Some subtleties of light-front quantization

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September 18, 2019

ABSTRACT: We give a survey of a few subtle mathematical points whose correct treatment is necessary for obtaining noncontradictory structure of the front form of the field theory and its physical predictions. First, we show that small imaginary parts in the exponents of the light-front

(LF) two-points functions (convergence factors) are mandatory for their correct equal - LF time limit and for vanishing of the surface terms in the Poincaré algebra. The same mechanism removes unwanted terms in the transformation law of the scalar LF field under some Poincaré generators. We also demonstrate that contrary to recent claims the LF Hamiltonian approach does not fail in the vacuum sector and actually yields results in agreement with the Feynman diagram method. The non-vanishing LF vacuum bubbles found recently within the Hamiltonian LF perturbation theory are shown to not violate conservation of the LF momentum. Their existence and analytic form agrees with the LF evaluation of the corresponding Feynman diagrams.

OUTLINE

1. Introduction
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4. Non-vanishing of vacuum bubbles in LF perturbation theory
5. SL and LF two-point functions in $x \to 0$ limit
6. Conclusions

I. INTRODUCTION

The development of the front form of quantum field theory (QFT) (Dirac 1949, Leutwyler, Klauder and Streit 1971, Rohrlich 1972, Kogut and Soper 1972, Yan 1973 ...) rather non-uniform and non-linear

We still do not have a compact formulation of LF field theory

In particular, it has been often claimed that the light front (LF) theory has certain drawbacks (Hagen, Yamawaki), not present in the usual (conventional) "instant" form (called "space-like" (SL) here),

that it violates some essential principles - causality (Heinzl, Kroeger and Scheu) and Lorentz invariance (Yamawaki)

even fails completely in some aspects (equal - LF time projection of two-point functions (Yamawaki) quatization of massless fields in two space-time dimensions (McCartor) very recently: Hamiltonian (Fock) formulation fails in the vacuum sector (Mannheim, Lowdon, Brodsky)

A possible optimistic explanation: structure of the LF QFT is completely consistent and we have just not found the correct formulation of these subtle points yet (being often led by an intuition and opatterns obtained in the usual SL form of QFT).

good reasons in favour of this attitude

For example, shown recently that the two-dimensional massless LF fields can be obtained as massless limits of the corresponding massive fields. Their quantization is canonical, no initialization on two surfaces necessary, and physical implications are both consistent and transparent (Martinovic and Grangé)

Another example – vacuum bubbles in perturbation theory

Naively, LF vacuum amplitudes vanish due to the conservation of the LF momentum p^+. Although it has been argued long time ago that the correct mathematical evaluation of the corresponding Feynman diagrams in terms of the LF variables yields non-vanishing vacuum amplitudes with the correct magnitude (Chang and Ma, T.-M. Yan)

direct LF calculation within the "old-fashioned" Hamiltonian LF perturbation theory was missing. The usual LF perturbation theory rules did not work in this case. The correct values of the vacuum bubbles were obtained only recently as the limits of the associated self-energy diagrams for vanishing external momentum (Collins, Martinovic and Dorokhov)

For completeness: some claims about an apparent inconsistency of the LF quantization scheme were simply a result of a conceptual or calculational mistake

Hagen argued that the Poincaré algebra of the free scalar LF field is violated due to the zero-mode term in the LF Hamiltonian. Such a term however does not exist as there are no dynamical zero modes in the case of the massive LF scalar field (crucial role of boundary conditions)

II. LF TWO-POINT FUNCTIONS AND THEIR ET LIMIT

one source of difficulties in the LF theory: improper mathematical treatment of certain type of integrals two-point correlation functions

the massive LF Fermion field, two-dimensional covariant Dirac equation decomposes into two equations

$$2i\partial_+\psi_2(x) = m\psi_-(x), \qquad 2i\partial_-\psi_-(x) = m\psi_2(x). \tag{1}$$

quantum solution of the dynamical equation

$$\psi_2(x) = \int_0^\infty \frac{dp^+}{4\pi}\left[b(p^+)e^{-\frac{i}{2}p^+x^- - \frac{i}{2}\frac{m^2}{p^+}x^+} + d^\dagger(p^+)e^{\frac{i}{2}p^+x^- + \frac{i}{2}\frac{m^2}{p^+}x^+}\right]. \tag{2}$$

solution of the constraint

$$\psi_1(x) = m\int_0^\infty \frac{dp^+}{4\pi p^+}\left[b(p^+)e^{-\frac{i}{2}p^+x^- - \frac{i}{2}\frac{m^2}{p^+}x^+} - d^\dagger(p^+)e^{\frac{i}{2}p^+x^- + \frac{i}{2}\frac{m^2}{p^+}x^+}\right], \tag{3}$$

where

$$\left\{\psi_2(0,x^-), \psi_2^\dagger(0,y^-)\right\} = \delta(x^- - y^-) \tag{4}$$

equivalent to

$$\left\{b(p^+), b^\dagger(q^+)\right\} = \left\{d(p^+), d^\dagger(q^+)\right\} = \delta(p^+ - q^+). \tag{5}$$

