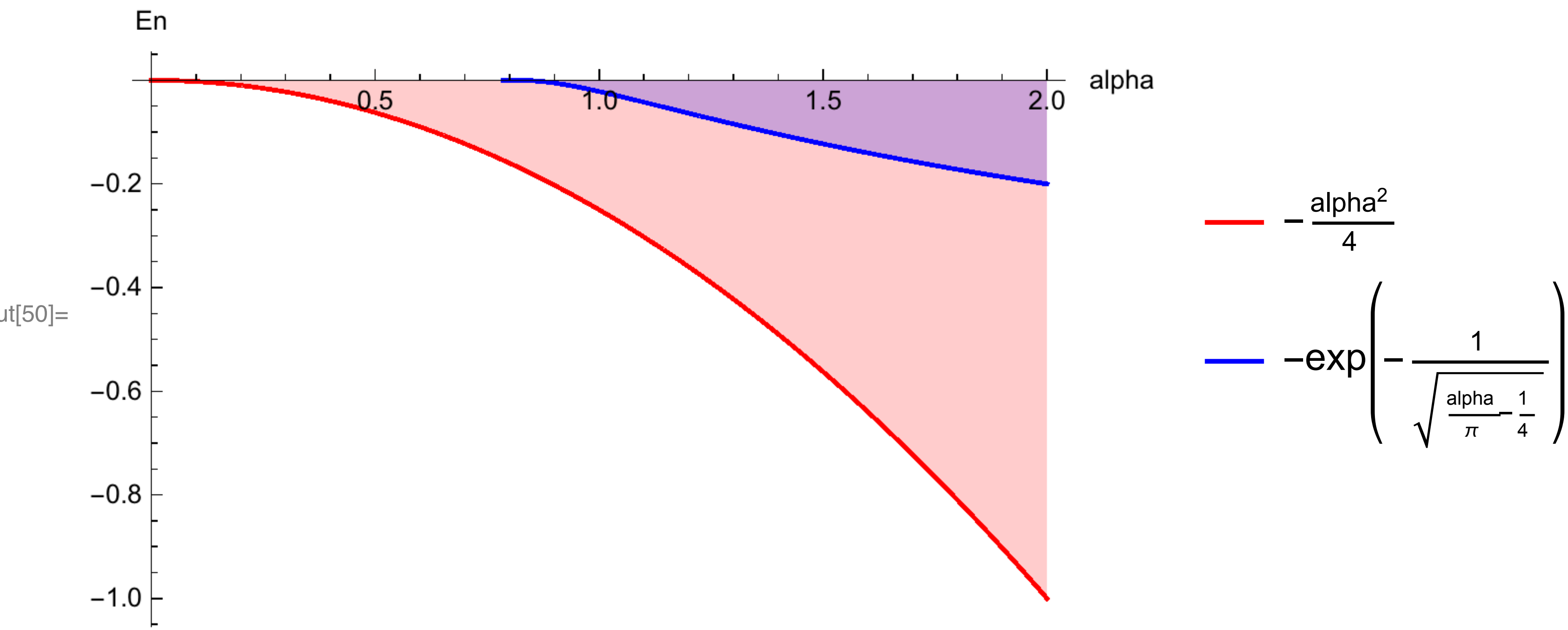
$$E_n = -\frac{\alpha^2 m}{4n^2}\left(1 + \frac{4}{\pi}\alpha \log \alpha\right)$$

However, for each given n - another (abnormal) series with $k = 1, 2, 3, \ldots$. For $n = 1$:

$$E_k = -m \exp\left(-\frac{2\pi k}{\sqrt{\frac{\alpha}{\pi} - \frac{1}{4}}}\right), \quad k = 1, 2, 3\ldots, \quad \alpha > \frac{\pi}{4}.$$

This *analytical* formula is valid when $\alpha \to \frac{\pi}{4}, \quad E_k \to 0$.

• Energy spectrum (still for $\mu = 0$)



The binding energies for normal and abnormal states.

Abnormal states are not predicted by the Schrödinger equation, but they are predicted by the BS one! They have purely relativistic origin.

• Limit $c \to \infty$

Normal: $B = \frac{\alpha^2}{c^2}\frac{mc^2}{4}\left(1 + \frac{4}{\pi}\frac{\alpha}{c}\log\frac{\alpha}{c}\right), \quad B = |E|$

$\Rightarrow$ solving relative to α

$$\alpha(c \to \infty) = \sqrt{\frac{4B}{m}} \to const$$

Abormal: $B = mc^2 \exp\left(-\frac{2\pi k}{\sqrt{\frac{\alpha}{c\pi} - \frac{1}{4}}}\right)$

$\Rightarrow$ solving relative to α

$$\alpha = \frac{\pi c}{4} + \frac{4\pi^3 c k^2}{\log^2 \frac{B}{mc^2}}$$

• What about the case $\mu \neq 0$?

J. Carbonell, V.A. Karmanov and H. Sazdjian, LC2018:

We solved the BS equation numerically for $\mu \neq 0$

and we found abnormal states.

They may exist in nature!

What are their properties?

Properties: the content and the EM form factors.

The content: from what are they made?

– The aim of the present talk.

• What is content?

"Two-body" BS amplitude is not the two-body one in terms of the Fock components!

$$|p\rangle = \sum_{n\geq 2}^{\infty} \int \psi_n(k_1,\ldots,k_n,p)\,|n\rangle$$

$$|n\rangle = \frac{1}{\sqrt{(n-2)!}} a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2)\ldots a^\dagger(\vec{k}_{n-2})b^\dagger(\vec{k}_1)b^\dagger(\vec{k}_2)\,|0\rangle\,,\ (n\geq 2)$$

$$\langle p'|p\rangle = 1 = \int \psi_2^2\ldots + \int \psi_3^2\ldots + \int \psi_4^2\ldots + \cdots$$

$$= N_2 + N_3 + N_4 + \cdots$$

$a^\dagger(\vec{k}_i)$ - the constituent particles, $b^\dagger(\vec{k}_{1,2})$ - the exchanged particles.

