STRUCTURE OF LIGHT-FRONT VACUUM SECTOR DIAGRAMS

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P. D. Mannheim, P. Lowdon and S. J. Brodsky, *Structure of light-front vacuum sector diagrams*, Phys. Lett. B in press. (arXiv:1904:05253 [hep-ph], SLAC-PUB-17393)

P. D. Mannheim, P. Lowdon and S. J. Brodsky, *Comparing light-front quantization with instant-time quantization*, SLAC-PUB-17390, in final preparation.

P. D. Mannheim, *Light-front quantization is instant-time quantization*, arXiv:1909:03458 [hep-ph].

1 TWO QUESTIONS

(1) Is light-front quantization at equal $x^+ = x^0 + x^3$ the same theory as instant-time quantization at equal x^0, or is it a different theory?

(2) Is there anything in quantum field theory that is not accounted for by the on-shell Light-Front Hamiltonian description of physics?

Two Answers

(1) **YES**, but it sure does not look like it.

(2) **YES**, but only in the vacuum sector. Because of zero modes with $p_- = 0$

Mass-shell conditions

$$\text{Instant} : p_0 = \pm[(p_1)^2 + (p_2)^2 + (p_3)^2 + m^2]^{1/2}, \quad \text{well–behaved at } \mathrm{p}_3 = 0.$$
$$\text{Front} : p_+ = \frac{(p_1)^2 + (p_2)^2 + (p_3)^2 + m^2}{4p_-}, \quad \text{singular at } \mathrm{p}_- = 0. \tag{1}$$

2 INSTANT-TIME AND LIGHT-FRONT FOCK SPACE EXPANSIONS

Instant-Time Scalar Field Fock Space Expansion

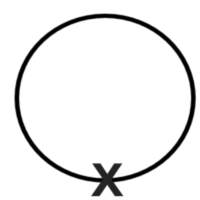
$$\phi(x^0,x^1,x^2,x^3)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{(2E_p)^{1/2}}[a(\vec{p})e^{-iE_pt+i\vec{p}\cdot\vec{x}}+a^\dagger(\vec{p})e^{+iE_pt-i\vec{p}\cdot\vec{x}}],\quad E_p=+[p^2+m^2]^{1/2}. \tag{2}$$

Contains $-\infty\leq p_3\leq\infty$, well-behaved at $p_3=0$.

Light-Front Scalar Field Fock Space Expansion

$$\phi(x^+,x^1,x^2,x^-)=\frac{2}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{(4p_-)^{1/2}}$$
$$\times\left[e^{-i(F_p^2x^+/4p_-+p_-x^-+p_1x^1+p_2x^2)}a(p_1,p_2,p_-)+e^{i(F_p^2x^+/4p_-+p_-x^-+p_1x^1+p_2x^2)}a_p^\dagger(p_1,p_2,p_-)\right],$$
$$F_p^2=(p_1)^2+(p_2)^2+m^2. \tag{3}$$

Singular at $p_-=0$, undefined at $x^+=0,\ p_-=0$.
Contains $p_-\geq 0$ only, Light-Front Hamiltonian approach restricts to $p_->0$.
Thus go beyond Light-Front-Hamiltonian if have processes with $p_-=0$.
This happens in vacuum sector where tadpole is $-i\langle\Omega|\phi(0)\phi(0)|\Omega\rangle$ with $x^+=0$.
So have to deal with indeterminacy of x^+/p_- at $x^+=0$, $p_-=0$.
Mannheim, Lowdon, Brodsky (2019): light-front tadpole = instant-time tadpole.



3 INSTANT-TIME AND LIGHT-FRONT COMMUTATORS & ANTICOMMUTATORS

Scalar instant-time commutators, $\Pi = \partial^0\phi = \partial_0\phi$

$$[\phi(x^0,x^1,x^2,x^3),\partial_0\phi(x^0,y^1,y^2,y^3)] = i\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3),$$
$$[\phi(x^0,x^1,x^2,x^3),\phi(x^0,y^1,y^2,y^3)] = 0. \tag{4}$$

Scalar light-front commutators, $\Pi = \partial^+\phi = 2\partial_-\phi$

$$[\phi(x^+,x^1,x^2,x^-),2\partial_-\phi(x^+,y^1,y^2,y^-)] = i\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^--y^-),$$
$$[\phi(x^+,x^1,x^2,x^-),\phi(x^+,y^1,y^2,y^-)] = -\frac{i}{4}\epsilon(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{5}$$

Gauge field instant-time commutators, $\Pi^\mu = -\partial^0 A^\mu = -\partial_0 A^\mu$

$$[A_\nu(x^0,x^1,x^2,x^3),-\partial_0 A_\mu(x^0,y^1,y^2,y^3)] = -ig_{\mu\nu}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3),$$
$$[A_\nu(x^0,x^1,x^2,x^3),A_\mu(x^0,y^1,y^2,y^3)] = 0. \tag{6}$$

Gauge field light-front commutators, $\Pi^\mu = -\partial^+ A^\mu = -2\partial_- A^\mu$

$$[A_\nu(x^+,x^1,x^2,x^-),2\partial_- A_\mu(x^+,y^1,y^2,y^-)] = ig_{\mu\nu}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^--y^-),$$
$$[A_\nu(x^+,x^1,x^2,x^-),A_\mu(x^+,y^1,y^2,y^-)] = -\frac{i}{4}g_{\mu\nu}\epsilon(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{7}$$

Fermion instant-time commutators, $\Pi = i\psi^\dagger$

$$\Big\{\psi_\alpha(x^0,x^1,x^2,x^3),\psi^\dagger_\beta(x^0,y^1,y^2,y^3)\Big\} = \delta_{\alpha\beta}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3). \tag{8}$$

Fermion light-front anticommutators, $\Pi = i\psi^\dagger\gamma^0(\gamma^0+\gamma^3) = 2i\psi^\dagger_{(+)}$

$$\big\{[\psi_{(+)}]_\alpha(x^+,x^1,x^2,x^-),[\psi^\dagger_{(+)}]_\beta(x^+,y^1,y^2,y^-)\big\} = \Lambda^+_{\alpha\beta}\delta(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{9}$$

$$\Lambda^\pm = \tfrac{1}{2}(\gamma^0\pm\gamma^3),\quad \Lambda^++\Lambda^- = I,\quad (\Lambda^+)^2 = \Lambda^+,\quad (\Lambda^-)^2=\Lambda^-,\quad \Lambda^+\Lambda^-=0,\quad \psi_{(\pm)} = \Lambda_\pm\psi. \tag{10}$$

non–invertible–projectors.

$$\psi_{(-)}(x^+,x^1,x^2,x^-) = \tfrac{1}{4i}\int dy^-\epsilon(x^--y^-)[-i\gamma^0(\gamma^1\partial_1+\gamma^2\partial_2)+m\gamma^0]\psi_{(+)}(x^+,x^1,x^2,y^-). \tag{11}$$

constrained variable.

Sure look different, but..(Mannheim 2019)....

4 UNEQUAL TIME COMMUTATORS AND ANTICOMMUTATORS

UNEQUAL TIME Scalar instant-time commutator

$$
\begin{aligned}
i\Delta(x-y) &= [\phi(x^0,x^1,x^2,x^3),\phi(y^0,y^1,y^2,y^3)] \\
&= \int \frac{d^3p d^3q}{(2\pi)^3(2p)^{1/2}(2q)^{1/2}}\Big([a(\vec{p}),a^\dagger(\vec{q})]e^{-ip\cdot x+iq\cdot y}+[a^\dagger(\vec{p}),a(\vec{q})]e^{ip\cdot x-iq\cdot y}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3 2p}\big(e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}\big) \\
&= -\frac{i}{2\pi}\frac{\delta(x^0-y^0-|\vec{x}-\vec{y}|)-\delta(x^0-y^0+|\vec{x}-\vec{y}|)}{2|\vec{x}-\vec{y}|} \\
&= -\frac{i}{2\pi}\epsilon(x^0-y^0)\delta[(x^0-y^0)^2-(x^1-y^1)^2-(x^2-y^2)^2-(x^3-y^3)^2]. \qquad (12)
\end{aligned}
$$

Since holds ALL times, also holds at EQUAL light front time.

Substitute $x^0=(x^++x^-)/2$**,** $x^3=(x^+-x^-)/2$**,** $y^0=(y^++y^-)/2$**,** $y^3=(y^+-y^-)/2$**:**

$$
i\Delta(x-y) = -\frac{i}{2\pi}\epsilon[\tfrac{1}{2}(x^++x^--y^+-y^-)]\delta[(x^+-y^+)(x^--y^-)-(x^1-y^1)^2-(x^2-y^2)^2]. \qquad (13)
$$

$$
i\Delta(x-y)\big|_{x^+=y^+} = [\phi(x^+,x^1,x^2,x^-),\phi(x^+,y^1,y^2,y^-)] = -\frac{i}{4}\epsilon(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \qquad (14)
$$

At $x^+=y^+$ **UNEQUAL instant-time commutator is EQUAL light-front time commutator**

Light-front quantization is instant-time quantization, and does not need to be independently postulated.

UNEQUAL TIME Abelian gauge field instant-time commutator

$$
\begin{aligned}
&[A_\nu(x^0,x^1,x^2,x^3),A_\mu(y^0,y^1,y^2,y^3)] = ig_{\mu\nu}\Delta(x-y)\\
&= -\frac{i}{2\pi}g_{\mu\nu}\epsilon(x^0-y^0)\delta[(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2]. \qquad (15)
\end{aligned}
$$

Leads to

$$
[A_\nu(x^+,x^1,x^2,x^-),A_\mu(x^+,y^1,y^2,y^-)] = -\frac{i}{4}g_{\mu\nu}\epsilon(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \qquad (16)
$$

At $x^+ = y^+$ UNEQUAL instant-time commutator is EQUAL light-front time commutator Similar result holds for non-Abelian gauge field.

UNEQUAL TIME fermion instant-time commutator

$$
\left\{\psi_\alpha(x^0,x^1,x^2,x^3),\psi^\dagger_\beta(y^0,y^1,y^2,y^3)\right\} = \left[(i\gamma^\mu\frac{\partial}{\partial x^\mu}+m)\gamma^0\right]_{\alpha\beta} i\Delta(x-y). \qquad (17)
$$

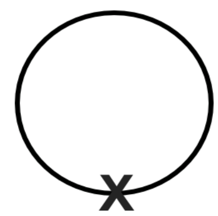
Apply projector

$$
\begin{aligned}
&\Lambda^+_{\alpha\gamma}\left\{\psi_\gamma(x^+,x^1,x^2,x^-),\psi_\delta(x^+,y^1,y^2,y^-)\right\}\Lambda^+_{\delta\beta}\\
&= \left\{[\psi_{(+)}(x^+,x^1,x^2,x^-)]_\alpha,[\psi^\dagger_{(+)}]_\beta(x^+,y^1,y^2,y^-)\right\} = \Lambda^+_{\alpha\beta}\delta(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \qquad (18)
\end{aligned}
$$

At $x^+ = y^+$ UNEQUAL instant-time anticommutator is EQUAL light-front time anticommutator.

Light-front quantization is instant-time quantization, and does not need to be independently postulated.
The seemingly different structure between equal instant-time commutators and anticommutators and equal light-front time commutators and anticommutators is actually due to the structure of unequal instant-time time commutators and anticommutators.