The two-point correlation functions $S^{(+)}_{\alpha\beta}(x-y)$

$$S^{(+)}_{22}(x-y) = \langle 0|\psi_2(x)\psi_2^\dagger(y)|0\rangle = \int_0^\infty \frac{dp^+}{4\pi} e^{-\frac{i}{2}p^+(x^- - y^- - i\epsilon^-) - \frac{i}{2}\frac{m^2}{p^+}(x^+ - y^+ - i\epsilon^+)}, \quad (6)$$

$$S^{(+)}_{11}(x-y) = \langle 0|\psi_1(x)\psi_1^\dagger(y)|0\rangle = \int_0^\infty \frac{dp^+}{4\pi}\frac{m^2}{p^{+2}} e^{-\frac{i}{2}p^+(x^- - y^- - i\epsilon^-) - \frac{i}{2}\frac{m^2}{p^+}(x^+ - y^+ - i\epsilon^+)},$$

$$S^{(+)}_{12}(x-y) = \langle 0|\psi_1(x)\psi_2^\dagger(y)|0\rangle = \int_0^\infty \frac{dp^+}{4\pi}\frac{m}{p^+} e^{-\frac{i}{2}p^+(x^- - y^- - i\epsilon^-) - \frac{i}{2}\frac{m^2}{p^+}(x^+ - y^+ - i\epsilon^+)}.$$

small imaginary parts (damping factors) in the plane-wave factors introduced, otherwise integrals would not exist mathematically, consequences for more physical aspects; explicitly ($z = x - y$)

$$\begin{aligned} S_{22}^{(+)}(z) = & - & \theta\big(z^2\big)\frac{m}{4}\sqrt{\frac{z^+}{z^-}}\Big[J_1\big(m\sqrt{z^2}\big) - i\ \mathrm{sgn}(z^+)N_1\big(m\sqrt{z^2}\big)\Big] + \\ & + & \theta\big(-z^2\big)\mathrm{sgn}(z^+)\frac{im}{2\pi}\sqrt{-\frac{z^+}{z^-}}K_1\big(m\sqrt{-z^2}\big), \\ S_{12}^{(+)}(z) = & - & \theta\big(z^2\big)\frac{m}{4}\Big[N_0\big(m\sqrt{z^2}\big) + i\ \mathrm{sgn}(z^+)J_0(m\sqrt{z^2})\Big] + \\ & + & \theta\big(-z^2\big)\frac{m}{2\pi}K_0\big(m\sqrt{-z^2}\big), \end{aligned} \quad (7)$$

We also have

$$S_{11}^{(+)}(z) = S_{22}^{+}(z^+ \leftrightarrow z^-). \quad (8)$$

Here $J_\nu(z), K_\nu(z)$ and $N_\nu(z)$ are the Bessel, modified Bessel and Neumann functions. The implicit small imaginary parts in the plane-wave factors crucial for the equal LF time limit, a self-consistency check - ET ACR and the mixed anticommutator

$$\{\psi_1(0,x^-),\psi_2^\dagger(0,y^-)\} = \frac{m}{4i}\mathrm{sgn}(x^- - y^-) \tag{9}$$

which is easily computed from the constraint (3) and the basic anticommutator (4). We will consider the fields at space-like separations and choose $x^+ - y^+ < 0, x^- - y^- > 0$ for definiteness. With all factors explicitly shown, we have

$$\Delta(x-y) \equiv \left\{\psi_2(x), \psi_2^\dagger(y)\right\} = S_{22}^{(+)}(x-y) + S_{22}^{(+)*}(x-y) =$$

$$= -\frac{im}{2\pi}\left[\sqrt{\frac{|x^+-y^+|+i\epsilon^-}{x^--y^--i\epsilon^-}}K_1\left(m\sqrt{(|x^+-y^+|+i\epsilon^+)(x^--y^--i\epsilon)^-}\right) - \right.$$

$$\left. -\sqrt{\frac{|x^+-y^+|-i\epsilon^+}{x^--y^-+i\epsilon^-}}K_1\left(m\sqrt{(|x^+-y^+|-i\epsilon^+)(x^--y^-+i\epsilon^-)}\right)\right]. \quad (10)$$