The "content" is the values $N_2, N_3, N_4, \ldots$

Two-body LFWF ψ_2 via BS amplitude

$$\Phi(x_1,x_2,p)=\langle 0|\,T(\varphi(x_1)\varphi(x_2))\,|p\rangle$$

Explicitly covariant version of LFD:

$$\omega\cdot x=\omega_0 t-\vec{\omega}\cdot\vec{x}=0,\quad \omega^2=0.$$

Standard version: $\omega=(\omega_0,\vec{\omega})=(\omega_0,\omega_x,\omega_y,\omega_z)=$
$(1,0,0,-1)\rightarrow \omega\cdot x=t+z=0$

Relation between ψ_2 and Φ:

$$\psi(\vec{k}_\perp,x)=\frac{(\omega\cdot k_1)(\omega\cdot k_2)}{\pi(\omega\cdot p)}\int_{-\infty}^{+\infty}\Phi(k+\beta\omega,p)d\beta\ \rightarrow\ dk_-\text{-integration}$$

• Nakanishi representation

for the BS amplitude:

$$\Phi(k,p) = -i\int_{-1}^{+1}\frac{g(z)dz}{(m^2 - M^2/4 - k^2 - zp{\cdot}k - i\epsilon)^3}$$

Two-body LFWF:

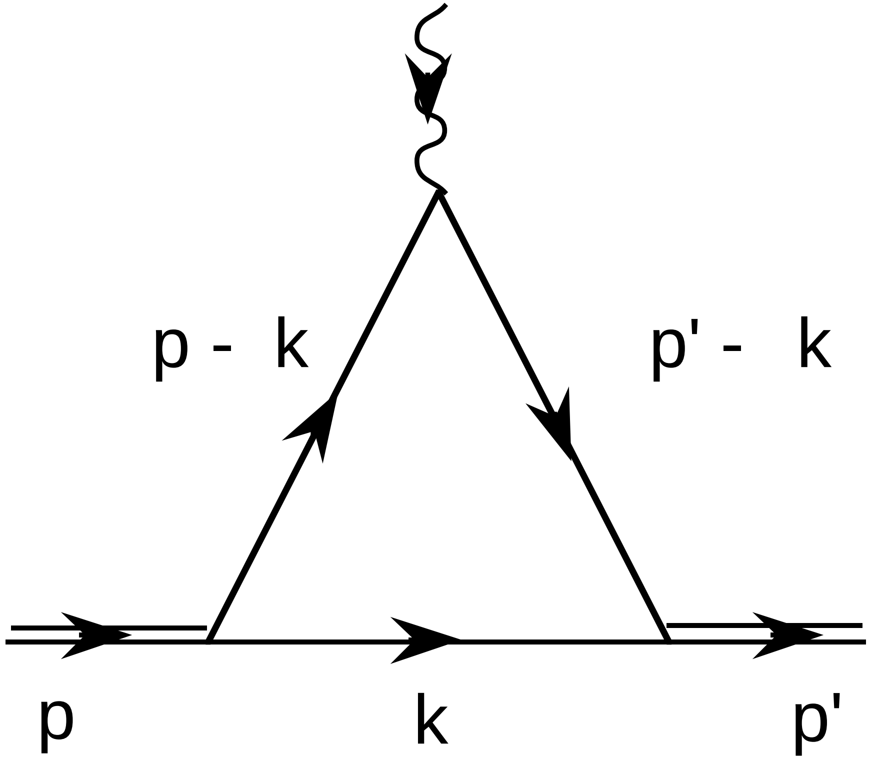
$$\psi(\vec{k}_\perp, x) = \frac{x(1-x)g(1-2x)}{\left(\vec{k}_\perp^2 + m^2 - x(1-x)M^2\right)^2}$$

Two-body contribution to norm:

$$N_2 = \frac{1}{(2\pi)^3}\int \psi^2(\vec{k}_\perp, x)\frac{d^2k_\perp dx}{2x(1-x)}$$

$$= \frac{1}{6\pi^2}\int_{-1}^{1}\frac{(1-z^2)g^2(z)dz}{[4m^2 - (1-z^2)M^2]^3}$$

• Form factor via BS amplitude



Feynman diagram for the EM form factor.

$$
\begin{aligned}
(p+p')^\mu F(Q^2) &= -i \int \frac{d^4k}{(2\pi)^4} \, (p+p'-2k)^\mu \, (m^2-k^2) \\
&\times \Phi\left(\frac{1}{2}p-k, p\right) \Phi\left(\frac{1}{2}p'-k, p'\right)
\end{aligned}
$$

• Form factor via Nakanishi $g(z)$

J. Carbonell, V.A. Karmanov, M. Mangin-Brinet, Eur. Phys. J. **A 39** (2009) 53.

$$F_{if}(Q^2) = -\frac{1}{32\pi^2}\int_{-1}^{1} dz\, g_i(z)\int_{-1}^{1} dz'\, g_f(z')\int_0^1 du\, u^2(1-u)^2\frac{f_{num}}{f_{den}^4},$$

$$\begin{aligned}
\xi &= \frac{1}{2}(1+z)u + \frac{1}{2}(1+z')(1-u). \\
f_{num} &= (6\xi-5)m^2 + 2M_i^2\xi(1-\xi) + \frac{1}{4}Q^2(1-u)u(1+z)(1+z') \\
&+ (M_f^2 - M_i^2)(1-u)(1-\xi)(1+z') \\
f_{den} &= m^2 - M_i^2(1-\xi)\xi + \frac{1}{4}Q^2(1-u)u(1+z)(1+z') \\
&- \frac{1}{2}(M_f^2 - M_i^2)(1-u)(1-\xi)(1+z')
\end{aligned}$$

M_i, M_f are the initial and final masses.

