Bound states and Perturbation theory

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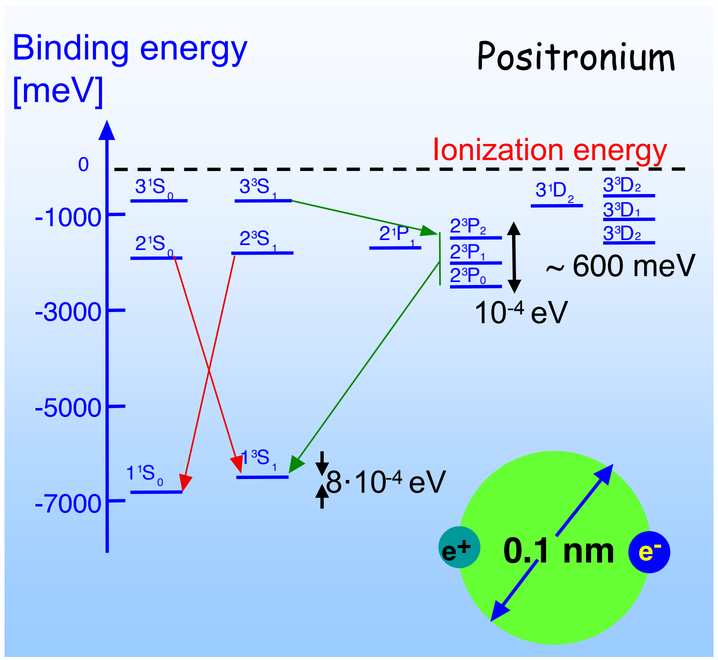
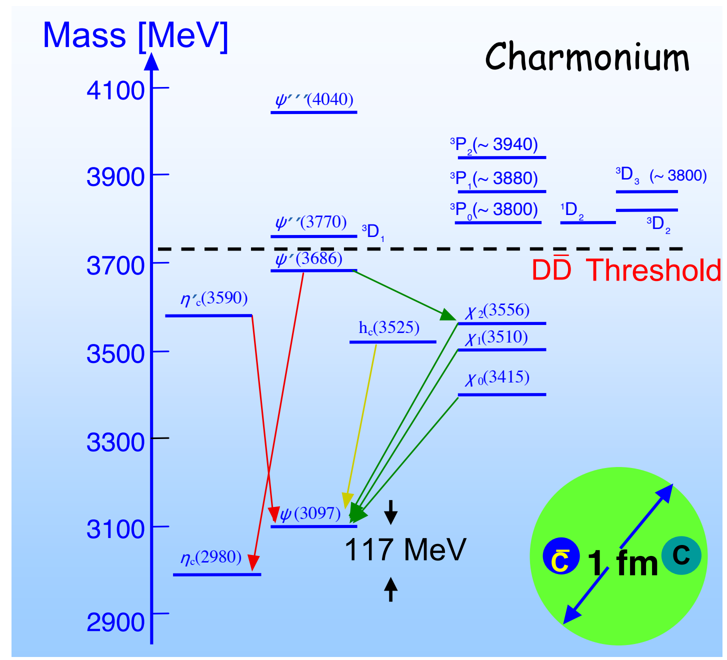




$$V(r) = -\frac{\alpha}{r}$$

$$V(r) = c\,r - \frac{4}{3}\frac{\alpha_s}{r}$$

The similarities between hadrons and atoms suggest that
atomic perturbation theory is relevant for hadrons.

PH 1807.05598v2 and refs. therein

The meaning of "non-perturbative"

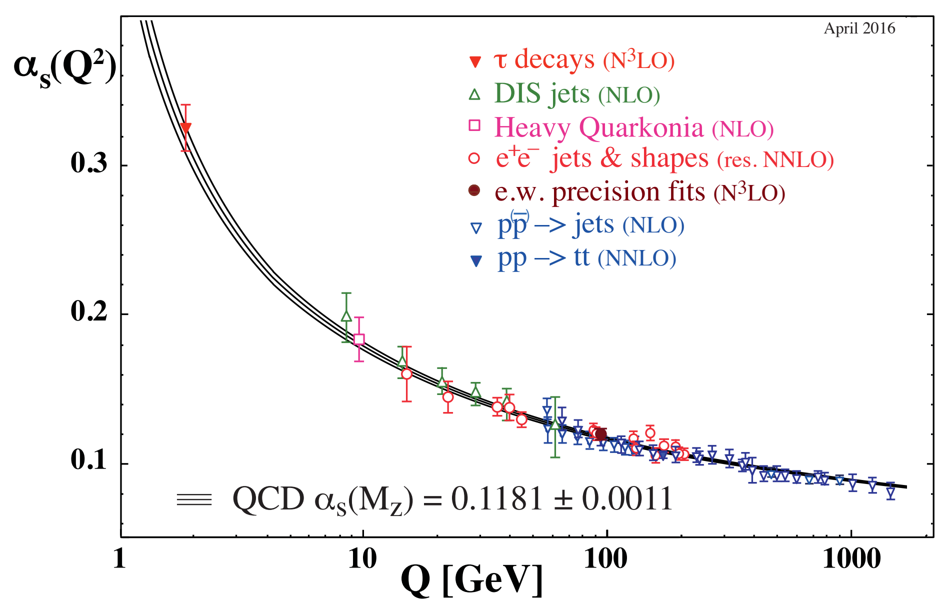
Perturbative expansion diverges

Feynman diagrams lack essential features

Common view for soft QCD: $\alpha_s \gg 1$ $\Rightarrow$ Use lattice QCD (or models)

Alternative possibility: Coupling freezes, remains perturbative $\alpha_s(0)/\pi \approx 0.14$