For $-(x^+-y^+) = \eta << 1$, we can use the expansion $K_1(z) \approx 1/z + O(z^2)$ to obtain

$$\{\psi_2(0,x^-), \psi_2^\dagger(0,y^-)\} = \frac{-i}{2\pi}\Big[\frac{1}{x^--y^--i\epsilon^-} - \frac{1}{x^--y^-+i\epsilon^-}\Big] = \delta(x^- - y^-), \quad (11)$$

because the time difference η canceled out. The relation $1/(x - i\epsilon) = \mathcal{P}\frac{1}{x} + i\pi\delta(x)$ used in the final step ($\mathcal{P}$ stands for the principal value). The same result for $x^+ - y^+ > 0, x^- - y^- < 0$. The presence of the $i\epsilon^\pm$ part essential, wrong (=0) result without it. But, generally speaking, the equal-time limit of the anticommutator function is not immediately the Dirac delta function $\delta(x^- - y^-)$ but the expression

$$\begin{aligned}\Delta(0, x^- - y^-) = & - \frac{im}{4\pi}\left[\sqrt{\frac{i\epsilon^-}{|x^- - y^-| - i\epsilon^-}} K_1\left(m\sqrt{i\epsilon^+(|x^- - y^-| - i\epsilon^-)}\right)\right. \\ & \left. - \sqrt{\frac{-i\epsilon^+}{|x^- - y^-| + i\epsilon^-}} K_1\left(m\sqrt{-i\epsilon^+(|x^- - y^-| + i\epsilon^-)}\right)\right] \quad (12)\end{aligned}$$

which for finite $x^- - y^-$ reduces to $\delta(x^- - y^-)$.

A similar conclusion for the $x^+ - y^+ = 0$ limit of the $S_{12}^{(+)}$ function, for

$x^+ - y^+ < 0,\ x^- - y^- > 0$

$$
\begin{aligned}
\{\psi_1(x), \psi_2^\dagger(y)\} &= S_{12}^{(+)}(x-y) - S_{12}^{(+)*}(x-y) \equiv m\Sigma(x^+ - y^+, x^- - y^-) \\
&= \frac{m}{4\pi}\Bigg[K_0\Big(m\sqrt{(|x^+ - y^+| + i\epsilon^+)(x^- - y^- - i\epsilon^-)}\Big) \\
&- K_0\Big(m\sqrt{(|x^+ - y^+| - i\epsilon^+)(x^- - y^- + i\epsilon^-)}\Big)\Bigg]. \qquad (13)
\end{aligned}
$$

The relative minus sign in the first line of the above expression is due to the negative sign of the second term in the Fock representation of $\psi_1(x)$. With the expansion $K_0(z) \approx -\gamma_E - \ln\frac{z}{2}$ we find in the $-(x^+ - y^+) = \eta \to 0$

limit

$$\left\{\psi_1(0,x^-),\psi_2^\dagger(0,y^-)\right\} = \frac{m}{2\pi}\Big[-\ln\Big(\frac{m}{2}\sqrt{i\epsilon^+(x^- - y^- - i\epsilon^-)}\Big) +$$

$$+\ln\Big(\frac{m}{2}\sqrt{-i\epsilon^+(x^- - y^- + i\epsilon^-)}\Big)\Big] = \frac{m}{2\pi}\Big[-\frac{1}{2}\ln(i) + \frac{1}{2}\ln(-i)\Big] = -i\frac{m}{4}. \quad (14)$$

For the case $x^- - y^- < 0$, one gets the above result with the opposite sign. Hence we recover the correct anticommutator, Eq.(9). The essential role played by the $i\epsilon^\pm$ factor is evident. Generally speaking however, the equal time limit of the considered anticommutator function is not the sign function but the expression

$$m\Sigma(0,z^-) = \frac{m}{4\pi}\left[K_0\Big(m\sqrt{i\epsilon^+(|z^-| - i\epsilon^-)}\Big) - K_0\Big(m\sqrt{-i\epsilon^+(|z^-| + i\epsilon^-)}\Big)\right]. \quad (15)$$

III. POINCARÉ ALGEBRA AND SURFACE TEMS

A necessary condition for the Lorentz invariance of a relativistic quantum field-theory model: the abstract algebra of the Poincaré generators has to be satisfied on quantum level using ETCR

in $D = 1 + 1$

$$\left[P^{+}, P^{-}\right] = 0, \quad \left[P^{+}, M^{+-}\right] = -2iP^{+}, \quad \left[P^{-}, M^{+-}\right] = 2iP^{-}. \quad (16)$$

The second condition: correct transformation of the quantum field under Lorentz transformations

Straightforward to calculate the commutators (16) using the anticommutation relations (4),(9): in the first two commutators the surface terms cancel; in the latter one, an unwanted surface term apparently present (Hagen)

A similar redundant term found in the four-dimensional theory in the commutator between the LF rotational generator M^{-i} and the scalar field.