Normalization of $g(z)$: $F_{ii}(Q^2=0)=1 \Rightarrow\ <p|p>=1$

• Equation for $g(z)$

Solved numerically:

$$g(z) = \frac{\alpha}{2\pi} \int_{-1}^{+1} \frac{R(z,z')}{[1-\eta^2(1-z'^2)]} g(z') dz',$$

$$R(z,z') = \begin{cases} \frac{1-z}{1-z'}, & \text{if } z' < z \\ \frac{1+z}{1+z'}, & \text{if } z' > z \end{cases}$$

$$\eta = \frac{M}{2m}, \quad M = 2m - B, \quad B = |E|$$

Solution: $g(z) = g_{nk}(z)$,

$k = 0, 1, 2, \ldots$ is the number of nodes of $g_{nk}(z)$ vs. z.

• Symmetry of $g(z)$

We will concentrate on the symmetric solutions

$$g(-z) = g(z).$$

The anti-symmetric solutions $g(-z) = -g(z)$ do not contribute in the S-matrix

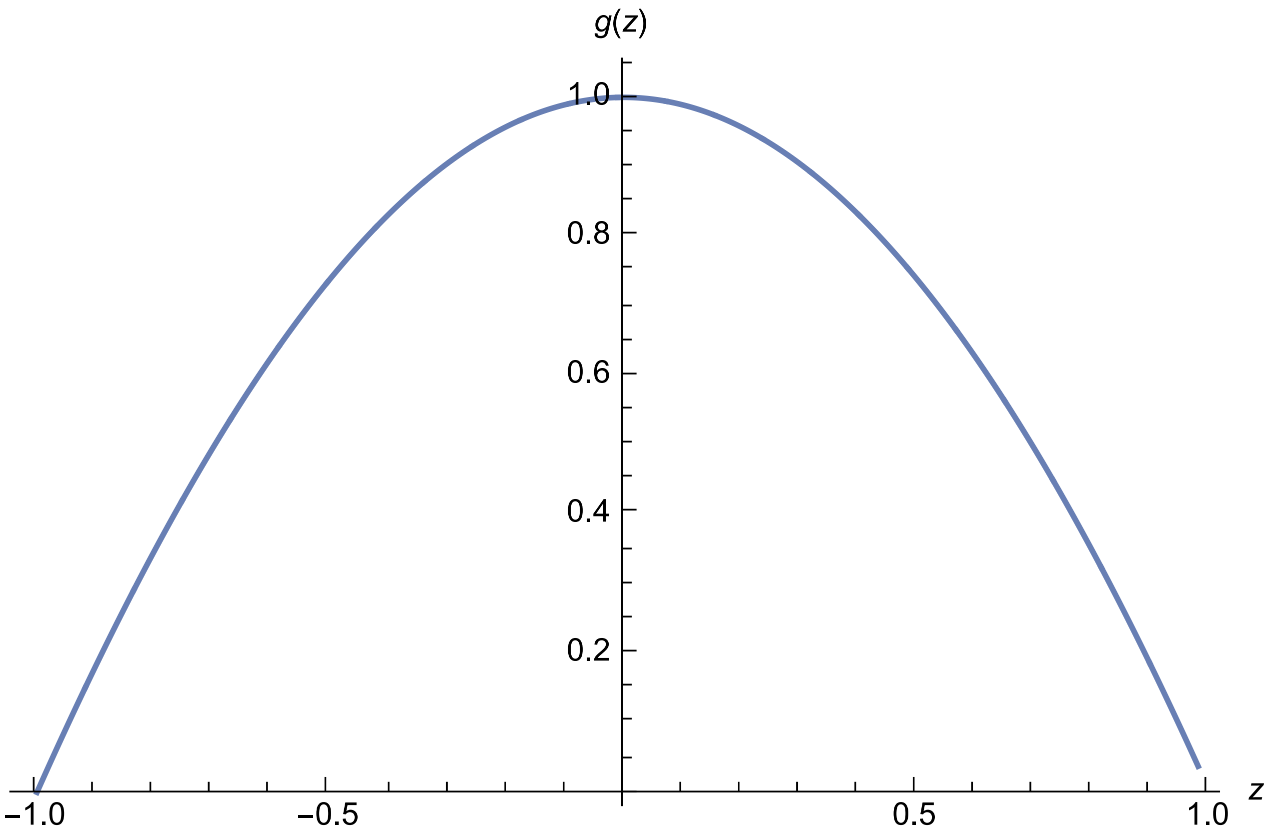
M. Ciafaloni and P. Menotti,