5 EQUIVALENCE OF INSTANT-TIME AND LIGHT-FRONT PROPAGATORS AND TADPOLES



Construct tadpole as $x^\mu \to 0$ limit of propagator, i.e., use x^μ as a regulator.

$$D(x^\mu) = -i\langle\Omega|[\theta(\sigma)\phi(x)\phi(0) + \theta(-\sigma)\phi(0)\phi(x)]|\Omega\rangle = \frac{1}{(2\pi)^4}\int d^4p \frac{e^{-ip\cdot x}}{p^2 - m^2 + i\epsilon}, \quad \sigma = \mathrm{x}^0 \text{ or } \sigma = \mathrm{x}^+. \tag{19}$$

$$D(x^\mu = 0) = -i\langle\Omega|\phi(0)\phi(0)|\Omega\rangle = \frac{1}{(2\pi)^4}\int d^4p \frac{1}{p^2 - m^2 + i\epsilon}. \tag{20}$$

$$\begin{aligned}
D(x^\mu, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{e^{-i(p_0x^0 + p_1x^1 + p_2x^2 + p_3x^3)}}{(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 - m^2 + i\epsilon}, \\
D(x^\mu, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{e^{-i(p_+x^+ + p_1x^1 + p_2x^2 + p_-x^-)}}{4p_+p_- - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon}, \\
D(x^\mu = 0, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{1}{(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 - m^2 + i\epsilon}, \\
D(x^\mu = 0, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{1}{4p_+p_- - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon}.
\end{aligned} \tag{21}$$

For all of these Feynman contours there are only poles, except $D(x^\mu = 0, \text{front})$, for which the circle at infinity in the complex p_+ plane is not suppressed.

6 THE NON-VACUUM INSTANT-TIME CASE

In the instant-time case the Feynman integral is readily performed since it is just pole terms and for the **forward** $D(x^0>0,\mathrm{instant})=-i\langle\Omega_I|\theta(x^0)\phi(x^0,x^1,x^2,x^3)\phi(0)|\Omega_I\rangle$ we obtain

$$\begin{aligned} D(x^0>0,\mathrm{instant}) &= D(x^0>0,\mathrm{instant},\mathrm{pole}) \\ &= -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p}e^{-iE_px^0+i\vec{p}\cdot\vec{x}}=\frac{1}{8\pi}\left(\frac{m^2}{x^2}\right)^{1/2}H_1^{(2)}(m(x^2)^{1/2}). \end{aligned} \tag{22}$$

Insertion of the Fock space expansion for $\phi(x^0,x^1,x^2,x^3)$ yields

$$D(x^0>0,\mathrm{instant},\ \mathrm{Fock})=-\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p}e^{-iE_px^0+i\vec{p}\cdot\vec{x}}. \tag{23}$$

We recognize (23) as (22), to thus establish the equivalence of the instant-time Feynman and Fock space prescriptions.

7 THE NON-VACUUM LIGHT-FRONT CASE

In the light-front case poles in the complex p_+ plane occur at

$$p_+=E_p'-\frac{i\epsilon}{4p_-},\quad E_p'=\frac{(p_1)^2+(p_2)^2+m^2}{4p_-}. \tag{24}$$

Poles with $p_-\geq 0^+$ thus all lie below the real p_+ axis and have positive E_p', while poles with $p_-\leq 0^-$ all lie above the real p_+ axis and have negative E_p'. For $x^+>0$, closing the p_+ contour below the real axis (which for $x^+>0$ suppresses the circle at infinity contribution) then restricts to poles with $E_p'>0$, $p_-\geq 0^+$. However, in order to evaluate the pole terms one has to deal with the fact that the pole at $p_-=0^+$ has $E_p'=\infty$.

Momentarily exclude the region around $p_- = 0$, and thus only consider poles below the real p_+ axis that have $p_- \geq \delta$. Evaluating the contour integral in the lower half of the complex p_+ plane thus gives

$$D(x^+ > 0, \text{front}, \text{pole}) = -\frac{2i}{(2\pi)^3}\int_\delta^\infty \frac{dp_-}{4p_-}\int_{-\infty}^\infty dp_1 \int_{-\infty}^\infty dp_2 e^{-i(E_p' x^+ + p_- x^- + p_1 x^1 + p_2 x^2) - \epsilon x^+/4p_-}$$
$$= -\frac{1}{4\pi^2 x^+}\int_\delta^\infty dp_- e^{-ip_- x^- + i[(x^1)^2 + (x^2)^2]p_-/x^+ - im^2 x^+/4p_- - \epsilon x^+/4p_-}$$
$$= -\frac{1}{4\pi^2 x^+}\int_\delta^\infty dp_- e^{-ip_- x^2/x^+ - im^2 x^+/4p_- - \epsilon x^+/4p_-}. \tag{25}$$

If we now set $\alpha = x^+/4p_-$, we obtain

$$D(x^+ > 0, \text{front}, \text{pole}) = -\frac{1}{16\pi^2}\int_0^{x^+/4\delta} \frac{d\alpha}{\alpha^2} e^{-ix^2/4\alpha - i\alpha m^2 - \alpha\epsilon}. \tag{26}$$

In (26) we can now take the limit $\delta \to 0$, $x^+/4\delta \to \infty$ without encountering any ambiguity **AS LONG AS x^+ IS NONZERO**, and with $x^+ > 0$ thus obtain

$$D(x^+ > 0, \text{front}, \text{pole}) = -\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2} e^{-ix^2/4\alpha - i\alpha m^2 - \alpha\epsilon} = \frac{1}{8\pi}\left(\frac{m^2}{x^2}\right)^{1/2} H_1^{(2)}(m(x^2)^{1/2}). \tag{27}$$

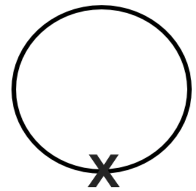
Comparing with (22) we see that $D(x^+ > 0, \text{instant})$ and $D(x^+ > 0, \text{front})$ are equal.

Inserting the Fock space expansion for $\phi(x^+, x^1, x^2, x^-)$ gives precisely the same result, and thus we obtain

$$D(x^+ > 0, \text{instant}) = D(x^+ > 0, \text{instant}, \text{pole}) = D(x^+ > 0, \text{instant}, \text{Fock}) =$$
$$D(x^+ > 0, \text{front}) = D(x^+ > 0, \text{front}, \text{pole}) = D(x^+ > 0, \text{front}, \text{Fock}). \tag{28}$$

General rule: the Feynman and Fock space prescriptions will coincide whenever the only contribution to Feynman contours is poles. Thus for $x^+ > 0$ the Feynman and Light-Front Hamiltonian approaches coincide. But what about $x^+ = 0$?

8 THE INSTANT-TIME VACUUM CASE



In the instant-time case one can readily set x^μ to zero, and obtain

$$\begin{aligned}D(x^\mu=0,\text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{1}{(p_0)^2-(p_1)^2-(p_2)^2-(p_3)^2-m^2+i\epsilon} \\ &= D(x^\mu=0,\text{instant},\text{pole}) = D(x^\mu=0,\text{instant},\text{Fock}) \\ &= -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p} = -\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2}e^{-i\alpha m^2-\alpha\epsilon}.\end{aligned} \tag{29}$$

9 THE LIGHT-FRONT VACUUM CASE - POLE AND FOCK SPACE CONTRIBUTIONS

In the light-front case we set x^μ to zero and evaluate

$$D(x^\mu=0,\text{front}) = \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{1}{4p_+p_- -(p_1)^2-(p_2)^2-m^2+i\epsilon}. \tag{30}$$

Again we need to take care of the $p_-=0$ region, so we again introduce the δ cutoff at small p_-. On closing below the real p_+ axis the only poles are those with $p_->0$, and for them we obtain a pole contribution of the form

$$D(x^\mu=0,\text{front},\text{pole}) = -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_\delta^\infty\frac{dp_-}{4p_-}. \tag{31}$$

Then on setting $p_-=1/\alpha$, we are able to let p_- go to zero, to obtain

$$D(x^\mu=0,\text{front},\text{pole}) = -\frac{i}{16\pi^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{1/\delta}\frac{d\alpha}{\alpha} = -\frac{i}{16\pi^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{d\alpha}{\alpha}. \tag{32}$$

For the Fock space prescription we set $x^\mu = 0$ in (3), viz.

$$\phi(0) = \frac{2}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{(4p_-)^{1/2}} [a_p + a_p^\dagger], \tag{33}$$

and on inserting $\phi(0)$ into $-i\langle\Omega|\phi(0)\phi(0)]|\Omega\rangle$ obtain

$$D(x^\mu = 0, \text{front}, \text{Fock}) = -\frac{2i}{(2\pi)^3} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{4p_-} = D(x^\mu = 0, \text{front}, \text{pole}). \tag{34}$$

Comparing with (31) we again see the equivalence of the pole and Fock space prescriptions.

However, something is wrong. We are evaluating the m-dependent $D(x^\mu = 0, \text{front})$ as given in (21), and yet we obtain an answer that does not depend on m at all. **What went wrong is that we left out the circle at infinity.**

10 THE LIGHT-FRONT VACUUM CASE - CIRCLE AT INFINITY CONTRIBUTION

To evaluate the circle at infinity contribution we introduce the regulator

$$\frac{1}{(A + i\epsilon)} = -i \int_0^{\infty} d\alpha e^{i\alpha(A+i\epsilon)}. \tag{35}$$

For $p_- > 0$ the regulator converges on the **UPPER** half circle, and there are no poles at all. We obtain

$$\begin{aligned}
&D(x^\mu = 0, p_- > 0, \text{front}, \text{upper circle}) \\
&= \frac{2i}{(2\pi)^4} \int_0^{\infty} dp_- \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\pi} iRe^{i\theta} d\theta \int_0^{\infty} d\alpha e^{i\alpha(4p_- Re^{i\theta} - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon)} \\
&= \frac{1}{8\pi^3} \int_0^{\infty} dp_- \int_0^{\infty} \frac{d\alpha}{\alpha} e^{-i\alpha m^2 - \alpha\epsilon} \int_0^{\pi} iRe^{i\theta} d\theta e^{4i\alpha p_- Re^{i\theta}} \\
&= \frac{1}{8\pi^3} \int_0^{\infty} dp_- \int_0^{\infty} \frac{d\alpha}{\alpha} e^{-i\alpha m^2 - \alpha\epsilon} \frac{(e^{4i\alpha p_- Re^{i\theta}} - e^{-4i\alpha p_- Re^{i\theta}})}{4i\alpha p_-} \Bigg|_0^{\pi} \\
&= \frac{1}{8\pi^3} \int_0^{\infty} dp_- \int_0^{\infty} \frac{d\alpha}{\alpha} e^{-i\alpha m^2 - \alpha\epsilon} \frac{(e^{-4i\alpha p_- R} - e^{4i\alpha p_- R})}{4i\alpha p_-} \\
&= -\frac{1}{4\pi^3} \int_0^{\infty} dp_- \int_0^{\infty} \frac{d\alpha}{\alpha} e^{-i\alpha m^2 - \alpha\epsilon} \frac{\sin(4\alpha p_- R)}{4\alpha p_-}.
\end{aligned} \tag{36}$$

Then, on letting R go to infinity we obtain

$$D(x^\mu=0,p_->0,\text{front},\text{upper circle})=-\frac{1}{4\pi^2}\int_0^\infty dp_-\int_0^\infty\frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\delta(4\alpha p_-)$$
$$=-\frac{1}{8\pi^2}\int_{-\infty}^\infty dp_-\int_0^\infty\frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\delta(4\alpha p_-)=-\frac{1}{32\pi^2}\int_0^\infty\frac{d\alpha}{\alpha^2}e^{-i\alpha m^2-\alpha\epsilon}. \quad (37)$$

We thus establish the centrality of $p_-=0$ modes.