★ ⟵ $\alpha_s^{crit} \approx 0.43$ Gribov



Divergence of perturbative expansion is due to low momentum transfers

This is the case for classical fields in QED

and for QED bound states $\alpha(0) \simeq 1/137$

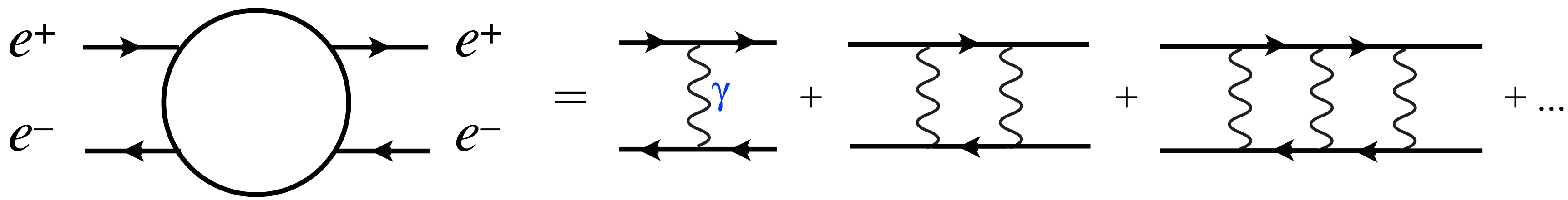
$$V(r) = -\alpha/r$$

Principle of bound state perturbation theory

QED calculations postulate an $\mathcal{O}(\alpha^\infty)$ (Schrödinger) wave function.

A generalization of QED bound state methods to QCD requires a derivation of the Schrödinger eq. from L_{QED}.

Summing ladder diagrams is not the answer: *E.g.*, for $e^+ e^- \to e^+ e^-$



The divergence of the ladder sum gives rise to Positronium poles.

But: The free *in* and *out* states of PQED lack overlap with Positronia.

QCD: Free quark states at $t = \pm\infty$ are incompatible with confinement.

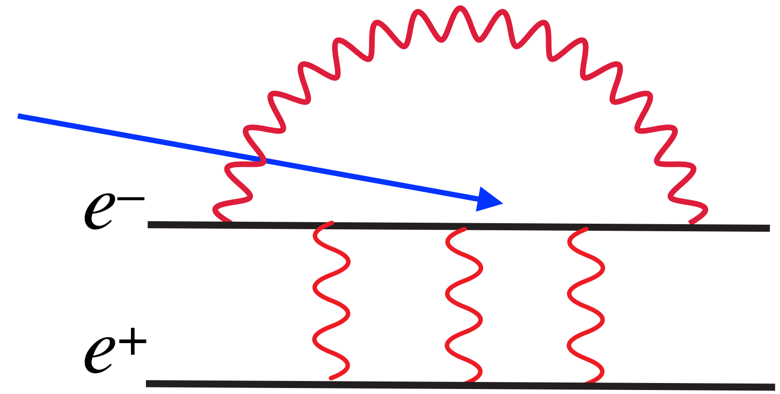
Beware of using Feynman diagrams, based on free propagators, for bound states! Confinement may not be recovered.

Bound state constituents propagate in a field

For QED lamb shift, need to calculate e^- propagator in the field of e^+

In an NR approximation, this can be described by a fixed $-\alpha/r$ potential.

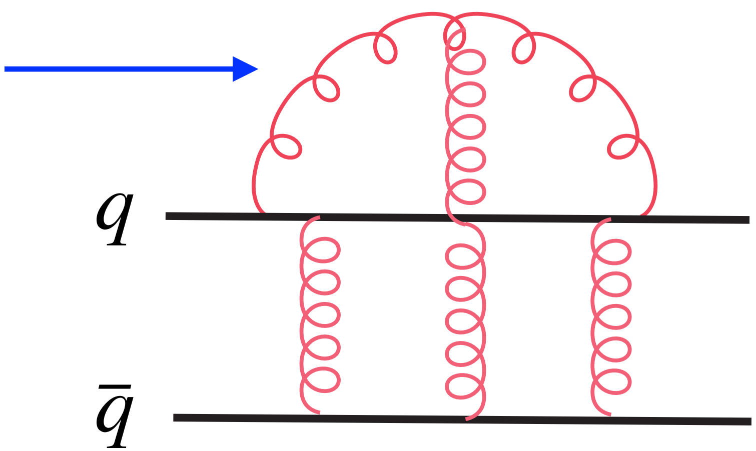
QED



Lamb shift

In QCD, colored gluons interact with relativistic quarks

QCD



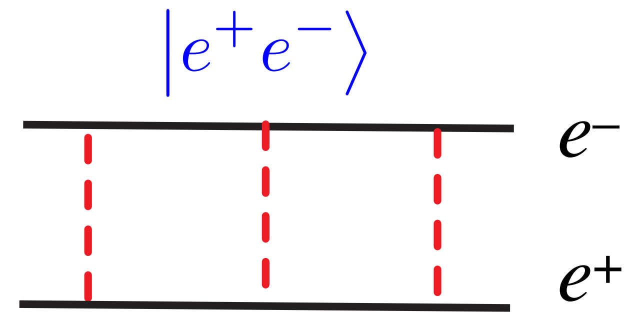
Gluon and quark propagators depend on the state in which they propagate.

Cannot build bound states with constituents that have predetermined propagators.