Phys. Rev. **140**, *No. 4B (1965) B929.*

• Finding $g(z)$, B and N_2

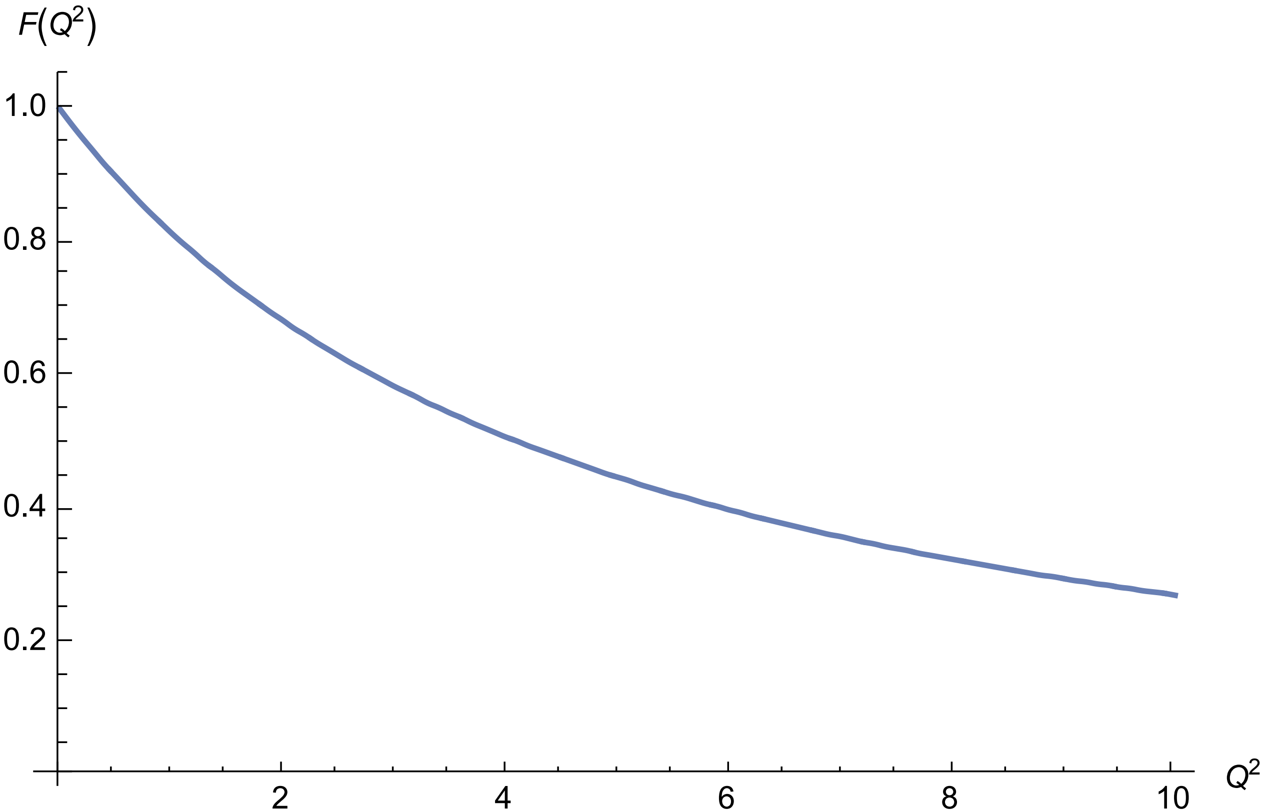
α	N_{nodes}	n/ab	B	N_2
0.02	0	normal	0.0001	0.992
1	0	normal	0.084203	0.737
2	0	normal	0.23634	0.695
2	2	abnormal	$1.2204 \cdot 10^{-5}$	$7.7 \cdot 10^{-3}$
3	0	normal	0.43224	0.674
3	2	abnormal	$2.3380 \cdot 10^{-4}$	$2.54 \cdot 10^{-2}$
4	0	normal	0.67743	0.661
4	2	abnormal	$1.21425 \cdot 10^{-3}$	$5.52 \cdot 10^{-2}$
5	0	normal	0.99925	0.651
5	2	abnormal	$3.5117 \cdot 10^{-3}$	$9.35 \cdot 10^{-2}$
5	4	abnormal	$0.2171803 \cdot 10^{-4}$	$8.55 \cdot 10^{-3}$

• **Normal** $g(z), \; N_{nodes} = 0$



The function $g(z)$, normal (ground) state, for $\alpha = 5, \; B = 0.99925, \; N_{nodes} = 0$.