Similarly, for $p_-<0$ close on the **LOWER** half circle, and again there are no poles. We obtain

$$D(x^\mu=0,p_->0,\text{front},\text{upper circle})=D(x^\mu=0,p_-<0,\text{front},\text{lower circle}), \quad (38)$$

and thus

$$D(x^\mu=0,\text{front})=D(x^\mu=0,p_->0,\text{front},\text{upper circle})+D(x^\mu=0,p_-<0,\text{front},\text{lower circle})$$
$$=-\frac{1}{16\pi^2}\int_0^\infty\frac{d\alpha}{\alpha^2}e^{-i\alpha m^2-\alpha\epsilon}. \quad (39)$$

Now not only is there now an m dependence, we obtain

$$D(x^\mu=0,\text{front})=D(x^\mu=0,\text{instant}). \quad (40)$$

So again, light-front quantization is instant-time quantization. And even though there is only a circle at infinity contribution in the light front case, it is this circle at infinity that enables the light-front and instant-time vacuum graphs to be the same.

11 RECONCILING THE FOCK SPACE AND FEYNMAN CALCULATIONS

To avoid $p_-=0$ difficulties we use the regulator on the real p_+ axis, and set

$$\begin{aligned}
&D(x^\mu,\text{front},\text{regulator})\\
&=-\frac{2i}{(2\pi)^4}\int_{-\infty}^{\infty}dp_+\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{-\infty}^{\infty}dp_-e^{-i(p_+x^++p_-x^-+p_1x^1+p_2x^2)}\int_0^\infty d\alpha e^{i\alpha(4p_+p_--(p_1)^2-(p_2)^2-m^2+i\epsilon)}\\
&=-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}dp_-e^{-i(p_-x^-+p_1x^1+p_2x^2)}\int_0^\infty d\alpha e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(4\alpha p_--x^+)\\
&-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{-\infty}^{0}dp_-e^{-i(p_-x^-+p_1x^1+p_2x^2)}\int_0^\infty d\alpha e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(4\alpha p_--x^+). \qquad (41)
\end{aligned}$$

On changing the signs of p_-, p_1 and p_2 in the last integral and setting F_p^2 equal to the positive $(p_1)^2+(p_2)^2+m^2$ we obtain

$$\begin{aligned}
&D(x^\mu,\text{front},\text{regulator})\\
&=-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{4p_-}e^{-i(p_-x^-+p_1x^1+p_2x^2)}\int_0^\infty d\alpha e^{ix^+(-F_p^2+i\epsilon)/4p_-}\delta(\alpha-x^+/4p_-)\\
&-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{4p_-}e^{i(p_-x^-+p_1x^1+p_2x^2)}\int_0^\infty d\alpha e^{ix^+(F_p^2-i\epsilon)/4p_-}\delta(\alpha+x^+/4p_-)\\
&=-\frac{2i\theta(x^+)}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{4p_-}e^{-i(F_p^2x^+/4p_-+p_-x^-+p_1x^1+p_2x^2+ix^+\epsilon/4p_-)}\\
&-\frac{2i\theta(-x^+)}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{4p_-}e^{i(F_p^2x^+/4p_-+p_-x^-+p_1x^1+p_2x^2-ix^+\epsilon/4p_-)}, \qquad (42)
\end{aligned}$$

and note that the structure of (42) is such that for $x^+>0$ (forward in time) one only has positive energy propagation, while for $x^+<0$ (backward in time) one only has negative energy propagation. With the insertion into $D(x^\mu)=-i\langle\Omega|[\theta(x^+)\phi(x)\phi(0)+\theta(-x^+)\phi(0)\phi(x)]|\Omega\rangle$ of the Fock space expansion for $\phi(x^\mu)$ given in (3) precisely leading to (42), we recognize (42) as the $x^\mu\neq 0$ $D(x^\mu,\text{front},\text{Fock})$.

Now if we set $x^\mu = 0$ in (42) we would appear to obtain the m-independent $D(x^\mu = 0, \text{front}, \text{Fock})$ given in (34). However, we cannot take the $x^+ \to 0$ limit since the quantity $x^+/4p_-$ is undefined if p_- is zero, and $p_- = 0$ is included in the integration range. Hence, just as discussed in regard to (26), the limit is singular.

To obtain a limit that is not singular we note that we can set x^μ to zero in (41) as there the limit is well-defined, and this leads to

$$
\begin{aligned}
&D(x^\mu = 0, \text{front}, \text{regulator}) \\
&= -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} dp_- \int_0^{\infty} d\alpha e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(4\alpha p_-) \\
&-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{0} dp_- \int_0^{\infty} d\alpha e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(4\alpha p_-) \\
&= -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_- \int_0^{\infty} \frac{d\alpha}{4\alpha} e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(p_-),
\end{aligned} \tag{43}
$$

and again see the **centrality of $p_- = 0$ modes**. If we do the momentum integrations we obtain the m-dependent

$$
D(x^\mu = 0, \text{front}, \text{regulator}) = -\frac{1}{16\pi^2}\int_0^{\infty} \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2-\alpha\epsilon}. \tag{44}
$$

We recognize (44) as being of the same form as the m-dependent $D(x^\mu = 0, \text{front})$ given in (39). We thus have to conclude that the limit $x^\mu \to 0$ of (42) is not (34) but is (44) instead, and that

$$
D(x^\mu = 0, \text{front}) = D(x^\mu = 0, \text{instant}) = -\frac{1}{16\pi^2}\int_0^{\infty} \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2-\alpha\epsilon}. \tag{45}
$$

Setting $p_- = 0$ and then $x^+ = 0$ is not the same as setting $x^+ = 0$ and then $p_- = 0$.

Thus because of singularities we first have to point split, and when we do so we find that it is the m-dependent (44) that is the correct value for the light-front vacuum graph. And it is equal to the instant-time vacuum graph.

12 THE MORAL OF THE STORY

When we let $p_- \to 0$ we are letting $p_+ = [(p_1)^1 + (p_2)^2 + m^2]/4p_- \to \infty$.
However x^+ is the conjugate of p_+, and thus as $p_+ \to \infty$, $x^+ \to 0$.
The $p_- \to 0$ and the $x^+ \to 0$ limits are thus intertwined.
If we stay away from $x^+ = 0$ and restrict to $x^+ > 0$ and thus $p_- > 0$ as in the Light-Front Hamiltonian approach, there is no difficulty as there are only poles and nothing is singular, with the forward scattering on-shell Light-Front Hamiltonian approach thus being validated.
However this does become a concern for tadpole graphs as they have $x^+ = 0$, since we need both $\theta(x^+)$ and $\theta(-x^+)$ time orderings in the limit, with $\langle\Omega|[\theta(x^+)\phi(x)\phi(0) + \theta(-x^+)\phi(0)\phi(x)]|\Omega\rangle \to \langle\Omega|[\theta(0^+)\phi(0)\phi(0) + \theta(0^-)\phi(0)\phi(0)]|\Omega\rangle = \langle\Omega|\phi(0)\phi(0)|\Omega\rangle$.
If we compare

$$\begin{aligned} D(x^\mu, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{e^{-i(p_0x^0+p_1x^1+p_2x^2+p_3x^3)}}{(p_0)^2-(p_1)^2-(p_2)^2-(p_3)^2-m^2+i\epsilon}, \\ D(x^\mu, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{e^{-i(p_+x^++p_1x^1+p_2x^2+p_-x^-)}}{4p_+p_--(p_1)^2-(p_2)^2-m^2+i\epsilon}, \end{aligned} \tag{46}$$

$$\begin{aligned} D(x^\mu=0, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{1}{(p_0)^2-(p_1)^2-(p_2)^2-(p_3)^2-m^2+i\epsilon}, \\ D(x^\mu=0, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{1}{4p_+p_--(p_1)^2-(p_2)^2-m^2+i\epsilon}, \end{aligned} \tag{47}$$

we can transform each instant-time graph into each corresponding light-front graph by a change of variable. Thus they must be equal. However, that does not mean that pole equals pole or that circle equals circle, only that pole plus circle equals pole plus circle, as it is only on the full closed contour that the integrals are equal.

The transformation $x^0 \to x^0 + x^3$, $x^3 \to x^0 - x^3$ is a spacetime-dependent general coordinate transformation (not a Lorentz transformation), and thus by the general coordinate invariance of the fundamental interactions it must be the case that

LIGHT-FRONT QUANTIZATION IS INSTANT-TIME QUANTIZATION, JUST ONE THEORY.

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Structure of light-front vacuum sector diagrams

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We study the structure of scalar field light-front quantization vacuum graphs. In instant-time quantization both non-vacuum and vacuum graphs can equivalently be described by either the off-shell four-dimensional Feynman diagram approach or the on-shell three-dimensional Fock space approach, with this being the case since the relevant Feynman diagrams are given entirely by pole terms. This is also the case for light-front quantization non-vacuum graphs. However this is not the case for light-front vacuum sector diagrams, since then there are also circle at infinity contributions to Feynman diagrams. These non-pole contributions cause light-front vacuum diagrams to be nonzero and to not be given by a Light-Front Hamiltonian Fock space analysis. The three-dimensional approach thus fails in the light-front vacuum sector. In consequence, the closely related infinite momentum frame approach also fails in the light-front vacuum sector.

I. INTRODUCTION

Since the original work of Dirac [1], there has been continuing interest in light-front (also known as “light-cone” or “front-form”) quantization of quantum field theories. Comprehensive reviews can be found in [2–5]. The light-front approach is based on 3-dimensional Hamiltonian field theory quantized at fixed light-front time $x^+ = x^0 + x^3$. The rules for calculations for Light-Front Hamiltonian QCD for both perturbative and nonperturbative applications are summarized in [6]. As is the case with the standard four-dimensional covariant Feynman Lagrangian theory, the light-front formalism is Poincaré invariant and causal. Observables in hadron physics such as form factors, structure functions, and distribution amplitudes are based on the nonperturbative light-front hadronic wave functions, the eigenfunctions of the QCD Light-Front Hamiltonian [7, 8]. In the case of scattering amplitudes, the covariant Feynman and the Light-Front Hamiltonian approaches give identical results. One can also replicate the calculation rules for light-front x^+-ordered perturbation theory using standard time-ordered perturbation theory based on quantization at fixed time (also known as instant-time or “instant-form” quantization) by choosing a Lorentz frame where the observer moves at infinite momentum [9–12].

While the light-front non-vacuum (i.e., scattering) sector is well understood, in the light-front literature there has been a spirited discussion as to the status of perturbative light-front vacuum graphs (see e.g. [2, 11–15]). In the light-front vacuum sector differing results have been obtained for the off-shell four-dimensional Feynman diagram approach and the on-shell three-dimensional Fock space approach, and the literature has not yet settled on which particular one might have fundamental validity, or identified what it is that causes differences between the various approaches. It is the purpose of this paper to address this issue in the scalar field theory case, and to show that because of circle at infinity contributions in four-dimensional light-front vacuum Feynman diagrams it is the Feynman approach that one must use as the light-front Fock space approach is equivalent to the pole term contribution to Feynman diagrams alone. Because of these non-pole circle at infinity contributions, light-front vacuum diagrams are not only nonzero, they are equal to instant-time vacuum diagrams, even though instant-time vacuum Feynman diagrams receive no circle at infinity contributions. Our result is initially surprising since the instant-time Fock space analysis correctly describes the instant-time vacuum sector, and in the infinite momentum frame the instant-time Fock space procedure transforms into the light-front Fock space description. However, even though circle at infinity contributions are suppressed in the instant-time case, when instant-time vacuum graphs are evaluated in the infinite momentum frame we find that instant-time circle contributions are no longer suppressed, to thus cause the light-front Fock space procedure to fail in the light-front vacuum sector. Thus one must use the off-shell four-dimensional Feynman diagram approach in order to correctly describe the light-front vacuum sector. Since circle at infinity issues are not of relevance in either instant-time non-vacuum or instant-time vacuum graphs, and since circle at infinity issues are not even of relevance in the light-front non-vacuum sector, the light-front vacuum sector has an intrinsic structure that is all its own, a structure that cannot be inferred from experience with the non-vacuum sector, and has to be treated independently.