$\Rightarrow$

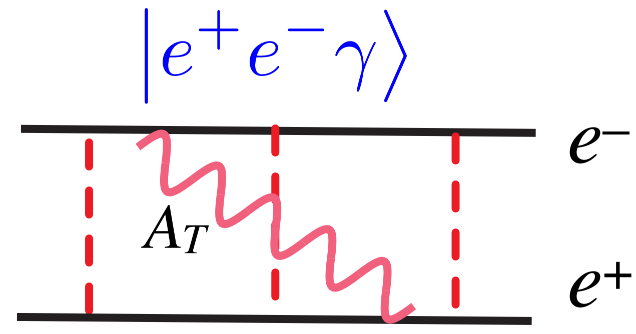
Constituents propagate in their instantaneous field, forming eigenstates of H.

Fock state expansion for Positronium (at rest)



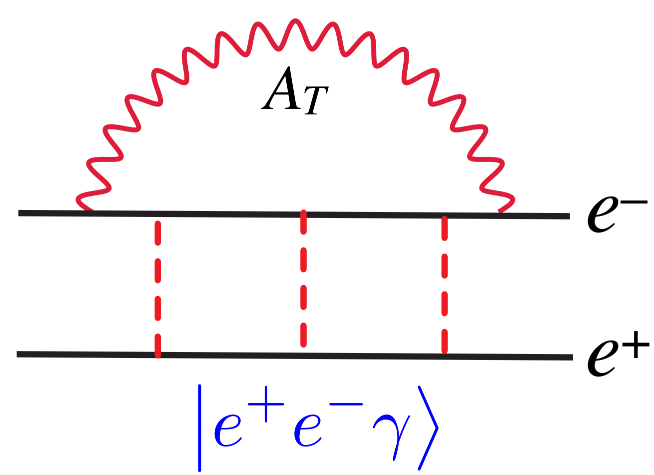
The $|e^+e^-\rangle$ Fock state is at $\mathcal{O}(\alpha^2)$ bound by the classical field of the constituents,

which is not suppressed by α.



Spin dependence arises at $\mathcal{O}(\alpha^4)$ from states with a transverse photon, $|e^+e^-\gamma\rangle$.

A_T exchange is suppressed by powers of α.



The Lamb shift also arises from $|e^+e^-\gamma\rangle$.

Perturbation theory for a bound state includes the instantaneous field of its Fock constituents.

This can be implemented in a Hamiltonian approach.

Canonical quantisation in temporal gauge: $A^0 = 0$

Avoids problem due to the missing conjugate field for A^0

$$E^i = F^{i0} = -\partial_0 A^i \quad \text{conjugate to } -A^i \qquad (i = 1,2,3)$$

$$\left[E^i(t,\boldsymbol{x}), A^j(t,\boldsymbol{y})\right] = i\delta^{ij}\delta(\boldsymbol{x}-\boldsymbol{y}) \qquad \left\{\psi^\dagger_\alpha(t,\boldsymbol{x}), \psi_\beta(t,\boldsymbol{y})\right\} = \delta_{\alpha\beta}\,\delta(\boldsymbol{x}-\boldsymbol{y})$$

$$H = \int d\boldsymbol{x}\left[\tfrac{1}{2}\boldsymbol{E}_L^2 + \tfrac{1}{2}\boldsymbol{E}_T^2 + \tfrac{1}{4}F^{ij}F^{ij} + \psi^\dagger(-i\alpha^i\partial_i - e\alpha^i A^i + m\gamma^0)\psi\right]$$

Gauss' operator: $G(x) \equiv \dfrac{\delta \mathcal{S}}{\delta A^0(x)} = \partial_i E^i_L(x) - e\psi^\dagger\psi(x)$

$G(x)$ generates *time*-independent gauge transformations, consistent with $A^0 = 0$

The gauge is fixed by constraining the physical states: $G(x)\,|phys\rangle = 0$

This determines $E_L(x)$ for each state, imposing Gauss' law.

J. D. Bjorken, SLAC Summer Institute (1979)
G. Leibbrandt, Rev. Mod. Phys. 59, 1067 (1987)

Schrödinger equation for Positronium

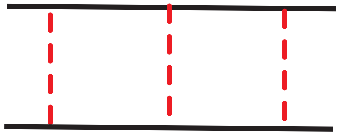
$$G(x)\,|phys\rangle = 0 \quad \Longrightarrow \quad \begin{aligned} &\partial_i E_L^i(t,\boldsymbol{x})\,|phys\rangle = e\psi^\dagger\psi(t,\boldsymbol{x})\,|phys\rangle \\ &E_L^i(t,\boldsymbol{x})\,|phys\rangle = -\partial_i^x \int d\boldsymbol{y}\,\frac{e}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\psi^\dagger\psi(t,\boldsymbol{y})\,|phys\rangle \end{aligned}$$

For the component of Positronium with an electron at $\boldsymbol{x}_1$ and a positron at $\boldsymbol{x}_2$:

$$\left|e^-(\boldsymbol{x}_1)e^+(\boldsymbol{x}_2)\right\rangle = \bar\psi_\alpha(\boldsymbol{x}_1)\psi_\beta(\boldsymbol{x}_2)\,|0\rangle$$

$$E_L^i\left|e^-(\boldsymbol{x}_1)e^+(\boldsymbol{x}_2)\right\rangle = -\partial_i^x\,\frac{e}{4\pi}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_2|}\right)\left|e^-(\boldsymbol{x}_1)e^+(\boldsymbol{x}_2)\right\rangle$$

The instantaneous Hamiltonian $H_V \equiv \frac{1}{2}\int d\boldsymbol{x} E_L^i E_L^i(\boldsymbol{x})$ gives the classical potential:



$$H_V\left|e^-(\boldsymbol{x}_1)e^+(\boldsymbol{x}_2)\right\rangle = -\frac{\alpha}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}\left|e^-(\boldsymbol{x}_1)e^+(\boldsymbol{x}_2)\right\rangle$$

The Schrödinger equation follows from $H\left|e^+e^-\right\rangle = (2m+E_b)\left|e^+e^-\right\rangle$

A Fock state expansion for QCD

The Fock expansion is compatible with the quark model of hadrons:

- Valence quantum numbers of mesons and baryons (lowest Fock state)
- Transverse gluon constituents contribute perturbatively, at $O(\alpha_s)$
- The E_L field is instantaneous even for relativistic constituents

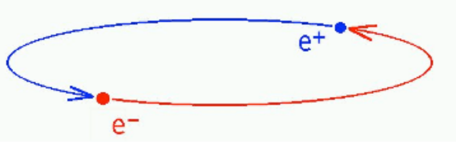
How can color confinement arise?