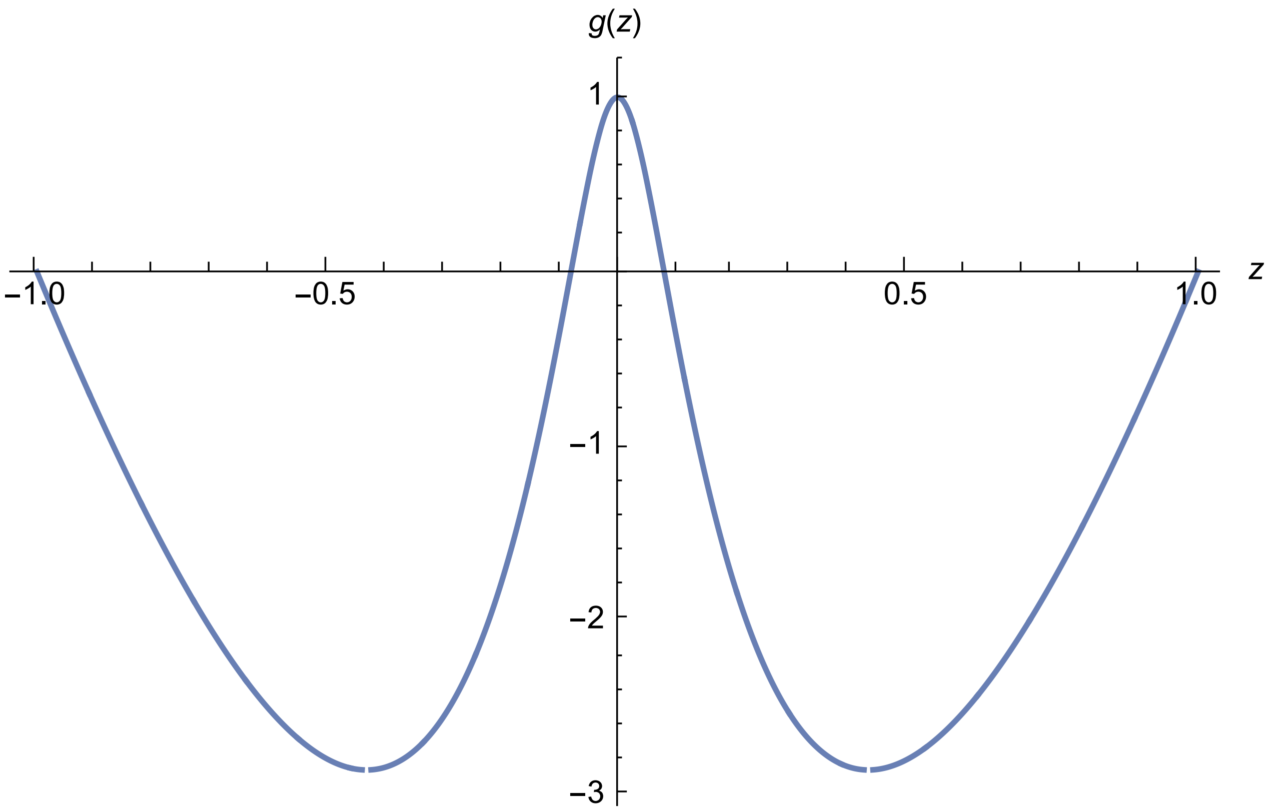
•Elastic (normal) EM form factor



Elastic form factors $F(Q^2)$ for the normal (ground) state

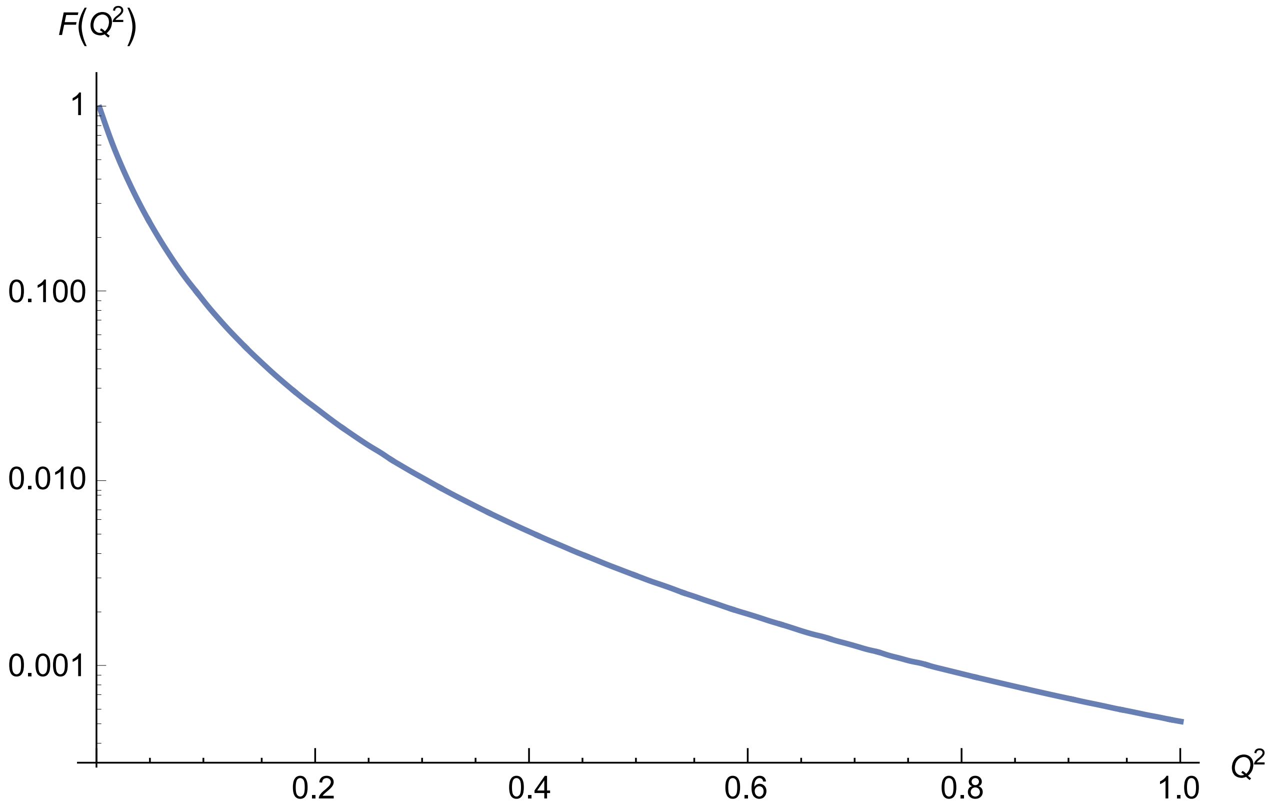
$\alpha = 5,\ B = 0.99925$

• **Abnormal** $g(z),\ N_{nodes} = 2$



The function $g(z)$, abnormal state,
$\alpha = 5,\ B = 3.5117 \cdot 10^{-3},\ N_{nodes} = 2$.