To address these issues we have found it instructive to study Feynman diagrams in coordinate space rather than in momentum space as this allows us to monitor the behavior of the theory when we bring spacetime points together, a procedure that we show to be singular in the light-front case. And in order to establish our results we only need to study the free propagators that contribute in a perturbative expansion, as they will prove rich enough for our purposes here. We shall thus study the structure of the scalar field $D(x^\mu) = -i\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle$ and its $x^\mu \to 0$ limit $D(x^\mu = 0) = -i\langle\Omega|\phi(0)\phi(0)|\Omega\rangle$, and this will enable us to use the spacetime coordinates as regulators when we take the limit. For the action $I_S = \int d^4x(-g)^{1/2}[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2]$, the free $D(x^\mu)$ propagator is given as a Feynman diagram as:

$$D(x^\mu) = -i\langle\Omega|[\theta(\sigma)\phi(x)\phi(0) + \theta(-\sigma)\phi(0)\phi(x)]|\Omega\rangle = \frac{1}{(2\pi)^4}\int d^4p\frac{e^{-ip\cdot x}}{p^2 - m^2 + i\epsilon}, \tag{1}$$

where σ is x^0 in the instant-time case and is $x^+ = x^0 + x^3$ in the light-front case. With $\theta(0) = 1/2$ the associated $x^\mu = 0$ vacuum bubble graph is given by

$$D(x^\mu = 0) = -i\langle\Omega|\phi(0)\phi(0)|\Omega\rangle = \frac{1}{(2\pi)^4}\int d^4p\frac{1}{p^2 - m^2 + i\epsilon}. \tag{2}$$

Generically, the vacuum bubble can be associated with either of the two graphs shown in Fig. 1, the pure circle graph and the graph with a cross. The pure circle graph is a disconnected vacuum bubble in which a propagating scalar field closes back on itself. The graph with the cross is associated with a propagating scalar field that closes back on itself and absorbs or emits a soft particle at the cross as it does so. It is a connected one loop tadpole graph of a $\lambda\phi^3$ theory, with an amputated external ϕ field bringing zero four-momentum into or taking zero four-momentum out of the cross with strength λ. In value the graph with the cross is λ times the graph without the cross, and thus a study of one is a study of both. The tadpole graph would also appear in a $g\phi\bar{\psi}\psi$ theory with the cross representing a fermion-antifermion insertion at the point where the scalar ϕ brings zero momentum into the loop with strength g. The tadpole graph appears in mass renormalization in theories such as $\lambda\phi^4$ or $g(\bar{\psi}\psi)^2$, in theories of dynamical symmetry breaking by fermion or scalar field bilinear composites, and in gravity theories where it can couple to the matter energy-momentum tensor, and thus be of relevance to cosmology and the cosmological constant problem. The computation of the pure circle diagram in Fig. 1 is ambiguous in Light-Front Hamiltonian theory since there is no starting nor ending point for the particles propagating in the loop as they can go round and round indefinitely. The same ambiguity appears for the tadpole diagram, since while the propagating particle could absorb or emit a soft particle, it could do so at any point in the loop. To resolve this ambiguity our paper uses the covariant four-dimensional Feynman definition of these diagrams.

For the purposes of monitoring the $x^\mu \to 0$ limit of interest to us here it will prove to be more instructive to treat both of the graphs as the $x^\mu \to 0$ limit of a Feynman time-ordered propagator rather than the limit $x^\mu \to 0$ of a two-point function such as $-i\langle\Omega|\phi(x)\phi(0)|\Omega\rangle$.

FIG. 1: Disconnected and connected $\langle\Omega|\phi(0)\phi(0)|\Omega\rangle$

When written out in detail the time-ordered Feynman propagators of interest to us are of the form

$$\begin{aligned}
D(x^\mu, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0dp_1dp_2dp_3\frac{e^{-i(p_0x^0+p_1x^1+p_2x^2+p_3x^3)}}{(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 - m^2 + i\epsilon},\\
D(x^\mu, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+dp_1dp_2dp_-\frac{e^{-i(p_+x^++p_1x^1+p_2x^2+p_-x^-)}}{4p_+p_- - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon},\\
D(x^\mu = 0, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0dp_1dp_2dp_3\frac{1}{(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 - m^2 + i\epsilon},\\
D(x^\mu = 0, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+dp_1dp_2dp_-\frac{1}{4p_+p_- - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon},
\end{aligned} \tag{3}$$

where $x^- = x^0 - x^3$ [16]. As long as either x^0 or x^+ is nonzero and positive, the circle at infinity contributions in the lower half of the complex p_0 or p_+ planes are suppressed by the $e^{-ip\cdot x}$ term, and the only contributions to the Feynman contours are the pole terms. Similarly, even if x^0 is zero, the circle at infinity contribution to the instant $D(x^\mu = 0, \text{instant})$ is still suppressed because there are two powers of p_0 in the denominator. However, the circle at infinity contribution to the front $D(x^\mu = 0, \text{front})$ is not suppressed because in that case there is only one power of p_- in the denominator. It is in this way then that $D(x^\mu = 0, \text{front})$ is conceptually different, and the purpose of this paper is to monitor the singular nature of the $x^\mu \to 0$ limit in this case.

That the limit is singular is due to the fact that θ functions (technically distributions) in the definition of time-ordered products can be written as contour integrals of the form

$$\theta(\sigma) = -\frac{1}{2\pi i}\oint_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\sigma}}{\omega + i\epsilon}, \tag{4}$$

(to thus cause Feynman diagrams to be off shell). And our ability to show that the complex ω plane contour integral is in fact a θ function resides in the fact that because of the $e^{-i\omega\sigma}$ term the circle at infinity contribution along a contour in the lower half complex ω plane is suppressed if $\sigma > 0$, with the pole term giving $-(1/2\pi i) \times (-2\pi i) = 1$ if $\sigma > 0$. However, suppose we drop the suppression factor by setting $\sigma = 0$. We now have a circle at infinity contribution and obtain

$$\theta(0) = -\frac{1}{2\pi i}\oint_{-\infty}^{\infty} d\omega \frac{1}{\omega + i\epsilon} = -\frac{1}{2\pi i}[-2\pi i + \pi i] = \frac{1}{2} \tag{5}$$

just as we should (i.e. our particular representation of $\theta(\sigma)$ as the contour integral given in (3) entails that $\theta(0) = 1/2$), and just as required in going from (1) to (2). Thus if we construct $D(x^\mu = 0, \text{front})$ as the $x^\mu = 0$ limit of $D(x^\mu, \text{front})$, then the circle at infinity term that had been suppressed when $x^\mu \neq 0$ is no longer suppressed, and thus needs to be taken into consideration, with the light-front $x^\mu \to 0$ limit thus being singular [17].

As can be seen from (3), poles in light-front Feynman diagrams are located at $p_+ = [(p_1)^2 + (p_2)^2 + m^2]/4p_-$, and thus become undefined at $p_- = 0$. As stressed in [11, 12] and [18], handling $p_- = 0$ singularities is one of the main challenges for light-front studies. The novelty of our current study is in how we handle the $p_- = 0$ region in a way that does not lead to singularities in the $p_- \to 0$ limit. Specifically, we develop the δ cutoff procedure for small p_- in (10) below, and then find a way to be able to take the $\delta \to 0$ limit in (11) without encountering any singularity. We use the $p_- = 1/\alpha$ substitution to handle the $p_- = 0$ region in (17). We use an exponential regulator technique on the p_+ circle at infinity to handle the $p_- = 0$ region in (20). Similarly, we use an exponential regulator technique on the real p_+ axis to handle the $p_- = 0$ region in (26) and (29). On thus being able to control the $p_- \to 0$ limit, we find that it is the contribution of the $p_- = 0$ region that enables light-front vacuum Feynman graphs to be both non-vanishing and equal to their instant-time counterparts. Moreover, since $p_- = 0$ entails that the energy p_+ is given by $p_+ = \infty$, in coordinate space this corresponds to the light-front time being given by $x^+ = 0$. In consequence, the limit $x^+ \to 0$ is singular, just as we had noted above. Singularities at $p_- = 0$ and at $x^+ = 0$ are thus correlated, and in this paper we study their interplay.

The light-front vacuum sector is of interest for another reason. As long as we stay away from $x^+ = 0$ we only need to deal with poles in light-front Feynman diagrams and everything can be described by on-shell physics, just as is done in the Light-Front Hamiltonian approach. However at $x^+ = 0$ circle at infinity contributions are present and an on-shell description (just poles) is inadequate. Thus for the light-front vacuum sector there is information in off-shell Feynman diagrams that is not accessible to, and thus goes beyond, the on-shell Hamiltonian approach.

II. THE NON-VACUUM INSTANT-TIME CASE

In the instant-time case the Feynman integral is readily performed since it is just pole terms and yields

$$\begin{aligned} D(x^0 > 0, \text{instant}) &= D(x^0 > 0, \text{instant}, \text{pole}) \\ &= -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty} \frac{d^3p}{2E_p} \exp(-iE_p x^0 + i\vec{p}\cdot\vec{x}) = \frac{1}{8\pi}\left(\frac{m^2}{x^2}\right)^{1/2} H_1^{(2)}(m(x^2)^{1/2}), \end{aligned} \tag{6}$$

where $E_p = +(\vec{p}^2 + m^2)^{1/2}$. In the instant-time case one can take an instant-time forward Green's function such as $D(x^0 > 0, \text{instant}) = -i\langle\Omega_I|\theta(x^0)\phi(x^0, x^1, x^2, x^3)\phi(0)|\Omega_I\rangle$ as evaluated in the instant-time no-particle (viz. vacuum) state $|\Omega_I\rangle$, and expand the field in terms of instant-time creation and annihilation operators that create and annihilate particles out of that vacuum state as

$$\phi(\vec{x}, x^0) = \int \frac{d^3p}{(2\pi)^{3/2}(2E_p)^{1/2}}[a(\vec{p})\exp(-iE_p t + i\vec{p}\cdot\vec{x}) + a^\dagger(\vec{p})\exp(+iE_p t - i\vec{p}\cdot\vec{x})], \tag{7}$$

where $[a(\vec{p}), a^\dagger(\vec{p}')] = \delta^3(\vec{p}-\vec{p}')$. The insertion of $\phi(\vec{x}, x^0)$ into $D(x^0 > 0, \mathrm{instant})$ immediately leads to the on-shell three-dimensional integral

$$D(x^0 > 0, \mathrm{instant},\ \mathrm{Fock}) = -\frac{i\theta(x^0)}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p}e^{-iE_px^0+i\vec{p}\cdot\vec{x}}. \tag{8}$$

We recognize (8) as (6), to thus establish the equivalence of the instant-time Feynman and Fock space prescriptions.