Gauss' law has no Λ_{QCD} scale

A crucial difference between QED and QCD

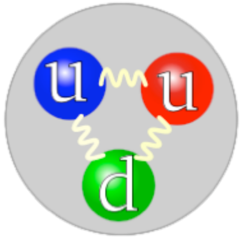
Global gauge invariance allows a classical gauge field for neutral atoms, but not a color octet gluon field for color singlet hadrons.

Positronium (QED)



$$E_L^i(\boldsymbol{x}) = -\frac{e}{4\pi}\,\partial_i^x\left(\frac{1}{\boldsymbol{x}-\boldsymbol{x}_1} - \frac{1}{\boldsymbol{x}-\boldsymbol{x}_2}\right)$$

Proton (QCD)



$$E_{L,a}^i(\boldsymbol{x}) = 0$$

However:
The classical gluon field is non-vanishing for each color component C of the state

$$E_{L,a}^i(\boldsymbol{x}, C) \neq 0$$

The blue quark feels the color field generated by the red and green quarks.

An external observer sees no field:
The gluon field generated by a color singlet state vanishes

$$\sum_C E_{L,a}^i(\boldsymbol{x}, C) = 0$$

Temporal gauge in QCD: $A_a^0 = 0$

Gauss' operator $$G_a(x) \equiv \frac{\delta \mathcal{S}}{\delta A_a^0(x)} = \partial_i E_a^i(x) + g f_{abc} A_b^i E_c^i - g\psi^\dagger T^a \psi(x)$$

generates time-independent gauge transformations, which keep $A_a^0 = 0$

The gauge is fully defined (in PT) by the constraint $G_a(x)\,|phys\rangle = 0$

$$\Longrightarrow \qquad \partial_i E_{L,a}^i(\boldsymbol{x})\,|phys\rangle = g\big[-f_{abc} A_b^i E_c^i + \psi^\dagger T^a \psi(\boldsymbol{x})\big]\,|phys\rangle$$

In QED one solves for $\boldsymbol{E}_L$ requiring $\boldsymbol{E}_L(\boldsymbol{x}) \to 0$ for $|\boldsymbol{x}| \to \infty$

In QCD, for (globally) color singlet bound states: $\sum_C E_{L,a}^i(\boldsymbol{x}, C) = 0$

For each color component C there are homogeneous solutions of Gauss' law for E_L, which do not vanish at spatial infinity.

Translation invariance requires a constant field energy density ($\Rightarrow$ scale Λ) .

The solution is unique, up to the magnitude of the energy density (Λ).

Including a homogeneous solution for $E^i_{L,a}$

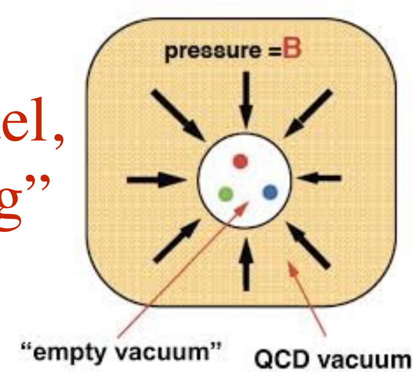
$$E^i_{L,a}(\boldsymbol{x})\,|phys\rangle = -\partial_i^x \int d\boldsymbol{y}\Big[\kappa\,\boldsymbol{x}\cdot\boldsymbol{y} + \frac{g}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\Big]\mathcal{E}_a(\boldsymbol{y})\,|phys\rangle$$

where $\quad \mathcal{E}_a(\boldsymbol{y}) = -f_{abc}A^i_b E^i_c(\boldsymbol{y}) + \psi^\dagger T^a \psi(\boldsymbol{y})$

$\kappa \neq \kappa(\boldsymbol{x},\boldsymbol{y})$ ensures $\partial_i \boldsymbol{E}^i(\boldsymbol{x}) = 0$ (a homogeneous solution)

The linear dependence on $\boldsymbol{x}$ makes $\boldsymbol{E}_L$ independent of $\boldsymbol{x}$, as required by translation invariance:
The field energy density is spatially constant.

"Bag model, but no bag"



The $\boldsymbol{E}_L$ contribution to the QCD Hamiltonian is

$$H_V = \int d\boldsymbol{y}d\boldsymbol{z}\Big\{\boldsymbol{y}\cdot\boldsymbol{z}\Big[\tfrac{1}{2}\kappa^2\int d\boldsymbol{x} + g\kappa\Big] + \tfrac{1}{2}\frac{\alpha_s}{|\boldsymbol{y}-\boldsymbol{z}|}\Big\}\mathcal{E}_a(\boldsymbol{y})\mathcal{E}_a(\boldsymbol{z})$$

The field energy $\propto$ volume of space is irrelevant only if it is universal.
This relates the normalisation $\varkappa$ of all Fock components, leaving an overall scale Λ as the single parameter.