• Elastic (abnormal) EM form factor

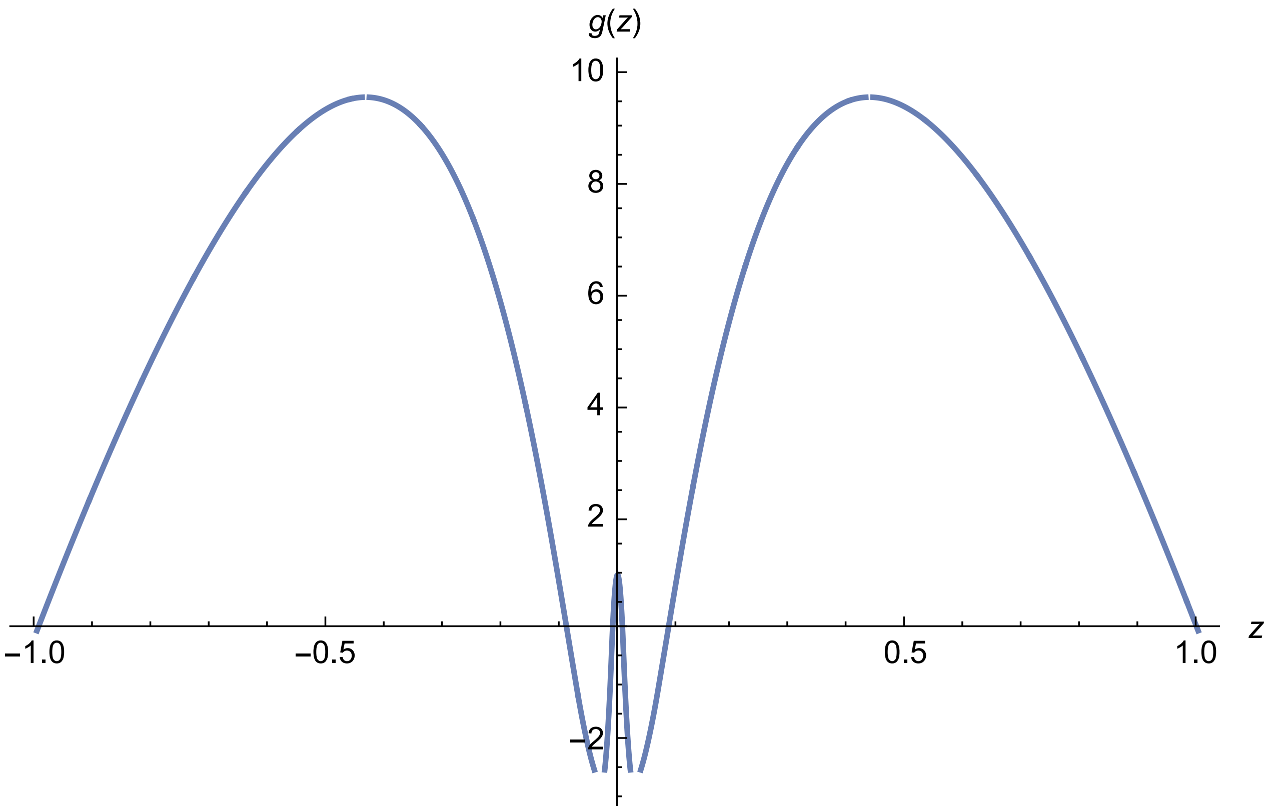


Elastic form factors $F(Q^2)$ for the abnormal state

$\alpha = 5,\ B = 3.5117 \cdot 10^{-3}$.

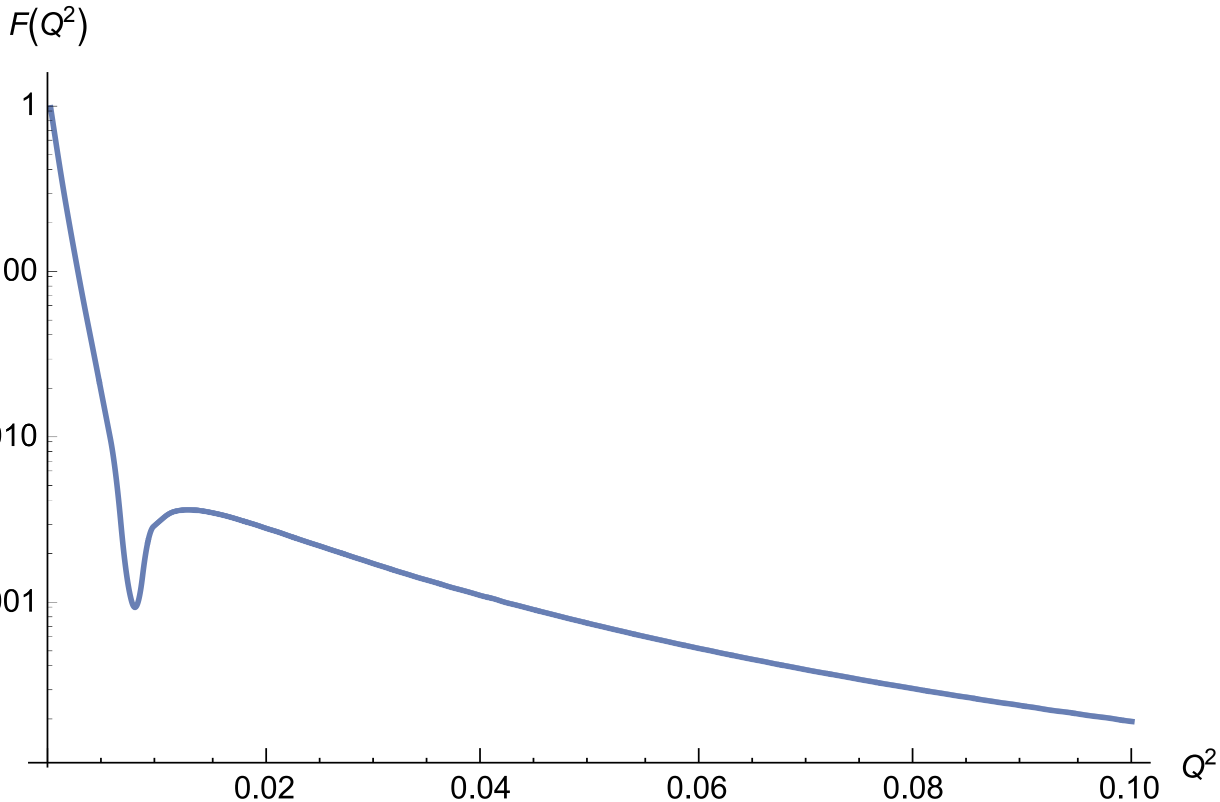
It crosses zero at $Q^2 = 26$.