If the above-mentioned violations of relativistic invariance were really true, the LF field theory would face a serious consistency problem

A careful mathematical analysis shows however that the claimed violation is an artifact and the surface terms actually vanish. The problematic surface term is equal to $s(L) = -8imL\psi_1^\dagger(L)\psi_2(L)$, where L is a value of x^- tending to infinity. Let us again use our careful definitions of the anticommutators and replace the $\delta(x^-)$ and $\epsilon(x^-)$ functions by $2\Delta(x)$ (10)

and $4i\Sigma(x)$ (13) for $x^+ = 0$. Then we find

$$\left[P^-, M^{+-}\right] = -\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}x^-}{2}\int\limits_{-\infty}^{+\infty}\mathrm{d}y^- y^-\left[\Theta^{+-}(x), \Theta^{++}(y)\right]$$

$$-4im\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}x^-}{2}\int\limits_{-\infty}^{+\infty}\mathrm{d}y^- y^-\Big\{\Sigma(x-y)\psi_2^\dagger(x)\partial_-\psi_2(y) - \partial_-^y\Delta(x-y)\psi_2^\dagger(y)\psi_1(x) +$$

$$+\Delta(x-y)\psi_1^\dagger(x)\partial_-\psi_2(y) - \frac{m}{2i}\Delta(x-y)\psi_2^\dagger(y)\psi_2(x)\Big\}. \tag{17}$$

Performing the partial integration and using the limiting values of the $\Delta(x)$ and $\Sigma(x)$ functions we obtain that the expression (17) equals to $2iP^-$ plus the surface term

$$s(i\epsilon^+, x^- - L) = 4imL \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2} \Big[\Sigma(i\epsilon^+, x^- - L)\psi_2^\dagger(L) + \Sigma(i\epsilon^+, x^- + L)\psi_2^\dagger(-L) \Big] \psi_1(x). \tag{18}$$

To proceed we have to look at the large-distance behaviour of the function $\Sigma(i\epsilon^+, x^- \pm L)$:

$$\Sigma(i\epsilon^+, x^- - L) = \frac{m}{2\pi} \Big\{ K_0 \Big(m\sqrt{(-i\epsilon^+)|x^- - L|} \Big) - \frac{m}{2\pi} \{ K_0 \Big(m\sqrt{i\epsilon^+|x^- - L|} \Big) \Big\}. \tag{19}$$

Using the value of the modified Bessel function $K_\nu(z)$ for large z

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} \exp(-z), \tag{20}$$

we get

$$K_0\big(m\sqrt{\pm i\epsilon^+ L}\big) \approx \sqrt{\frac{\pi}{2m\sqrt{\pm i\epsilon^+ L}}}\exp\left(-m\sqrt{\pm i\epsilon^+ L}\right)$$
$$\approx L^{-1/4}\exp\big[-\frac{m}{\sqrt{2}}(1\pm i)\sqrt{\epsilon^+ L}\big] \qquad (21)$$

and similarly for other surface terms $\Rightarrow$ exponential suppression for $L \to \infty$, the correct Poincaré algebra recovered. A possibility of this mechanism in 4D mentioned by Nakanishi and Yabuki, 1977

IV. NON-VANISHING VACUUM BUBBLES IN LF PERTURBATION THEORY

Discussed by LM at LC 2018, a few more details here

positivity of the LF momentum p^+ plus its conservation $\Rightarrow$ the ground state of any dynamical model cannot contain quanta carrying $p^+ \neq 0$. Only a tiny subset of all field modes, namely those carrying $p^+ = 0$ - the (dynamical) LF zero modes (ZM) - can contribute in principle

NB: some field modes (ZM of the scalar field) which appear as dynamical ones in the conventional (”space-like”, SL for short) theory become constrained (non-dynamical) in the LF form of the theory $\Rightarrow$ cannot contribute to vacuum processes directly

The LF/SL EQUIVALENCE ISSUE studied already in the pioneering papers on LF perturbative S-matrix by Chang and Ma (1969) and by T.-M. Yan (1973) includig the vacuum problem at the perturbation theory level