Examples: Fock state potentials (I)

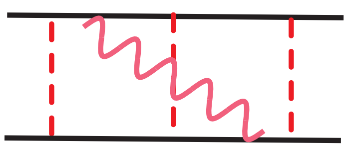
q$\bar{\text{q}}$: $$H_V\,|q(\boldsymbol{x}_1)\bar{q}(\boldsymbol{x}_2)\rangle = V_{q\bar{q}}\,|q(\boldsymbol{x}_1)\bar{q}(\boldsymbol{x}_2)\rangle$$

$$V_{q\bar{q}} = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| - C_F\frac{\alpha_s}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}$$

"Cornell potential" also for relativistic quarks

qg$\bar{\text{q}}$: $$V^{(0)}_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) = \frac{\Lambda^2}{\sqrt{C_F}}\,d_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2)$$

(universal Λ)



$$d_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) \equiv \sqrt{\tfrac{1}{4}(N-2/N)(\boldsymbol{x}_1-\boldsymbol{x}_2)^2 + N(\boldsymbol{x}_g-\tfrac{1}{2}\boldsymbol{x}_1-\tfrac{1}{2}\boldsymbol{x}_2)^2}$$

$$V^{(1)}_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) = \tfrac{1}{2}\,\alpha_s\Big[\frac{1}{N}\,\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} - N\Big(\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_g|}+\frac{1}{|\boldsymbol{x}_2-\boldsymbol{x}_g|}\Big)\Big]$$

When q and g coincide:

$$V^{(0)}_{qgq}(\boldsymbol{x}_1=\boldsymbol{x}_g,\boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| = V^{(0)}_{q\bar{q}}$$

$$V^{(1)}_{qgq}(\boldsymbol{x}_1=\boldsymbol{x}_g,\boldsymbol{x}_2) = V^{(1)}_{q\bar{q}}$$

Fock state potentials (II)

qqq :

$$V_{qqq} = \Lambda^2 d_{qqq}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) - \frac{2}{3}\,\alpha_s\Big(\frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} + \frac{1}{|\boldsymbol{x}_2 - \boldsymbol{x}_3|} + \frac{1}{|\boldsymbol{x}_3 - \boldsymbol{x}_1|}\Big)$$

$$d_{qqq}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \equiv \frac{1}{\sqrt{2}}\sqrt{(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 + (\boldsymbol{x}_2 - \boldsymbol{x}_3)^2 + (\boldsymbol{x}_3 - \boldsymbol{x}_1)^2}$$

gg : $$V_{gg} = \sqrt{\frac{N}{C_F}}\,\Lambda^2\,|\boldsymbol{x}_1 - \boldsymbol{x}_2| - N\,\frac{\alpha_s}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}$$

This agrees with the $qg\bar{q}$ potential where the quarks coincide:

$$V_{gg}(\boldsymbol{x}, \boldsymbol{x}_g) = V_{qg\bar{q}}(\boldsymbol{x}, \boldsymbol{x}_g, \boldsymbol{x})$$

It is straightforward to work out the instantaneous potential for any Fock state.

Thus: A perturbative approach to soft QCD

- The instantaneous $\mathcal{O}\left(\alpha_s^0\right)$ field binds the lowest Fock states
- The higher Fock states given by the Hamiltonian H_{QCD} are of $\mathcal{O}\left(\alpha_s\right)$
- Makes bound state calculations less of an art

For the approach to be viable the $\mathcal{O}\left(\alpha_s^0\right)$ dynamics must have:

Poincaré symmetry
Unitarity
Confinement
Chiral Symmetry Breaking (CSB)
Reasonable mass spectrum
. . .

Some of these have been demonstrated, and the prospects seem promising (see extra slides, my home page and 1807.05598v2).

Extra slides

$\mathcal{O}\left(\alpha_s^0\right)$ $q\bar{q}$ bound states

The $\mathcal{O}\left(\alpha_s^0\right)$ meson is a superposition of $q\bar{q}$ Fock states with wave function Φ,

$$|M\rangle = \sum_{A,B;\alpha,\beta} \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\, \bar{\psi}_\alpha^A(t=0,\boldsymbol{x}_1)\delta^{AB}\Phi_{\alpha\beta}(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi_\beta^B(t=0,\boldsymbol{x}_2)\,|0\rangle$$

The bound state condition $H\,|M\rangle = M\,|M\rangle$ gives

$$\left[i\gamma^0\boldsymbol{\gamma}\cdot\overrightarrow{\boldsymbol{\nabla}}+m\gamma^0\right]\Phi(\boldsymbol{x})+\Phi(\boldsymbol{x})\left[i\gamma^0\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{\nabla}}-m\gamma^0\right]=\left[M-V(|\boldsymbol{x}|)\right]\Phi(\boldsymbol{x})$$

where $\boldsymbol{x} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$ and $V(|\,\boldsymbol{x}\,|) = V'|\,\boldsymbol{x}\,| = \Lambda^2|\,\boldsymbol{x}\,|$.

In the non-relativistic limit ($m \gg \Lambda$) this reduces to the Schrödinger equation, and we may add the instantaneous gluon exchange potential.

$\Rightarrow$ The successful quarkonium phenomenology with the Cornell potential.