• **Abnormal** $g(z),\ N_{nodes} = 4$



The function $g(z)$, 2nd abnormal state,
$\alpha = 5,\ B = 2.171803 \cdot 10^{-5},\ N_{nodes} = 4$.

• Elastic (abnormal) EM form factor



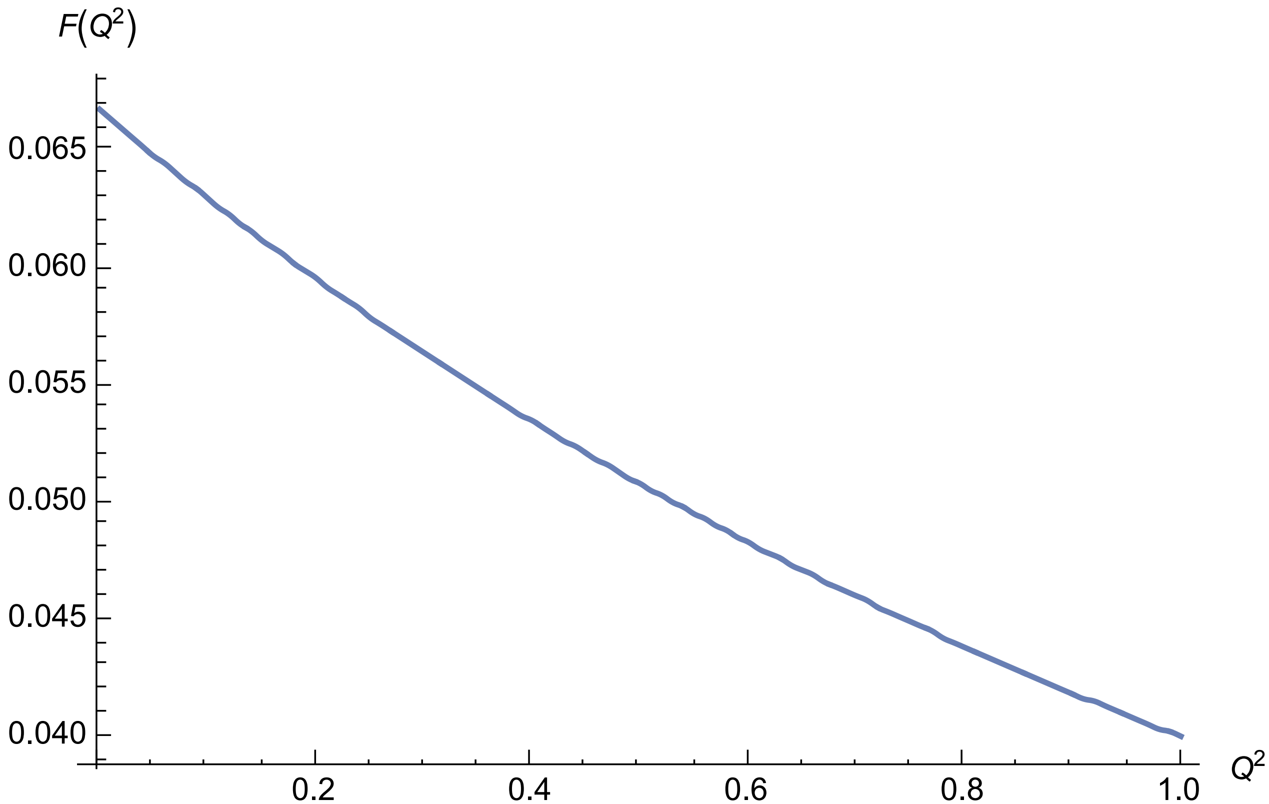
Elastic form factors $F(Q^2)$ for the 2nd abnormal state

$\alpha = 5,\ B = 2.171803 \cdot 10^{-5}$.