III. THE NON-VACUUM LIGHT-FRONT CASE

In the light-front case poles in the complex p_+ plane occur at

$$p_+ = E'_p - \frac{i\epsilon}{4p_-}, \tag{9}$$

where $E'_p = ((p_1)^2 + (p_2)^2 + m^2)/4p_-$. Poles with $p_- \geq 0^+$ thus all lie below the real p_+ axis and have positive E'_p, while poles with $p_- \leq 0^-$ all lie above the real p_+ axis and have negative E'_p. For $x^+ > 0$, closing the p_+ contour below the real axis (which for $x^+ > 0$ suppresses the circle at infinity contribution) then restricts to poles with $E'_p > 0$, $p_- \geq 0^+$. However, in order to evaluate the pole terms one has to deal with the fact that the pole at $p_- = 0^+$ has $E'_p = \infty$. We shall thus momentarily exclude the region around $p_- = 0$, and thus only consider poles below the real p_+ axis that have $p_- \geq \delta$ where δ is a small positive number. Evaluating the contour integral in the lower half of the complex p_+ plane thus gives

$$\begin{aligned} D(x^+ > 0, \mathrm{front}, \mathrm{pole}) &= -\frac{2i}{(2\pi)^3}\int_\delta^\infty \frac{dp_-}{4p_-}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2 e^{-i(E'_px^+ + p_-x^- + p_1x^1 + p_2x^2) - \epsilon x^+/4p_-} \\ &= -\frac{1}{4\pi^2 x^+}\int_\delta^\infty dp_- e^{-ip_-x^- + i[(x^1)^2 + (x^2)^2]p_-/x^+ - im^2x^+/4p_- - \epsilon x^+/4p_-} \\ &= -\frac{1}{4\pi^2 x^+}\int_\delta^\infty dp_- e^{-ip_-x^2/x^+ - im^2x^+/4p_- - \epsilon x^+/4p_-}. \end{aligned} \tag{10}$$

If we now set $\alpha = x^+/4p_-$, we obtain

$$D(x^+ > 0, \mathrm{front}, \mathrm{pole}) = -\frac{1}{16\pi^2}\int_0^{x^+/4\delta}\frac{d\alpha}{\alpha^2}e^{-ix^2/4\alpha - i\alpha m^2 - \alpha\epsilon}. \tag{11}$$

In (11) we can now take the limit $\delta \to 0$, $x^+/4\delta \to \infty$ without encountering any ambiguity as long as x^+ is nonzero, and with $x^+ > 0$ thus obtain

$$D(x^+ > 0, \mathrm{front}, \mathrm{pole}) = -\frac{1}{16\pi^2}\int_0^{\infty}\frac{d\alpha}{\alpha^2}e^{-ix^2/4\alpha - i\alpha m^2 - \alpha\epsilon}. \tag{12}$$

This integral is readily done and yields

$$D(x^+ > 0, \mathrm{front}) = D(x^+ > 0, \mathrm{front}, \mathrm{pole}) = \frac{1}{8\pi}\left(\frac{m^2}{x^2}\right)^{1/2} H_1^{(2)}(m(x^2)^{1/2}). \tag{13}$$

Comparing with (6) we see that $D(x^+ > 0, \mathrm{instant})$ and $D(x^+ > 0, \mathrm{front})$ are equal. As discussed in [19], where details of our work may be found, this is not the case just for this particular Green's function, as it actually holds for the instant-time and light-front evaluations of any scalar field Green's function. Specifically, while one ordinarily tries to relate instant-time and light-front graphs by a Lorentz boost to the infinite momentum frame (something we discuss below), the transformation $x^0 \to x^+ = x^0 + x^3$, $x^3 \to x^- = x^0 - x^3$ is actually a spacetime-dependent translation, i.e. a general coordinate transformation. Since Feynman diagrams are just integrals over c-number momentum variables, and when written in coordinate space are just functions of c-number coordinates, Feynman diagrams are general coordinate invariant, and thus instant-time and light-front evaluations of any scalar field Feynman diagram must be equal [20]. However, that does not mean that one can transform equal instant-time canonical commutators into equal front time ones as well. And in fact one cannot [19], and as noted in [19] it is this that enables the instant-time and light-front scalar field Green's functions to be equal.

IV. NON-VACUUM LIGHT-FRONT FOCK SPACE TREATMENT

In the light-front case the Fock space expansion of modes that obey $[4\partial_-\partial_+ - (\partial_1)^2 - (\partial_2)^2 + m^2]\phi(x^+, x^1, x^2, x^-) = 0$ is of the form

$$\phi(x^+, x^-, x^1, x^2) = \frac{2}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{(4p_-)^{1/2}} \bigg[e^{-i(F_p^2 x^+/4p_- + p_- x^- + p_1 x^1 + p_2 x^2)} a_p + e^{i(F_p^2 x^+/4p_- + p_- x^- + p_1 x^1 + p_2 x^2)} a_p^\dagger \bigg], \tag{14}$$

where $F_p^2 = (p_1)^2 + (p_2)^2 + m^2$, and where the light-front $[a_p, a_{p'}^\dagger]$ commutator is normalized to $[a_p, a_{p'}^\dagger] = (1/2)\delta(p_- - p'_-)\delta(p_1 - p'_1)\delta(p_2 - p'_2)$ so as to impose the canonical commutator $[\phi(x^+, x^1, x^2, x^-), 2\partial_-\phi(x^+, y^1, y^2, y^-)] = i\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^- - y^-)$, a derivation for which may be found in [21] and more recently in [19]. In (14) we note that the p_- integration is only over nonnegative p_- [22].

With this normalization we can then insert this on-shell form for $\phi(x)$ into $D(x^\mu, \text{front}) = -i\langle\Omega_F|[\theta(x^+)\phi(x^\mu)\phi(0) + \theta(-x^+)\phi(0)\phi(x^\mu)]|\Omega_F\rangle$ as evaluated in the light-front no-particle state $|\Omega_F\rangle$ that the light-front a_p annihilate. With the insertion of this $\phi(x)$ then precisely giving the first line in (10), we establish the equivalence of the Feynman and Fock space prescriptions in the non-vacuum light-front case. As we see, just as with the instant-time case, since there are only pole terms and no circle at infinity contributions in the non-vacuum case, we are able to establish the equivalence of the light-front Fock space and Feynman diagram prescriptions in such case. In the non-vacuum (i.e. $x^+ \neq 0$) sector we are thus able to validate the standard non-vacuum light-front on-shell Fock space prescription that is widely used in light-front studies. And in addition we see a general rule emerge, namely that Feynman and Fock space prescriptions will coincide whenever the only contribution to Feynman contours is poles. However, as we shall now see, in the light-front vacuum sector there are circle at infinity contributions, to thus cause the on-shell Fock space description to become invalid.

V. THE INSTANT-TIME VACUUM CASE

In the instant-time case one can readily set x^μ to zero in (6), (7) and (8), to obtain

$$D(x^\mu = 0, \text{instant}) = D(x^\mu = 0, \text{instant}, \text{pole}) = D(x^\mu = 0, \text{instant}, \text{Fock}) = -\frac{i}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d^3p}{2E_p} = -\frac{1}{16\pi^2} \int_0^{\infty} \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2 - \alpha\epsilon}. \tag{15}$$

VI. THE LIGHT-FRONT VACUUM CASE – POLE CONTRIBUTION

In the light-front case we set x^μ to zero and evaluate $D(x^\mu = 0, \text{front})$ as given in (3). Just as above we again need to take care of the $p_- = 0$ region, so we again introduce the δ cutoff at small p_-. On closing below the real p_+ axis the only poles are those with $p_- > 0$, and for them we obtain a pole contribution of the form

$$D(x^\mu = 0, \text{front}, \text{pole}) = -\frac{2i}{(2\pi)^3} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{\delta}^{\infty} \frac{dp_-}{4p_-}. \tag{16}$$

Then on setting $p_- = 1/\alpha$, we are able to let p_- go to zero, to obtain

$$D(x^\mu = 0, \text{front}, \text{pole}) = -\frac{i}{16\pi^3} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{1/\delta} \frac{d\alpha}{\alpha} = -\frac{i}{16\pi^3} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{d\alpha}{\alpha}. \tag{17}$$

For the Fock space prescription we set $x^\mu = 0$ in (14), viz.

$$\phi(0) = \frac{2}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{(4p_-)^{1/2}} [a_p + a_p^\dagger], \tag{18}$$

and on inserting $\phi(0)$ into $-i\langle\Omega|\phi(0)\phi(0)]|\Omega\rangle$ obtain

$$D(x^\mu = 0, \text{front}, \text{Fock}) = -\frac{2i}{(2\pi)^3} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{4p_-}. \tag{19}$$

Comparing with (16) we again see the equivalence of the pole and Fock space prescriptions.

However, there is something wrong with both prescriptions. We are evaluating the m-dependent $D(x^\mu=0,\text{front})$ as given in (3), and yet we obtain an answer that does not depend on m at all. Moreover, the theorem presented in [19] and [20] would require that $D(x^\mu=0,\text{front})$ and $D(x^\mu=0,\text{instant})$ be equal, and neither $D(x^\mu=0,\text{front},\text{pole})$ nor $D(x^\mu=0,\text{front},\text{Fock})$ is equal to $D(x^\mu=0,\text{instant})$ as given in (15), and indeed they could not be since $D(x^\mu=0,\text{instant})$ is m-dependent. Thus something must have gone wrong.

VII. THE LIGHT-FRONT VACUUM CASE – CIRCLE AT INFINITY CONTRIBUTION

What went wrong is that there is a circle at infinity contribution. To evaluate the circle at infinity contribution we have found it convenient to use an exponential regulator on the circle of the type usually used on the real frequency axis as this enables us to set $1/(A+i\epsilon)=-i\int_0^\infty d\alpha\exp(i\alpha(A+i\epsilon))$ for any $A+i\epsilon$ on the circle that is such that there is convergence at $\alpha=\infty$. On setting

$$D(x^\mu=0,\text{front},\text{circle})=-\frac{2i}{(2\pi)^4}\int_{-\infty}^{\infty}dp_+\int_{-\infty}^{\infty}dp_-\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^\infty d\alpha e^{i\alpha(4p_+p_--(p_1)^2-(p_2)^2-m^2+i\epsilon)}, \tag{20}$$

we see that on setting $p_+=Re^{i\theta}$ on a circle at infinity of radius R we can get convergence at $\alpha=\infty$ if $4i\alpha p_-R(\cos\theta+i\sin\theta)=4i\alpha p_-R\cos\theta-4\alpha p_-R\sin\theta$ converges, i.e. if $p_-\sin\theta$ is positive. With positive p_- this would then require that $\sin\theta$ be positive, while negative p_- would require that $\sin\theta$ be negative. Now $\sin\theta$ is positive for $0<\theta<\pi$, and negative for $\pi<\theta<2\pi$. Thus in order to use the exponential regulator on the circle we must close above the real p_+ axis in a counter-clockwise direction for positive p_-, while we must close below the real p_+ axis in a clockwise direction for negative p_-. However, for positive p_- the poles in p_+ are below the real axis, while for negative p_- the poles in p_+ are above the real axis. Thus in applying the exponential regulator on the circle at infinity we always have to close the contour so that we do not encounter any poles at all, to thereby show that the circle contribution cannot be ignored. As we see from (20), the great utility of the use of the exponential regulator in the light-front case is that it is well-defined at $p_-=0$.