Relativistic $q\bar{q}$ bound states

$$i\boldsymbol{\nabla}\cdot\left\{\gamma^0\boldsymbol{\gamma},\Phi(\boldsymbol{x})\right\}+m\left[\gamma^0,\Phi(\boldsymbol{x})\right]=\left[M-V(\boldsymbol{x})\right]\Phi(\boldsymbol{x})$$

Expanding the 4×4 wave function in a basis of 16 Dirac structures $\Gamma_i(\boldsymbol{x})$

$$\Phi(\boldsymbol{x})=\sum_i \Gamma_i(\boldsymbol{x})F_i(r)Y_{j\lambda}(\hat{\boldsymbol{x}})$$

we may use rotational, parity and charge conjugation invariance to determine which $\Gamma_i(\boldsymbol{x})$ may occur for a state of given j^{PC}:

0^{-+} trajectory $[s=0,\ \ell=j]$:	$-\eta_P=\eta_C=(-1)^j$	$\gamma_5,\ \gamma^0\gamma_5,\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L}$
0^{--} trajectory $[s=1,\ \ell=j]$:	$\eta_P=\eta_C=-(-1)^j$	$\gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \boldsymbol{\alpha}\cdot\boldsymbol{L},\ \gamma^0\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$
0^{++} trajectory $[s=1,\ \ell=j\pm 1]$:	$\eta_P=\eta_C=+(-1)^j$	$1,\ \boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma^0\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \gamma^0\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$
0^{+-} trajectory [exotic] :	$\eta_P=-\eta_C=(-1)^j$	$\gamma^0,\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$

$\Rightarrow$ There are no solutions for quantum numbers that would be exotic in the quark model (despite the relativistic dynamics)

Example: 0⁻⁺ trajectory wf's

$$\Phi_{-+}(\boldsymbol{x}) = \Big[\frac{2}{M-V}(i\boldsymbol{\alpha}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0) + 1\Big]\gamma_5\, F_1(r)Y_{j\lambda}(\hat{\boldsymbol{x}})$$

$\eta_P = (-1)^{j+1}$

$\eta_C = (-1)^j$

Radial equation: $F_1'' + \Big(\frac{2}{r} + \frac{V'}{M-V}\Big)F_1' + \Big[\frac{1}{4}(M-V)^2 - m^2 - \frac{j(j+1)}{r^2}\Big]F_1 = 0$

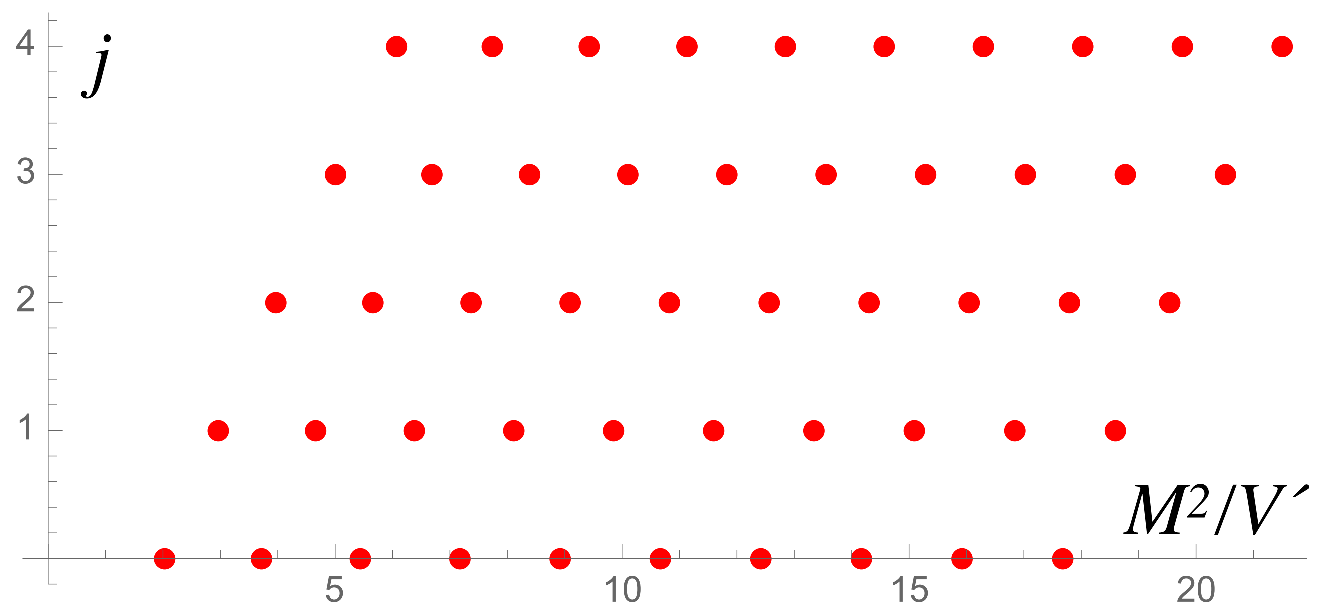
Local normalizability at $r = 0$ and at $V(r) = M$ determines the discrete M

Mass spectrum:

Linear Regge trajectories with daughters

Spectrum similar to dual models

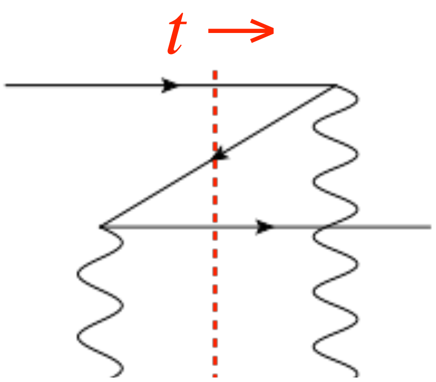
$m = 0$



Sea quark contributions

Quark states in a strong field have E<0 components

Bogoliubov transformation, cf. Dirac states.



In time-ordered PT, these correspond to Z-diagrams, and interpreted as contributions from $q\bar{q}$ pairs.

This effect is manifest in the behavior of the wave function Φ for large V = V´|$\boldsymbol{x}$| :

$$\lim_{\boldsymbol{x}\to\infty} |\Phi(\boldsymbol{x})|^2 = const.$$

The asymptotically constant norm reflects, via duality, pair production as the linear potential $V(|\,\boldsymbol{x}\,|)$ increases.