It crosses zero at $Q^2 = 1.1{\cdot}10^{-4}$ and $Q^2 = 0.75{\cdot}10^{-2}$

• Transition EM form factor

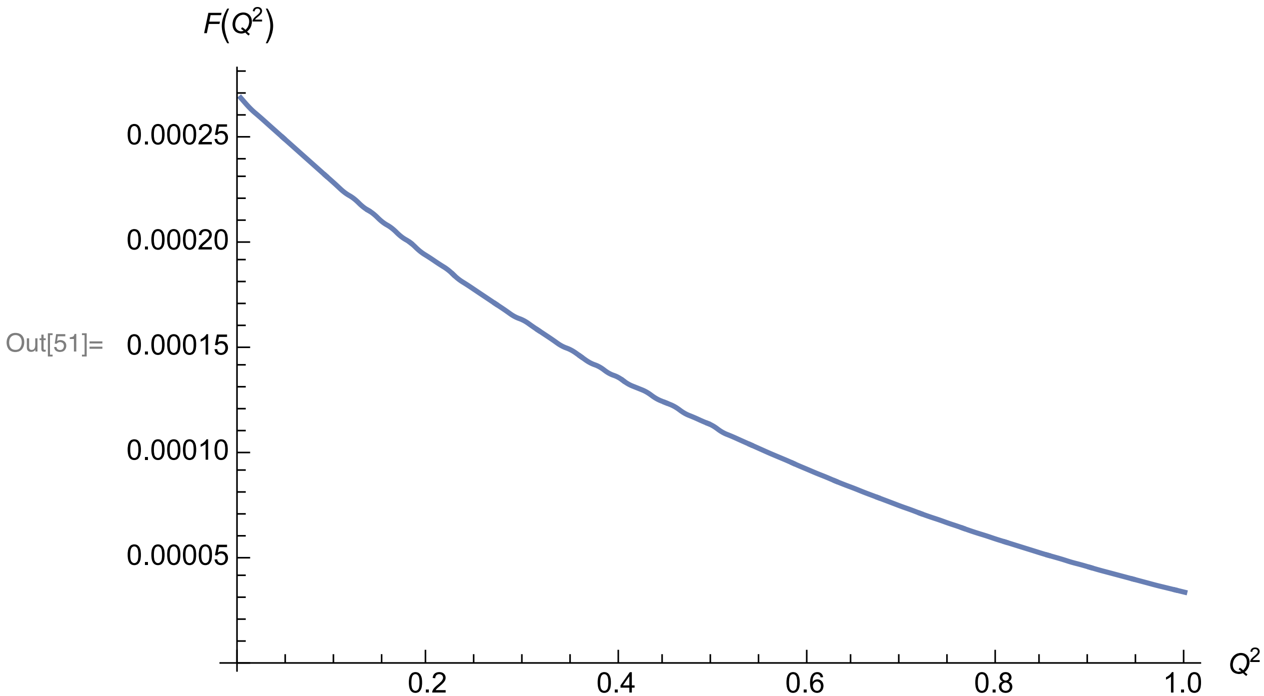
normal $\rightarrow$ 1st abnormal



Transition form factors $F(Q^2)$ between the normal state $B = 0.99925$ and the 1st abnormal one, $B = 3.5117 \cdot 10^{-3}$

• Transition EM form factor

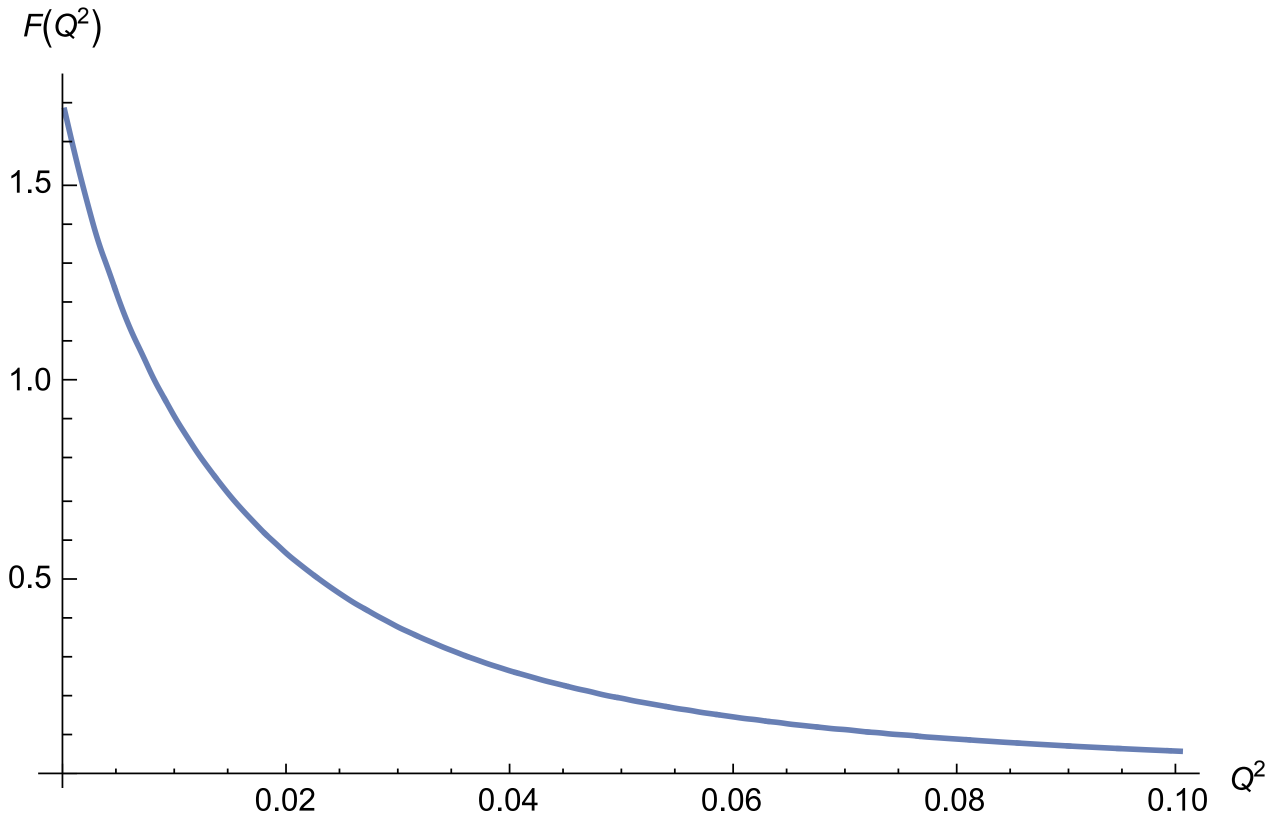
normal $\rightarrow$ 2nd abnormal



Transition form factors $F(Q^2)$ between the normal state $B = 0.99925$ and the 2nd abnormal one, $B = 2.171803 \cdot 10^{-5}$

• Transition EM form factor

1st abnormal $\rightarrow$ 2nd abnormal



Transition form factors $F(Q^2)$ between the 1st abnormal state $B = 3.5117 \cdot 10^{-3}$ and the 2nd abnormal one, $B = 2.171803 \cdot 10^{-5}$

• Conclusions

- BS equation predicts the states having pure relativistic origin (not given by the Schrödinger equation), both for massless (Wick-Cutkosky, 1954) and massive exchanges.
  Analogy: Dirac equation predicts antiparticles.
- For massless exchange, the abnormal states are dominated by the many-body sectors.
- Abnormal elastic EM ff's vs. Q^2 decrease much faster than the normal ones. The transition ff's *normal* $\leftrightarrow$ *abnormal* are small ($\sim 10^{-2} - 10^{-3}$). The transition ff's *abnormal* $\leftrightarrow$ *abnormal* are "normal" (~ 1).
- It is interesting to analyze, from this point of view, the properties of particles.

Thank you!