Symbolically we can set

$$\int_{-\infty}^{\infty}dp_+=\int_{-\infty}^{\infty}dp_+(p_->0)+\int_{-\infty}^{\infty}dp_+(p_-<0)=-\int_0^\pi iRe^{i\theta}d\theta(p_->0)-\int_{2\pi}^\pi iRe^{i\theta}d\theta(p_-<0). \tag{21}$$

And thus for $p_->0$ first we obtain an upper circle contribution to $D(x^\mu=0,\text{front})$ of the form

$$\begin{aligned}&D(x^\mu=0,p_->0,\text{front},\text{upper circle})\\&=\frac{2i}{(2\pi)^4}\int_0^\infty dp_-\int_{-\infty}^\infty dp_1\int_{-\infty}^\infty dp_2\int_0^\pi iRe^{i\theta}d\theta\int_0^\infty d\alpha e^{i\alpha(4p_-Re^{i\theta}-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\\&=\frac{1}{8\pi^3}\int_0^\infty dp_-\int_0^\infty\frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\int_0^\pi iRe^{i\theta}d\theta e^{4i\alpha p_-Re^{i\theta}}\\&=\frac{1}{8\pi^3}\int_0^\infty dp_-\int_0^\infty\frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\frac{(e^{4i\alpha p_-Re^{i\theta}}-e^{-4i\alpha p_-Re^{i\theta}})}{4i\alpha p_-}\Bigg|_0^\pi\\&=\frac{1}{8\pi^3}\int_0^\infty dp_-\int_0^\infty\frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\frac{(e^{-4i\alpha p_-R}-e^{4i\alpha p_-R})}{4i\alpha p_-}\\&=-\frac{1}{4\pi^3}\int_0^\infty dp_-\int_0^\infty\frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\frac{\sin(4\alpha p_-R)}{4\alpha p_-}.\end{aligned} \tag{22}$$

Then, on letting R go to infinity we obtain

$$\begin{aligned}&D(x^\mu=0,p_->0,\text{front},\text{upper circle})=-\frac{1}{4\pi^2}\int_0^\infty dp_-\int_0^\infty\frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\delta(4\alpha p_-)\\&=-\frac{1}{8\pi^2}\int_{-\infty}^\infty dp_-\int_0^\infty\frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\delta(4\alpha p_-)=-\frac{1}{32\pi^2}\int_0^\infty\frac{d\alpha}{\alpha^2}e^{-i\alpha m^2-\alpha\epsilon}.\end{aligned} \tag{23}$$

As the presence of the $\delta(4\alpha p_-)$ term shows, the key region is $p_-=0$, something also noted in [11, 12, 23] in their light-front studies [24]. In an on-shell approach states obey $4p_+p_--(p_1)^2-(p_2)^2=m^2$, and thus states with $p_-=0$ would be missed, and without them one would otherwise have had to conclude [2] that $D(x^\mu=0,\text{front})=0$.

Moreover, since $[\exp(4i\alpha p_- Re^{i\theta}) - \exp(-4i\alpha p_- Re^{i\theta})]\big|_0^{\pi} = [\exp(4i\alpha p_- Re^{i\theta}) - \exp(-4i\alpha p_- Re^{i\theta})]\big|_{2\pi}^{\pi}$ and since $\delta(4\alpha p_-)$ is even under $p_- \to -p_-$, it follows that $D(x^\mu = 0, p_- > 0, \text{front}, \text{upper circle})$ and $D(x^\mu = 0, p_- < 0, \text{front}, \text{lower circle})$ must be equal. Thus finally we obtain

$$
\begin{aligned}
D(x^\mu = 0, \text{front}) &= D(x^\mu = 0, p_- > 0, \text{front}, \text{upper circle}) + D(x^\mu = 0, p_- < 0, \text{front}, \text{lower circle}) \\
&= -\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2 - \alpha\epsilon}. \qquad (24)
\end{aligned}
$$

As we see, $D(x^\mu = 0, \text{front})$ is dependent on m after all. We recognize (24) as (15), and thus by direct evaluation confirm that the instant-time and light-front vacuum bubbles are equal, with both being nonzero.

Now, instead of having to deal with pole or circle contributions, we can also evaluate $D(x^\mu = 0, \text{front})$ by using the exponential regulator directly on the real p_+ axis. We do this below to obtain $D(x^\mu = 0, \text{front}, \text{regulator})$ as given in (30) below, and recognize (30) as being none other than (24). We thus confirm the validity of (24).

That we were able to avoid pole terms altogether in deriving (24) is because we used different complex p_+ plane contours for $p_- > 0$ (upper half p_+ plane) and $p_- < 0$ (lower half p_+ plane). However to make contact with the Fock space evaluation we must restrict the discussion to just the one contour that is closed below the real p_+ axis. Then, since we can set $D(x^\mu = 0, \text{front}) = D(x^\mu = 0, \text{front}, \text{pole}) + D(x^\mu = 0, \text{front}, \text{lower circle})$, we see that, even without evaluating it explicitly, not only must the circle contribution be nonvanishing, it must restore the dependence on m. Thus the correct determination of $D(x^\mu = 0, \text{front})$ is its m-dependent value given in (24), and this determination is nonzero.

VIII. RECONCILING THE FOCK SPACE AND FEYNMAN CALCULATIONS

Now while we have seen that $D(x^\mu = 0, \text{front})$ is not given by $D(x^\mu = 0, \text{front}, \text{Fock})$, this is nonetheless puzzling since on the face of it there would not appear to be anything wrong with the Fock space calculation that we have presented above. However, the quantity $-i\langle\Omega|\phi(0)\phi(0)]|\Omega\rangle$ involves the product of two fields at the same spacetime point and is thus ill-defined. To define it we must first split the points and then carefully monitor the limit in which the point splitting is set to zero. We thus use $D(x^\mu \neq 0, \text{front}) = -i\langle\Omega|[\theta(x^+)\phi(x)\phi(0) + \theta(-x^+)\phi(0)\phi(x)]|\Omega\rangle$ as a point-splitting regulator [25]. Since we have seen that there are issues with both the $p_- = 0$ region and the circle at infinity, we shall avoid them both by evaluating $D(x^\mu \neq 0, \text{front})$ directly on the real p_+ axis via the exponential regulator technique, and then monitor the limit in which we set x^μ to zero. This will enable us to develop a single formalism in which we can realize both Fock and Feynman prescriptions simultaneously, so that we can then see exactly where the Fock space approach breaks down.

In order to do this we will need to represent time-ordering theta functions in a form that also does not involve closing a Feynman contour. We shall thus employ the real frequency axis exponential regulator for the theta function as well, and set

$$
\begin{aligned}
\theta(x^+) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_0^\infty d\alpha e^{-i\omega x^+} e^{i\alpha(\omega + i\epsilon)} \\
&= \int_0^\infty d\alpha e^{-\alpha\epsilon}\delta(\alpha - x^+) = \int_0^\infty d\alpha e^{-x^+\epsilon}\delta(\alpha - x^+) = \int_0^\infty d\alpha \delta(\alpha - x^+), \qquad (25)
\end{aligned}
$$

with the $i\epsilon$ providing convergence at $\alpha = \infty$. That $\int_0^\infty d\alpha\delta(\alpha - x^+)$ indeed is $\theta(x^+)$ follows since $\int_0^\infty d\alpha\delta(\alpha - x^+) = 1$ if x^+ is positive, and $\int_0^\infty d\alpha\delta(\alpha - x^+) = 0$ if x^+ is negative. With this representation of the θ function we can now write the entire Feynman diagram as an integral on the real p_+ axis alone.

To specifically evaluate $D(x^\mu, \text{front}, \text{regulator})$ with $x^\mu \neq 0$ and with no restriction on the sign of x^+, we introduce a parameter α with the dimension of length squared, and set

$$
\begin{aligned}
&D(x^\mu, \text{front}, \text{regulator}) \\
&= -\frac{2i}{(2\pi)^4}\int_{-\infty}^{\infty} dp_+ \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_- e^{-i(p_+x^+ + p_-x^- + p_1x^1 + p_2x^2)} \int_0^\infty d\alpha e^{i\alpha(4p_+p_- - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon)} \\
&= -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^\infty dp_- e^{-i(p_-x^- + p_1x^1 + p_2x^2)} \int_0^\infty d\alpha e^{i\alpha(-(p_1)^2 - (p_2)^2 - m^2 + i\epsilon)}\delta(4\alpha p_- - x^+) \\
&- \frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^0 dp_- e^{-i(p_-x^- + p_1x^1 + p_2x^2)} \int_0^\infty d\alpha e^{i\alpha(-(p_1)^2 - (p_2)^2 - m^2 + i\epsilon)}\delta(4\alpha p_- - x^+), \qquad (26)
\end{aligned}
$$

with suppression of the α integration at $\alpha=\infty$ again being supplied by the $i\epsilon$ term. On changing the signs of p_-, p_1 and p_2 in the last integral and setting F_p^2 equal to the positive $(p_1)^2+(p_2)^2+m^2$ we obtain

$$\begin{aligned}
&D(x^\mu,\text{front},\text{regulator})\\
&=-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{4p_-}e^{-i(p_-x^-+p_1x^1+p_2x^2)}\int_0^{\infty}d\alpha e^{ix^+(-F_p^2+i\epsilon)/4p_-}\delta(\alpha-x^+/4p_-)\\
&-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{4p_-}e^{i(p_-x^-+p_1x^1+p_2x^2)}\int_0^{\infty}d\alpha e^{ix^+(F_p^2-i\epsilon)/4p_-}\delta(\alpha+x^+/4p_-).
\end{aligned}\tag{27}$$

Then, using (25), and with the sign of p_- not being negative we obtain

$$\begin{aligned}
&D(x^\mu,\text{front},\text{regulator})\\
&=-\frac{2i\theta(x^+)}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{4p_-}e^{-i(F_p^2x^+/4p_-+p_-x^-+p_1x^1+p_2x^2+ix^+\epsilon/4p_-)}\\
&-\frac{2i\theta(-x^+)}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{4p_-}e^{i(F_p^2x^+/4p_-+p_-x^-+p_1x^1+p_2x^2-ix^+\epsilon/4p_-)},
\end{aligned}\tag{28}$$

and note that the structure of (28) is such that for $x^+>0$ (forward in time) one only has positive energy propagation, while for $x^+<0$ (backward in time) one only has negative energy propagation. With the insertion into $D(x^\mu)=-i\langle\Omega|[\theta(x^+)\phi(x)\phi(0)+\theta(-x^+)\phi(0)\phi(x)]|\Omega\rangle$ of the Fock space expansion for $\phi(x^\mu)$ given in (14) precisely leading to (28), we recognize (28) as the $x^\mu\neq 0$ $D(x^\mu,\text{front},\text{Fock})$ [26].

Now if we set $x^\mu=0$ in (28) we would appear to obtain the m-independent $D(x^\mu=0,\text{front},\text{Fock})$ given in (19). However, we cannot take the $x^+\to 0$ limit since the quantity $x^+/4p_-$ is undefined if p_- is zero, and $p_-=0$ is included in the integration range. Hence, just as discussed in regard to (11), the limit is singular.