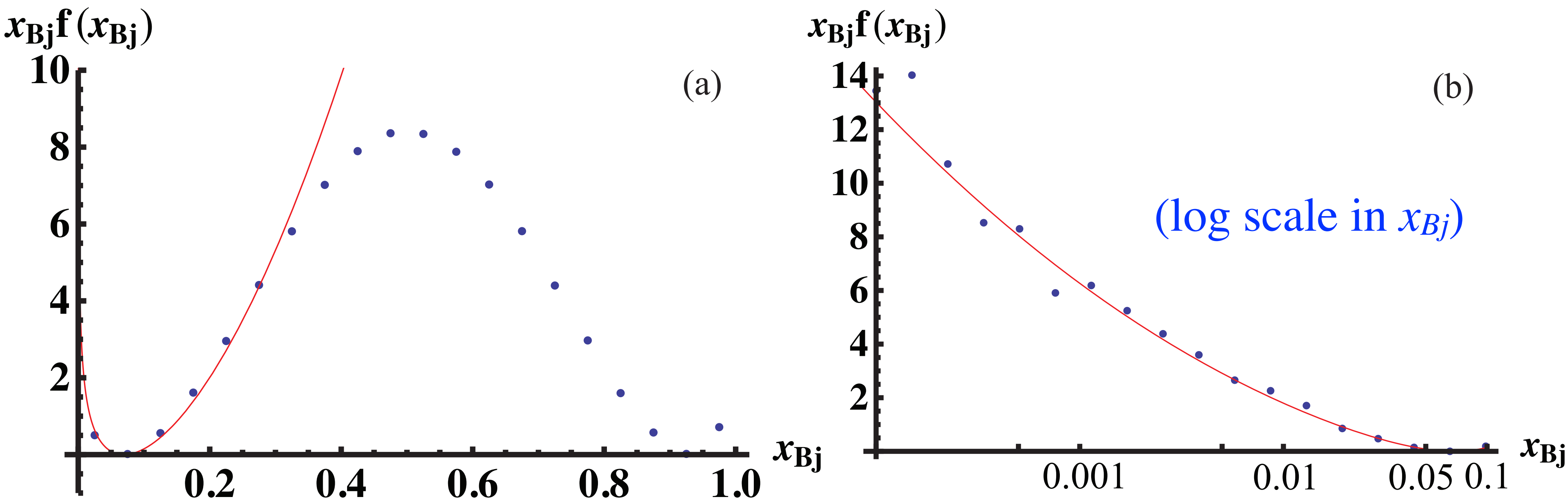
These sea quarks show up in the parton distribution measured in DIS.

Parton distributions have a sea component

In D=1+1 dimensions the sea component is prominent at low *m/e* :

$m/e = 0.1$

D. D. Dietrich, PH, M. Järvinen
arXiv 1212.4747



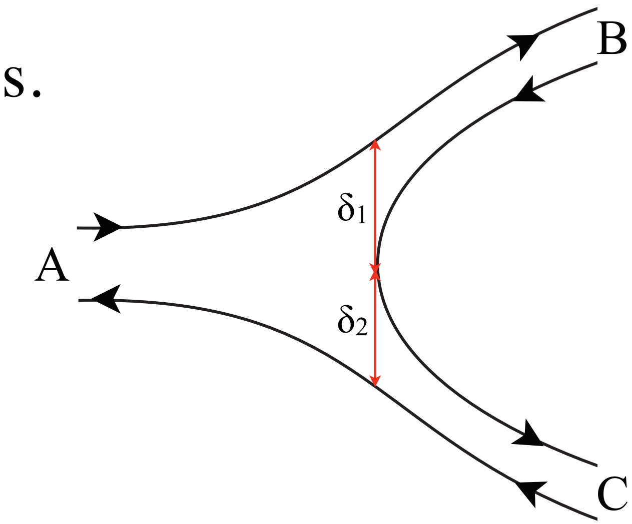
The red curve is an analytic approximation, valid in the $x_{Bj} \to 0$ limit.

Note: Enhancement at low x is due to bd (sea), not to $b^\dagger d^\dagger$ (valence) component.

To be calculated in D=3+1 (and in various frames!)

Decays and hadron loops

The bound state equation determines zero-width states.

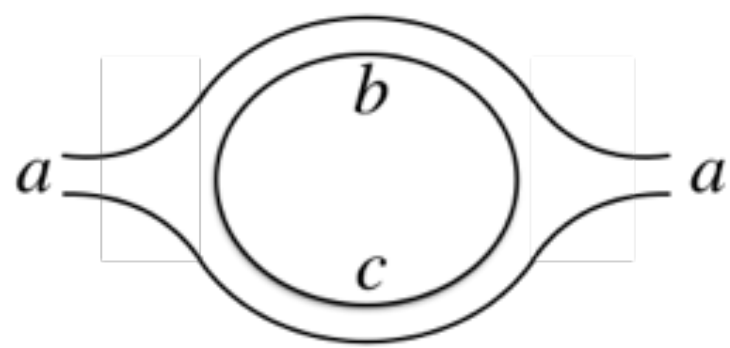


There is an $\mathcal{O}\left(1/\sqrt{N_C}\right)$ coupling between the states: string breaking

$$\langle B, C|A\rangle =$$

$$-\frac{(2\pi)^3}{\sqrt{N_C}}\delta^3(\boldsymbol{P}_A-\boldsymbol{P}_B-\boldsymbol{P}_C)\int d\boldsymbol{\delta}_1 d\boldsymbol{\delta}_2\, e^{i\boldsymbol{\delta}_1\cdot\boldsymbol{P}_C/2-i\boldsymbol{\delta}_2\cdot\boldsymbol{P}_B/2}\text{Tr}\left[\gamma^0\Phi_B^\dagger(\boldsymbol{\delta}_1)\Phi_A(\boldsymbol{\delta}_1+\boldsymbol{\delta}_2)\Phi_C^\dagger(\boldsymbol{\delta}_2)\right]$$

When squared, this gives a $1/N_C$ hadron loop unitarity correction:



Unitarity should be satisfied at hadron level at each order of $1/N_C$.