Method: covariant Feynman amplitudes in terms of LF variables

delicate step: integration in p^- variable, the propagators in 2D behave as $(k^+k^- - m^2 + i\epsilon)^{-1}$ instead of $(k_0^2 - k_1^2 - m^2 + i\epsilon)^{-1}$ - convergence

T.-M. Yan, PRD 7, 1780 (1973):

$$I = \int d^4p \frac{1}{(p^2 - \mu^2 + i\epsilon)^3} = \frac{\pi^2}{2i\mu^2} \tag{22}$$

Here $d^4p = dp^0 dp^1 dp^2 dp^3$ and $p^0 \to idp^4$. In LF variables,

$$\begin{aligned} I &= \int dp^+ dp^- d^2p_\perp \frac{1}{(p^+p^- - p_\perp^2 - \mu^2 + i\epsilon)^3} \\ &= -\frac{\pi}{4} \int dp^+ dp^- \frac{1}{(p^+p^- - \mu^2 + i\epsilon)^2}. \end{aligned} \tag{23}$$

A double pole at $p^- = \frac{\mu^2 - i\epsilon}{p^+}$, at infinity for $p^+ = 0$, a careful treatment:

$$I = -\frac{\pi}{4} \int\limits_{-\infty}^{+\infty} dp^+ \lim_{\Lambda \to \infty} \int\limits_{-\Lambda}^{+\Lambda} dp^- \frac{1}{(p^+ p^- - \mu^2 + i\epsilon)^2} =$$

$$= \frac{\pi}{4} \int\limits_{-\infty}^{+\infty} \frac{dp^+}{p^+} \lim_{\Lambda \to \infty} \Big(\frac{1}{p^+ \Lambda - \mu^2 + i\epsilon} - \frac{1}{-p^+ \Lambda - \mu^2 + i\epsilon} \Big). \quad (24)$$

Using the identity

$$\frac{1}{p^+} \Big(\frac{1}{p^+ \Lambda - \mu^2 + i\epsilon} - \frac{1}{-p^+ \Lambda - \mu^2 + i\epsilon} \Big) = \frac{1}{\mu^2} \Big(\frac{\Lambda}{p^+ \Lambda - \mu^2 + i\epsilon} - \frac{\Lambda}{p^+ \Lambda + \mu^2 - i\epsilon} \Big), \quad (25)$$

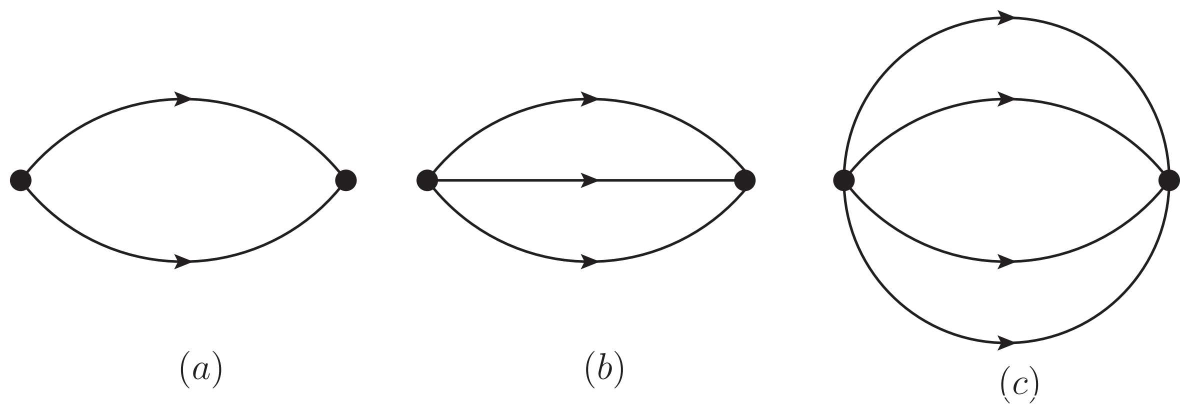
for $\Lambda \to \infty$, one gets

$$I = \frac{\pi}{4\mu^2} \int\limits_{-\infty}^{+\infty} dp^+ \Big(\frac{1}{p^+ + i\epsilon} - \frac{1}{p^+ - i\epsilon} \Big) = \frac{\pi}{4\mu^2} \int\limits_{-\infty}^{+\infty} dp^+ \big[-2i\pi\delta(p^+) \big] = \frac{\pi^2}{2i\mu^2}. \tag{26}$$

Same result with the exponential α-representation $(D^{-1} = -i \int\limits_0^{+\infty} d\alpha e^{i\alpha(D+i\epsilon)})$.

complete agreement with covariant Feynman results

The rules of the LF perturbation theory imply that the vacuum amplitudes (bubbles) are mathematically ill-defined (Yan 1973) as the corresponding integrals contain $\delta(p_1^+ + p_2^+ ... + p_n^+)$ which can be satisfied only if all of them vanish, leading to singular integrands.