To obtain a limit that is not singular we note that we can set x^μ to zero in (26) as there the limit is well-defined, and this leads to

$$\begin{aligned}
&D(x^\mu=0,\text{front},\text{regulator})\\
&=-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}dp_-\int_0^{\infty}d\alpha e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(4\alpha p_-)\\
&-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{-\infty}^{0}dp_-\int_0^{\infty}d\alpha e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(4\alpha p_-)\\
&=-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{-\infty}^{\infty}dp_-\int_0^{\infty}\frac{d\alpha}{4\alpha}e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(p_-)
\end{aligned}\tag{29}$$

If we do the momentum integrations we obtain the m-dependent

$$D(x^\mu=0,\text{front},\text{regulator})=-\frac{1}{16\pi^2}\int_0^{\infty}\frac{d\alpha}{\alpha^2}e^{-i\alpha m^2-\alpha\epsilon}.\tag{30}$$

We recognize (30) as being of the same form as the m-dependent $D(x^\mu=0,p_->0,\text{front},\text{upper circle})+D(x^\mu=0,p_-<0,\text{front},\text{lower circle})$ given in (24). We thus have to conclude that the limit $x^\mu\to 0$ of (28) is not (19) but is (30) instead. The technical difference between (30) and (28) is that to obtain (28) we did the α integration first, while to obtain (30) we did the p_- integration first. Only the latter procedure takes care of the $p_-=0$ contribution. Thus to conclude, we see that because of singularities we first have to point split, and when we do so we find that it is the m-dependent (30) that is the correct value for the light-front vacuum graph.

IX. INFINITE MOMENTUM FRAME CONSIDERATIONS

The infinite momentum frame is a very convenient frame to use in quantum field theory since many Feynman diagrams are suppressed if an observer makes a Lorentz boost with a velocity at or close to the velocity of light. Under a Lorentz boost with velocity u in the 3-direction the contravariant and covariant components of a general four-vector A^μ transform as

$$A^0\to\frac{A^0+uA^3}{(1-u^2)^{1/2}},\quad A^3\to\frac{A^3+uA^0}{(1-u^2)^{1/2}},\quad A_0\to\frac{A_0-uA_3}{(1-u^2)^{1/2}},\quad A_3\to\frac{A_3-uA_0}{(1-u^2)^{1/2}}.\tag{31}$$

If we set $(1-u)=\epsilon^2/2$, then with ϵ small, to leading order we obtain

$$A^0 \to \frac{A^0+A^3}{\epsilon}+O(\epsilon), \quad A^3 \to \frac{A^3+A^0}{\epsilon}+O(\epsilon), \quad A_0 \to \frac{A_0-A_3}{\epsilon}+O(\epsilon), \quad A_3 \to \frac{A_3-A_0}{\epsilon}+O(\epsilon),$$
$$(A^0)^2-(A^3)^2 \to A^+A^- + O(\epsilon). \tag{32}$$

This leads to

$$p^3 \to \frac{p^+}{\epsilon}=\frac{2p_-}{\epsilon}, \quad E_p \to \frac{2p_-}{\epsilon}, \quad \frac{dp^3}{E_p} \to \frac{dp_-}{p_-}, \tag{33}$$

where $E_p=((p_3)^2+(p_1)^2+(p_2)^2+m^2)^{1/2}$.

As well as transforming energies and momenta we also have to transform the ranges of integration in Feynman graphs. To this end we recall that under a Lorentz boost the velocity transforms as

$$v \to \frac{v+u}{1+vu}=\frac{v+1-\epsilon^2/2}{1+v-v\epsilon^2/2}. \tag{34}$$

Thus with $u=1-\epsilon^2/2$, $v=1-\epsilon^2/2$ transforms into $v'=1$, while $v=-1+\epsilon^2/2$ transforms into $v'=-1$. With the quantity p^3+p^0 being given by $m(v+1)/(1-v^2)^{1/2}=m(1+v)^{1/2}(1-v)^{1/2}$, the range $p^3=-\infty$ to $p^3=+\infty$, viz. $v=-1+\epsilon^2/2$ to $v=1-\epsilon^2/2$, transforms into the $p^+=2p_-$ range $m\epsilon^2/4$ to ∞, and thus to the range 0 to ∞ when we set $\epsilon=0$.

In the instant-time vacuum sector we had found that $D(x^\mu=0,\text{instant},\text{Fock})$ and $D(x^\mu=0,\text{instant},\text{pole})$ are equal, with both being given by (15). On transforming (15) to the infinite momentum frame and comparing with (17) and (19) we obtain

$$D(x^\mu=0,\text{instant},\text{Fock})=D(x^\mu=0,\text{instant},\text{pole})=-\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p}$$
$$\to -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{2p_-}=D(x^\mu=0,\text{front},\text{Fock})=D(x^\mu=0,\text{front},\text{pole}). \tag{35}$$

As such, the infinite momentum frame is doing what it is supposed to do, namely it is transforming an instant-time on-shell graph into a light-front on-shell graph. However, we have seen that the light-front mass-independent on-shell evaluation of the vacuum graph does not agree with correct mass-dependent value provided by the off-shell light-front vacuum Feynman diagram. Thus in this respect not only is the on-shell prescription failing for light-front vacuum graphs, so is the infinite momentum frame prescription. (Technically, one ordinarily applies the infinite momentum frame approach to instant-time Feynman diagrams, but as long as they only receive pole contributions this is equivalent to applying the infinite momentum frame approach to the instant-time on-shell Fock space amplitude.)

There is an oddity in (35), one peculiar to the infinite momentum frame. Since the mass-dependent quantity $d^3p/2E_p$ is Lorentz invariant, under a Lorentz transformation with a velocity less than the velocity of light it must transform into itself and thus must remain mass dependent. However, in the infinite momentum frame it transforms into a quantity $dp_1dp_2dp_-/2p_-$ that is mass independent. This is because velocity less than the velocity of light and velocity equal to the velocity of light are inequivalent, since an observer that is able to travel at less than the velocity of light is not able to travel at the velocity of light. Lorentz transformations at the velocity of light are different than those at less than the velocity of light, and at the velocity of light observers (viz. observers on the light cone) can lose any trace of mass.

Moreover, (35) also raises a puzzle. Specifically, while the instant-time on-shell evaluation of the vacuum $D(x^\mu=0,\text{instant},\text{Fock})$ does coincide with the instant-time evaluation of the vacuum off-shell Feynman diagram $D(x^\mu=0,\text{instant},\text{regulator})$ [27], and while the instant-time evaluation of the off-shell Feynman diagram $D(x^\mu=0,\text{instant},\text{regulator})$ does coincide with the light-front evaluation of the off-shell Feynman diagram $D(x^\mu=0,\text{front},\text{regulator})$, nonetheless, the light-front on-shell evaluation of the vacuum $D(x^\mu=0,\text{front},\text{Fock})$ does not coincide with the light-front evaluation of the off-shell Feynman diagram $D(x^\mu=0,\text{front},\text{regulator})$.

The resolution of this puzzle lies in the contribution of circle at infinity to the Feynman contour. In the instant-time case the integral $\int dp_0dp_3/[(p_0)^2-(p_3)^2-(p_1)^2-(p_2)^2-m^2+i\epsilon]$ is suppressed on the circle at infinity in the complex p_0 plane (p_3 being finite), and only poles contribute. However, when one goes to the infinite momentum frame the instant-time dp_3 also becomes infinite ($p^3=mv/(1-v^2)^{1/2}$) and the circle contribution is no longer suppressed. Specifically, on the instant-time circle at infinity the term that is of relevance behaves as $\int Rie^{i\theta}d\theta dp_3/(R^2e^{2i\theta}-(p_3)^2)$, and on setting $\epsilon=1/R$ in the infinite momentum frame limit it behaves as the nonvanishing $\int Rie^{i\theta}d\theta R dp_-/(R^2e^{2i\theta}-R^2p_-^2)$.Thus in the instant-time case one cannot ignore the circle at infinity in the infinite momentum frame even though one can ignore it for observers moving with finite momentum, a point that appears to have been missed in prior infinite momentum frame studies. Consequently, the initial reduction from the instant-time Feynman diagram to the on-shell instant-time Fock space prescription is not valid in the infinite momentum frame, and one has to do the full four-dimensional Feynman contour integral.

We had noted earlier a general rule that the on-shell evaluation always coincides with the pole term evaluation, and that if the pole is not the only contributor to the Feynman contour then the Feynman and Fock space prescriptions cannot agree and one must use the Feynman prescription. We can now add that if we ignore the effect of an infinite Lorentz boost on the instant-time circle at infinity, the instant-time infinite momentum frame evaluation always coincides with the light-front pole term evaluation, and if the light-front pole is not the only contributor to the light-front Feynman contour then the Feynman and infinite momentum frame evaluations cannot agree and one must use the light-front Feynman contour or exponential regulator prescription.

X. DRESSING THE VACUUM GRAPH

FIG. 2: Disconnected and connected dressed $\langle\Omega|\phi(0)\phi(0)|\Omega\rangle$

The first dressings to the graphs in Fig. 1 are shown in Fig. 2. These graphs are actually self-energy graphs within a vacuum loop. To see this for the tadpole graph for instance, momentarily separate the lines at the cross. This then becomes a $\Sigma(p)$ self-energy renormalization graph. However, this renormalization comes with a δm and a Z, and the graph can be replaced by a dressed propagator. To calculate

$$I = \int \frac{d^4k d^4p}{(4p_+p_- - F_p^2 + i\epsilon)(4k_+k_- - F_k^2 + i\epsilon)[4(k_+ + p_+)(k_- + p_-) - F_{p+k}^2 + i\epsilon]} \tag{36}$$

(viz. the Fig. 2 tadpole), we first do the d^4k integration with p_μ held fixed. Up to irrelevant factors this is

$$\Sigma(p) = \int \frac{dk_+dk_1dk_2dk_-}{(4k_+k_- - k_1^2 - k_2^2 - m^2 + i\epsilon)[4(k_+ + p_+)(k_- + p_-) - (k_1 + p_1)^2 - (k_2 + p_2)^2 - m^2 + i\epsilon]}. \tag{37}$$

There is no circle at infinity contribution as the denominator has two powers of k_+. The graph diverges as a single logarithm, i.e. as $\log(\Lambda^2/p^2)$ at large p^2. Introducing a mass renormalization counter term $\delta m = -\log(\Lambda^2/m^2)$ gives $\log(m^2/p^2)$. The original graph is thus

$$I = \int dp_+dp_1dp_2dp_- \frac{\log(m^2/p^2)}{(4p_+p_- - p_1^2 - p_2^2 - m^2 + i\epsilon)} \tag{38}$$

and the circle at infinity is not suppressed. The concerns raised in this paper thus carry over to dressed light-front vacuum graphs as well and cannot be ignored.

XI. SHORTCOMINGS OF NORMAL ORDERING

In our study of light-front vacuum graphs we have studied a point of principle, namely the appropriate way to evaluate the vacuum graphs. Now a reader might regard the issue as being somewhat academic since vacuum graphs can be normal ordered away. However, since (perturbative) normal ordering involves moving all annihilation operators to the right and all creation operators to the left in a vacuum matrix element, it does not encompass the circle at infinity contributions that occur in light-front vacuum graphs. For light-front vacuum graphs we thus need to deal with circle at infinity contributions and such contributions are foreign to standard renormalization techniques, and indeed in their presence one cannot effect a Wick rotation to Euclidean momenta. To get round this we note that for renormalization one does not actually need to consider circle at infinity contributions per se since one can evaluate Feynman diagrams as real frequency axis integrals by using the exponential regulator on the real frequency axis, just as was done for the light-front case in (26). Then one can introduce a second field with a regulator mass M and subtract off its contribution, the Pauli-Villars prescription, and use the Pauli-Villars prescription to regulate the ultraviolet behavior of light-front vacuum graphs [28].