Bound states in motion

An $\mathcal{O}\left(\alpha_s^0\right)$ $q\bar{q}$ bound state with CM momentum $\boldsymbol{P}$ may be expressed as

$$|M,\boldsymbol{P}\rangle = \int dx_1\, dx_2\, \bar{\psi}(t=0,x_1)\, e^{i\boldsymbol{P}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2}\, \Phi^{(\boldsymbol{P})}(x_1-x_2)\, \psi(t=0,x_2)\, |0\rangle$$

The instantaneous potential is $\boldsymbol{P}$-independent, $V(\boldsymbol{x}) = V'|\boldsymbol{x}|$, hence the BSE:

$$i\boldsymbol{\nabla}\cdot\left\{\boldsymbol{\alpha},\Phi^{(\boldsymbol{P})}(\boldsymbol{x})\right\} - \tfrac{1}{2}\boldsymbol{P}\cdot\left[\boldsymbol{\alpha},\Phi^{(\boldsymbol{P})}(\boldsymbol{x})\right] + m\left[\gamma^0,\Phi^{(\boldsymbol{P})}(\boldsymbol{x})\right] = \left[E - V(\boldsymbol{x})\right]\Phi^{(\boldsymbol{P})}(\boldsymbol{x})$$

The solution for $\Phi^{(P)}(\boldsymbol{x})$ is not simply Lorentz contracting in $\boldsymbol{x}$.

States with general $\boldsymbol{P}$ are needed for:

- $\boldsymbol{P}$-dependence of angular momentum ($P \to \infty$ frame).
- EM form factors (gauge invariance has been verified)
- Parton distributions
- Hadron scattering
- . . .

States with M = 0

PRELIMINARY

We required the wave function to be normalizable at $r = 0$ and $V'r = M$

For $M = 0$ the two points coincide. Regular, massless solutions are found.

The massless 0^{++} meson "σ"

$$|\sigma\rangle = \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, \bar\psi(\boldsymbol{x}_1)\, \Phi_\sigma(\boldsymbol{x}_1 - \boldsymbol{x}_2)\, \psi(\boldsymbol{x}_2)\, |0\rangle \equiv \hat\sigma\, |0\rangle$$

For $m = 0$ and $V' = 1$:

$$\Phi_\sigma(\boldsymbol{x}) = N_\sigma\Big[J_0(\tfrac{1}{4}r^2) + \boldsymbol{\alpha}\cdot\boldsymbol{x}\frac{i}{r}\, J_1(\tfrac{1}{4}r^2)\Big]$$

J_0 and J_1 are Bessel functions.

$\hat P^\mu\, |\sigma\rangle = 0$

State has *vanishing four-momentum* in any frame.
It may mix with the perturbative vacuum.
This spontaneously breaks chiral invariance.

A chiral condensate (m = 0)

PRELIMINARY

Since | σ ⟩ has vacuum quantum numbers and zero momentum it can mix with the perturbative vacuum without violating Poincaré invariance

Consider: $|\chi\rangle = \exp(\hat{\sigma})\,|0\rangle$ for which $\langle\chi|\bar{\psi}\psi\,|\chi\rangle = 4N_\sigma$

An infinitesimal chiral rotation of the condensate generates a pion

$$U_\chi(\beta) = \exp\left[i\beta\int d\boldsymbol{x}\,\psi^\dagger(\boldsymbol{x})\gamma_5\psi(\boldsymbol{x})\right] \qquad U_\chi(\beta)\,|\chi\rangle = (1-2i\beta\,\hat{\pi}\,|\chi\rangle$$

where $\hat{\pi}$ is the massless 0^{-+} state with wave function $\Phi_\pi = \gamma_5\Phi_\sigma$

This seems to provide an explicit example of chiral condensate.

Small quark mass: m > 0

PRELIMINARY

When $m \neq 0$ the massless ($M_\sigma = 0$) sigma 0^{++} state has wave function

$$\Phi_\sigma(\boldsymbol{x}) = f_1(r) + i\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\,f_2(r) + i\,\boldsymbol{\gamma}\cdot\boldsymbol{x}\,g_2(r)$$

Radial functions are Laguerre fn's

An $M_\pi > 0$ pion 0^{-+} state has rest frame wave function

$$\Phi_\pi(\boldsymbol{x}) = \left[F_1(r) + i\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\,F_2(r) + \gamma^0\,F_4(r)\right]\gamma_5 \qquad F_4(0) = \frac{2m}{M}F_1(0)$$

$$F_1'' + \Big(\frac{2}{r} + \frac{1}{M-r}\Big)F_1' + \big[\tfrac{1}{4}(M-r)^2 - m^2\big]F_1 = 0$$

$$\langle\chi|j_5^\mu(x)\hat{\pi}\,|\chi\rangle = iP^\mu f_\pi\,e^{-iP\cdot x} \quad \Rightarrow \quad F_4(0) = \tfrac{1}{4}iM_\pi f_\pi$$

$$\langle\chi|\bar{\psi}(x)\gamma_5\psi(x)\,\hat{\pi}\,|\chi\rangle = -i\,\frac{M^2}{2m}\,f_\pi\,e^{-iP\cdot x} \quad \Rightarrow \quad F_1(0) = i\,\frac{M^2}{8m}\,f_\pi$$

CSB relations are satisfied for any P.