Vacuum diagrams for (a) the free theory (Green function of $\phi^2(x)\phi^2(0)$ composite operator), (b) the ϕ^3 and (c) for ϕ^4 model. The arrows indicate the momentum flow.

Consider instead the associated **self-energy** diagram, $D = 1 + 1$ sufficient. The simplest - the $\lambda\phi^3$ theory (Fig. 1 (a) with two external lines attached). The corresponding Feynman amplitude is

$$-i\Sigma_3(p^2) = \frac{1}{2}\frac{(-i\lambda)^2}{(2\pi)^2}\int d^2k\ G(k)G(p-k), \tag{27}$$

The vacuum bubble ($p = 0$, set $\lambda = 1$) in terms of the LF variables

$$V_2(\mu) = \frac{i}{16\pi^2}\int\limits_{-\infty}^{\infty} dk^+ \int\limits_{-\infty}^{\infty} dk^- \frac{1}{(k^+k^- - \mu^2 + i\epsilon)^2}. \tag{28}$$

Same as the second line of (26): to correctly evaluate the integral over k^-, one has to impose a cutoff Λ leading to ($c = -i/16\pi^2$)

$$V_2(\mu) = c\int\limits_{-\infty}^{\infty} \frac{dk^+}{k^+}\Big[\frac{1}{\Lambda k^+ - \mu^2 + i\epsilon} - \frac{1}{-\Lambda k^+ - \mu^2 + i\epsilon}\Big].$$

Same trick (identity) – for $\Lambda \to \infty$ find

$$V_2(\mu) = \frac{c}{\mu^2} \int_{-\infty}^{\infty} dk^+ \Big[\frac{1}{k^+ + i\epsilon} - \frac{1}{k^+ - i\epsilon}\Big] =$$

$$= \frac{c}{\mu^2}(-2\pi i) \int_{-\infty}^{\infty} dk^+ \delta(k^+) = -\frac{1}{8\pi}\frac{1}{\mu^2}. \qquad (29)$$

This simplest diagram sheds light upon the mechanism at work in the genuine LF case. The LFPT formula for the self-energy Σ_3 (27) is

$$\Sigma_3(p) = \frac{\lambda^2}{8\pi} \int_0^{p^+} \frac{dk^+}{k^+(p^+ - k^+)} \frac{1}{p^- - \frac{\mu^2}{k^+} - \frac{\mu^2}{p^+ - k^+} + i\epsilon}. \qquad (30)$$

Going over to the variable $x = k^+/p^+$, the denominator becomes $x(1-x)p^2 - \mu^2 + i\epsilon$ and for $p = 0$ (29) reproduced. Alternatively, work directly with the form (30). Taking $p^+ = p^- = \eta$ for simplicity, we have

$$\Sigma_3(\eta) = \frac{1}{8\pi} \int_0^\eta \frac{dk^+}{k^+(\eta - k^+)} \frac{1}{\eta - \frac{\mu^2}{k^+} - \frac{\mu^2}{\eta - k^+} + i\epsilon}. \tag{31}$$

The integral can be evaluated exactly with the result

$$\Sigma_3(\eta) = -\frac{1}{4\pi}\big(G(\eta) - G(0)\big), \; G(k) = \frac{\arctan\left(\frac{2k-\eta}{\sqrt{4\mu^2-k^2}}\right)}{\eta\sqrt{4\mu^2 - \eta^2}}. \tag{32}$$

The expansion for infinitesimal η gives

$$\Sigma_3(\eta) = -\frac{1}{8\pi}\frac{1}{\mu^2}\Big[1 + \frac{\eta^2}{4\mu^2} + \mathcal{O}(\eta^4)\Big]. \tag{33}$$

The correct result recovered for $\eta = 0$. Simple reason: the integrand in (31) is $\eta^{-1}[k^+(\eta - k^+) - \mu^2]^{-1}$. For very small η the expression in the brackets has practically a constant value very close to $(-\mu^2)$ at the interval $(0, \eta)$, while the diverging η^{-1} factor is canceled by the length η of the integration domain. However, setting $\eta = 0$ from very beginning yields the wrong (ill-defined) result