Moreover, we also note that certain vacuum graphs are actually observable and cannot in fact be normal ordered away anyway, namely those associated with dynamical symmetry breaking or those that couple to gravity. When a symmetry is, for instance, broken dynamically by a fermion bilinear composite one is interested in evaluating the expectation value $\langle S|\bar{\psi}\psi|S\rangle$ where $|S\rangle$ is a spontaneously broken vacuum and then comparing it with the expectation value $\langle N|\bar{\psi}\psi|N\rangle$ where $|N\rangle$ is a normal vacuum. Now dynamical symmetry breaking is a long range order infrared effect while normal ordering or Pauli-Villars is a way of dealing with ultraviolet divergences. Since dynamical symmetry breaking is an infrared effect the short distance behaviors of $\langle S|\bar{\psi}\psi|S\rangle$ and $\langle N|\bar{\psi}\psi|N\rangle$ are the same. Thus even if we were to normal order $\langle N|\bar{\psi}\psi|N\rangle$ by setting $\langle N| : \bar{\psi}\psi : |N\rangle$ equal to zero, we would still need to evaluate $\langle S| : \bar{\psi}\psi : |S\rangle$ where the normal ordering is done with respect to $|N\rangle$, with $\langle S| : \bar{\psi}\psi : |S\rangle$ being nonzero in the broken symmetry case.

For gravity, consider a free massive fermion with energy-momentum tensor $T_{\mu\nu} = i\bar{\psi}\gamma_\mu\partial_\nu\psi$. The Einstein equations are of the form $R_{\mu\nu} - (1/2)g_{\mu\nu}R = -8\pi G T_{\mu\nu}$ (here R is the Ricci scalar), with trace $R = 8\pi Gm\langle\Omega|\bar{\psi}\psi|\Omega\rangle$. The fermion vacuum bubble thus couples to gravity. Moreover, here one is not free to normal order at all, since the hallmark of Einstein gravity is that gravity couples to energy and not to energy difference. Since in analog to $\langle\Omega|\phi^2|\Omega\rangle$ the light-front circle at infinity contribution to $\langle\Omega|\bar{\psi}\psi|\Omega\rangle$ is nonzero, in light-front quantization the circle at infinity contributes to the cosmological constant [29].

XII. CONCLUSIONS

If one starts with a field equation and an equal instant-time or equal light-front time commutator one can make a Fock space expansion of a field in terms of creation and annihilation operators as multiplied by on-shell solutions to the field equation. (For an instant-time free field for instance the solutions are labelled by a three-vector $\vec{p}$ with the energy fixed to the on-shell $E_p = (p^2 + m^2)^{1/2}$). In constructing a Feynman diagram one has the same field equations and the same canonical commutators but one in addition has a time ordering. It is this time ordering that takes the Feynman diagram off shell, with the energy being replaced by a contour integration in a complex frequency plane as in (4). In the literature there are four approaches to dealing with the light-front vacuum sector: the Feynman diagram approach, the light-front Fock space approach, the light-front Fock space approach as restricted to states with $p_- > 0$, and the light-front sector as derived by writing the instant-time vacuum sector in the infinite momentum frame. In this paper we have analyzed all of these different approaches and identified why they differ and identified which approach (viz. the Feynman diagram approach) is to be the valid one.

In the non-vacuum sector all of these approaches lead to the same outcome because in the Feynman diagrams there are only pole contributions (to thus recover both the Fock space and infinite momentum frame approaches), and when x^+ is non-zero and positive (the scattering situation) the $\delta(4\alpha p_- - x^+)$ term in (26) ensures that only $p_- > 0$ terms are relevant. However, in the light-front vacuum sector there are circle at infinity contributions in the Feynman diagrams to thus make the Fock space and infinite momentum frame approaches incorrect, while the replacement of the $\delta(4\alpha p_- - x^+)$ term by $\delta(4\alpha p_-)$ when x^+ is zero now permitting a contribution from $p_- = 0$. Thus as noted in going from (4) with $x^+ \neq 0$ to (5) with $x^+ = 0$ one has to include a circle at infinity contribution that one previously had not needed to include. It is this circle at infinity contribution that is then paramount in the light-front vacuum sector, to thus make the off-shell Feynman diagram approach with its non-zero value for light-front vacuum graphs the correct one.

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[16] In light-front coordinates the line element is given by $ds^2 = x^+x^- - (x^1)^2 - (x^2)^2$, to thus correspond to a metric with $g_{+-} = g_{-+} = 1/2$, $g_{11} = g_{22} = -1$, $\det[g_{\mu\nu}] = -1/4$, and $x_+ = x^-/2$, $x_- = x^+/2$. The invariant measure is given by $(1/2)\int dx^+dx^1dx^2dx^- = 2\int dx_+dx_1dx_2dx_-$.
[17] With the lowest order contribution to the fermionic $\langle\Omega|\bar{\psi}\psi|\Omega\rangle$ being given by $(-4i/(2\pi)^4)\int d^4p\ m/(p^2 - m^2 + i\epsilon)$, and with (5) being a generic contour integral representation of $\theta(0)$ that holds in any field theory bosonic or fermionic, for the light-front $\langle\Omega|\bar{\psi}\psi|\Omega\rangle$ there again is a circle at infinity contribution.
[18] Modes with $p_- = 0$ play a central role in light-front studies, and have been discussed in various contexts in papers such as G. McCartor Z. Phys. C **41**, 271 (1988) (demonstrates how to include zero modes through the choice of two null quantization surfaces, $x^+ = 0$ and $x^- = 0$, in $1+1$ dimensions, with this being necessary for massless fields); T. Heinzl, *Light-cone zero modes revisited*, arXiv:hep-th/0310165 (discusses vacuum graphs in both instant-time form and light-front form); C.-R. Ji and S.-J. Rey, Phys. Rev. D **53**, 5815 (1996) (demonstrates the necessity of including zero modes in the calculation of the axial anomaly); T. N. Tomaras, N. C. Tsamis and R. P. Woodard, JHEP **11**, 008 (2001) (demonstrates the necessity of including zero modes in the calculation of $1+1$ dimensional Schwinger pair production); A. Ilderton, J. High Energ. Phys. (2014) 2014: 166 (also Schwinger pair production); and P. P. Srivastava and S. J. Brodsky, Phys. Rev. D **66**, 045019 (2002) (connects light-front zero modes to the Higgs mechanism).
[19] P. D. Mannheim, P. Lowdon and S. J. Brodsky, *Comparing light-front quantization with instant-time quantization*, SLAC-PUB-17390, February, 2019.
[20] Equations obeyed by spacetime-dependent scalar field Green's functions such as $[g^{-1/2}\partial_\nu g^{1/2}\partial^\nu + m^2]D(x^\mu) = -g^{-1/2}\delta^4(x)$ are general coordinate invariant c-number equations, and one can directly transform them from instant-time coordinates to light-front coordinates. An equivalent argument is made in [19] for the scalar field c-number path integral formulation, and this enables us to extend the general coordinate invariance argument to the spacetime-independent vacuum sector.
[21] R. A. Neville and F. Rohrlich, Il Nuovo Cimento A **1**, 625 (1971).
[22] In passing, we note that it is often found advantageous in the light-front parton literature to restrict the Fock space expansion to hypersurfaces with $x^+ = 0$. Here we make no such restriction.
[23] B. L. G. Bakker, M. A. DeWitt, C.-R. Ji and Y. Mishchenko, Phys. Rev. D **72**, 076005 (2005).
[24] As discussed in [11, 12, 23], in light-front Feynman diagram studies one of the challenges is that the pole at $p_- = 0$ has $p_+ = \infty$, and thus lies on the circle at infinity in the complex p_+ plane, from which it then has to be disentangled. By using the exponential regulator given in (20) this issue is avoided since we close the contour in such a way that there are no pole contributions at all.
[25] We note that do not use the standard point-splitting technique here. The standard procedure is based on the limit $x^\mu \to 0$ of the two-point function $\langle\Omega|\phi(x)\phi(0)|\Omega\rangle$, and has been applied to light-front studies in M. Burkardt, S. S. Chabysheva and J. R. Hiller, Phys. Rev. D **94**, 065006 (2016), M. Burkardt, F. Lenz and M. Thies, Phys. Rev. D **65**, 125002 (2002). Rather, we use the time ordered product $\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle$. Our reason for doing this is that we are not just interested in regularizing ultraviolet infinities in the vacuum graph (one also needs to regularize them in the instant-time case), but are interested in exploring why the $x^\mu \to 0$ limit in which the two points coincide does not give the Fock space answer, with this being the reason why the Fock space answer is not the correct one.
[26] While the on-shell old-fashioned Hamiltonian perturbation theory formalism actually predates quantum field theory, as (28) shows, even with quantum field theory it continues to hold in the light-front case for forward in time scattering processes with $x^+ > 0$, since for them the energy $F_p^2/4p_-$ is positive. However, the on-shell approach fails for the light-front vacuum graphs that appear in quantum field theory, with the $\theta(-x^+)$ term making a contribution in the $x^+ \to 0$ limit, with the $x^+ \to 0$ limit being singular, and with the intuition acquired for $x^+ > 0$ not carrying over to the light-front $x^+ = 0$ vacuum sector. From the perspective of the on-shell light-front $4p_+p_- - (p_1)^2 - (p_2)^2 = m^2$ condition, $p_- = 0$ corresponds to

$p_+ = \infty$, and thus to $x^+ = 0$, so that the $p_- \to 0$ and the $x^+ \to 0$ limits are intrinsically related. In contrast, we should note that in the instant-time case the $x^0 \to 0$ limit of the free particle propagator is not singular, with the structure of the vacuum sector in the light-front case thus being conceptually different from its structure in the instant-time case. And yet despite this (and in fact because of it) $D(x^\mu = 0, \text{instant})$ and $D(x^\mu = 0, \text{front})$ are equal.

[27] In [19] the instant-time $D(x^\mu = 0, \text{instant}, \text{regulator})$ is evaluated using the exponential regulator on the real p_0 axis. Its value is found to be equal to $D(x^\mu = 0, \text{instant})$ as given in (15), just as it should be.

[28] In the Pauli-Villars regularization procedure one uses a propagator of the form $D_{\rm PV}(k^2) = 1/(k^2 - M_1^2) - 1/(k^2 - M_2^2)$ as it is more convergent in the ultraviolet than $1/(k^2 - M_1^2)$ itself. If the two terms in $D_{\rm PV}(k^2)$ are to be associated with different fields then the M_2 field must be quantized with a negative metric. However if the two terms arise from one and the same field, a field that must then obey a fourth-order equation of motion, it was noted in C. M. Bender and P. D. Mannheim, Phys. Rev. Lett. **100**, 110402 (2008);Phys. Rev. D **78**, 025022 (2008) that then the Hamiltonian is not Hermitian but is instead PT symmetric. Because of this one must use as a norm the overlap of a state with its PT conjugate rather than that with its Hermitian conjugate, and this PT conjugate norm is both positive definite and unitary.

[29] For scalar fields with action $I_S = \int d^4x(-g)^{1/2}[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2]$ the energy-momentum tensor is given by $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}[\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - \frac{1}{2}m^2\phi^2]$. Consequently, its vacuum expectation value is given by $\langle\Omega|T_{\mu\nu}|\Omega\rangle = \frac{1}{2}g_{\mu\nu}m^2\langle\Omega|\phi^2|\Omega\rangle$, to thus be non-zero. Thus in both the scalar and the fermion field cases the light-front circle at infinity contributes non-trivially to the cosmological constant. And given the equality of light-front quantized scalar graphs and instant-time quantized scalar graphs noted in [19] and [20], the standard Einstein gravity cosmological constant problem is thus identical to (and thus just as severe as) the standard one that appears in instant-time quantization. We should also note that in evaluating vacuum graphs with massive fields such as in (1), the only thing that matters is that the fields have mass. It is thus immaterial whether their masses are elementary or dynamical, with our results thus holding even if the scalar field is a Higgs field (a field that would give mass to both the fermion and Higgs fields).