NOTE: for arbitrarily small external p^μ the momentum is exactly conserved, in the limiting sense also for $p = 0$

V. SL AND LF TWO-POINT FUNCTION IN $x \to 0$ LIMIT

Recent study (P. Mannheim, P. Lowdon, S. Brodsky), comparing the (time-ordered) two-point function in SL/LF formulations within the Feynman/Hamiltonian (Fock) treatment for $x \neq$ and $x = 0$

$$D(x^\mu) = -i\langle 0|\theta(\sigma)\phi(x)\phi(0) + \theta(-\sigma)\phi(0)\phi(x)|0\rangle, \tag{34}$$

$$= \frac{1}{(2\pi)^4}\int d^4k \frac{e^{-ik.x}}{k^2 - \mu^2 + i\epsilon}, \quad \sigma = t \text{ or } x^+. \tag{35}$$

CONCLUSION: the Fock (Hamiltonian) approach breaks down, one must use the Feynman diagram approach to correctly describe the light front vacuum secor, the non-pole (circle at infinity contributions) are not captured by the Hamiltonian method

Previous section: evaluation of Feynman vacuum amplitudes in terms of the LF variables poduced correct results, Hamiltonian LFPTh reproduced them as well $\Rightarrow$ check the above conclusions in $D = 1 + 1$

Fig.2 Simple tadpole and a generalized tadpole in ϕ^4 model

The simplest example: **the tadpole diagram**

$$T(x) = \int d^2k \frac{e^{-ik.x}}{k^2 - \mu^2 + i\epsilon} \quad \text{for} \quad x^\mu = 0. \tag{36}$$

SL case: $k.x = k^0 x^0 - k^1 x^1, \quad k^2 = (k^0)^2 - (k^1)^2$

LF case: $k.x = \frac{1}{2} k^+ x^- + \frac{1}{2} k^- x^+, \quad k^2 = k^+ k^-$

SL calculation: $k_0 \to ik_2, \quad T = const \log \frac{\Lambda^2}{\mu^2}$.

In $D = 3+1$, $d^4k \to k^3 dk \sin\theta d\theta d\varphi \Rightarrow T \sim \Lambda^2$

LF evaluation: $T = -i\pi \log \frac{\Lambda}{i\delta\mu^2}$ ("Ligterink's method")
Bakker, DeWitt, C.-R. Ji, Mischenko, PRD 2005

LFPTh: tadpole arises when we normal-order the Hamiltonian

$$a^\dagger(k_1^+)a(k_2^+)a^\dagger(k_3^+)a(k_4^+) = \delta(k_2^+ - k_3^+)a^\dagger(k_1^+)a(k_4^+) + ::\equiv V_T + ::,$$

$$S_{fi}^{(1)} = -\frac{i}{2}\int_{-\infty}^{+\infty} dx^+ \langle 0|a(p_f^+)e^{\frac{i}{2}p_f^- x^+} V_T e^{-\frac{i}{2}p_i^- x^+} a^\dagger(p_i^+)|0\rangle$$

$$\Rightarrow M_T = \frac{\lambda}{8\pi}\int_0^{+\infty} \frac{dk^+}{k^+} \to \frac{\lambda}{8\pi}\int_\epsilon^\Lambda \frac{dk^+}{k^+}. \tag{37}$$

Changing the variable $k^+ \to \frac{\mu^2}{k^+}$:

$$M_T = \frac{\lambda}{8\pi} \int\limits_{\frac{\mu^2}{\Lambda}}^{\Lambda} \frac{dk^-}{k^-} = \frac{\lambda}{8\pi} \log \frac{\Lambda^2}{\mu^2}. \tag{38}$$

Another view: 2D LF scalar field and its 2-point functions:

$$\phi(x) = \int\limits_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \left[a(k^+) e^{-\frac{i}{2}k^+x^- - \frac{i}{2}\frac{\mu^2}{k^+}x^+} + a^\dagger(k^+) e^{\frac{i}{2}k^+x^- + \frac{i}{2}\frac{\mu^2}{k^+}x^+} \right], \tag{39}$$

where small imaginary parts for $x^\pm$ are understood as discussed for the case of fermions. The corresponding two-point function

$$D^{(+)}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int\limits_0^\infty \frac{dp^+}{4\pi p^+} e^{-\frac{i}{2}p^+(x^- - y^- - i\epsilon^-) - \frac{i}{2}\frac{\mu^2}{p^+}(x^+ - y^+ - i\epsilon^+)} \tag{40}$$

can easily be obtained using the commutation relation

$$\left[a(k^+), a^\dagger(l^+)\right] = \delta(k^+ - l^+). \tag{41}$$

With the help of known integral formulae, we again find

$$\begin{aligned} D^{(+)}(z) &= -\frac{1}{4}\theta\big(z^2\big)\Big(N_0(\mu\sqrt{z^2}) - i\mathrm{sgn}(z^+)J_0(\mu\sqrt{z^2})\Big) \\ &+ \frac{1}{2\pi}\theta\big(-z^2\big)K_0(\mu\sqrt{-z^2}). \end{aligned} \tag{42}$$

In the time-like region and for $x^+ >$, we explicitly have

$$D^{(+)}(x) = -\frac{1}{4}\big[N_0(\mu\sqrt{(x^- - i\epsilon^-)(x^+ - i\epsilon^+)}) \\ + iJ_0(\mu\sqrt{(x^- - i\epsilon^-)(x^+ - i\epsilon^+)})\big]. \tag{43}$$

For small z

$$J_0(z) = 1 + O(z^2), \quad N_0(z) = \frac{2}{\pi}\gamma_E + \frac{2}{\pi}\log(z/2) + O(z^2). \tag{44}$$

For $x = 0$, ϵ^+, ϵ^- serve as regulators, otherwise $D^{(+)}(0)$ ill-defined:

$$D^{(+)}(x = 0) \sim \frac{2}{\pi}\log(\mu\sqrt{-\epsilon^+\epsilon^-/4}). \tag{45}$$

Manifestation of the logaritmic divergence of the tadpole diagram. We could have set $x^+ = x^- = 0$ directly in (40), the same result. The LF

Hamiltonian approach correctly predicts the "vacuum sector diagram", no circle at infinity needed

The situation similar in $D = 3 + 1$:

$$\phi(x) = \frac{1}{\sqrt{16\pi^3}} \int_0^{+\infty} \frac{dk^+}{k^+} \int_{-\infty}^{+\infty} d^2k_\perp \Big[a(k^+, k_\perp) e^{-i\hat{k}\cdot x} + a^\dagger(k^+, k_\perp) e^{i\hat{k}\cdot x} \Big] \quad (46)$$

where $\hat{k} \cdot x \equiv \frac{1}{2}k^+ x^- + \frac{1}{2}\frac{\mu^2 + k_\perp^2}{k^+} x^+ - k_\perp x_\perp$.

The two-point function is

$$D^{(+)}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle \tag{47}$$
$$= \frac{1}{16\pi^3}\int\limits_0^\infty \frac{dk^+}{k^+}\int\limits_\infty^{+\infty} d^2k_\perp e^{-\frac{i}{2}k^+(x^- - y^- - i\epsilon^-) - \frac{i}{2}\frac{\mu^2+k_\perp^2}{k^+}(x^+ - y^+ - i\epsilon^+) - k_\perp x_\perp}.$$

In the time-like region for $x^+ > 0$

$$D^{(+)}(x) = \frac{1}{8\pi}\frac{\mu}{\sqrt{x^2}}\Big[J_1(\mu\sqrt{x^2}) - iN_1(\mu\sqrt{x^2})\Big], x^2 = (x^+ + i\epsilon^+)(x^- + i\epsilon^-) - x_\perp^2. \tag{48}$$

For small z

$$J_1(z) \sim z/2 + O(z^3), \quad N_1(z) \sim -\frac{2}{\pi z}. \tag{49}$$

Therefore for small x

$$D^{(+)}(x) \sim \frac{1}{x^2} \Rightarrow D^{(+)}(0) \sim \frac{1}{\epsilon^+ \epsilon^-}. \tag{50}$$

Indicates $\sim \Lambda^2$ divergence of the tadpole in $D = 3 + 1$.

The α-representation

$$D^{(+)}(x) = -\frac{1}{16\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-ix^2/\alpha - i\alpha m^2 - \alpha\epsilon} \tag{51}$$

is OK for $x^2 \neq 0$ but is singular for $x = 0$ if ϵ^+, ϵ^- not present (see Eq.(6) of the paper).

VI. CONCLUSIONS

- The careful definition of the LF two-point functions

  1. implies their correct equal-LF time limit, i.e. correct ET (anti)commutators
  2. makes the unwanted surface terms in the LF Poincaré algebra vanish
  3. regularizes their $x^\mu \to 0$ limit in the Fock (Hamiltonian) approach, making it equivalent to the Feynman approach

- The limit of external momentum $p^\mu \to 0$ for the simplest scalar LF self-energy diagram with two internal lines clarifies how the correct value of the LF vacuum bubble is obtained, p^+ is conserved