Ward-Takahashi Identity on Ligh-Front

"LC2019-QCD on the Light Cone: from Hadrons to Heavy Ions"

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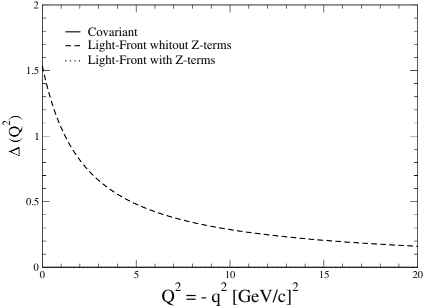
5. Non-Valence contribuition to J^+ electromagnetic current
6. Ward-Takahashi Identity

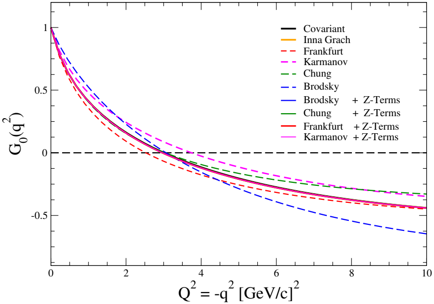
⇒ **Some references:**

• Ref. H. Naus, J.de Melo and T. Frederico
"Ward-Takahashi identity on the light front," Few Body Syst. 24, 99 (1998)
• J.de Melo, J. Sales, T. Frederico and P. Sauer,
"Pairs in the light front and covariance," Nucl. Phys. A 631, 574C (1998)
• J. de Melo, H. Naus and T. Frederico,
"Pion electromagnetic current in the light cone formalism," Phys. Rev. C 59, 2278 (1999)
• J. de Melo, T. Frederico, H. W. L. Naus and P. U. Sauer, Nucl. Phys. A 660, 219 (1999)
• J. de Melo, T. Frederico, E. Pace and G. Salme,
"Pair term in the electromagnetic current within the front form dynamics: Spin-0 case," Nucl. Phys. A 707, 399 (2002)
• J. de Melo / "Unambiguous Extraction of the Electromagnetic Form Factors for Spin-1 Particles on the Light-Front," Phys. Lett. B 788, 152 (2019)

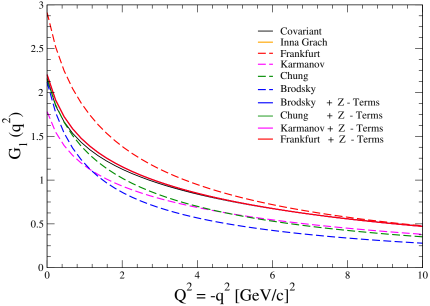
Motivation: S=1 / Rho Meson

- **J. P. B. C. de Melo and T. Frederico, Phys. Rev. C 55, 2043 (1997)**

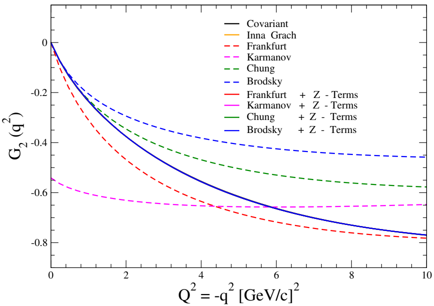




$$\begin{aligned}\Delta(q^2) &= (1+2\eta)I^+_{11} + I^+_{1-1} - \sqrt{8\eta}I^+_{10} - I^+_{00} = 0 \\ &= (1+\eta)(J^{+V}_{yy} - J^+_{zz}) = 0\end{aligned}$$



3
2.5
2
1.5
1
0.5
0
0
2
4
6
8
10
$G_1(q^2)$
$Q^2 = -q^2$ [GeV/c]^2
Covariant
Inna Grach
Frankfurt
Karmanov
Chung
Brodsky
Brodsky + Z - Terms
Chung + Z - Terms
Karmanov + Z - Terms
Frankfurt + Z - Terms



0
-0.2
-0.4
-0.6
-0.8
0
2
4
6
8
10
$G_2(q^2)$
$Q^2 = -q^2$ [GeV/c]^2
Covariant
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Brodsky
Frankfurt + Z - Terms
Karmanov + Z - Terms
Chung + Z - Terms
Brodsky + Z - Terms

- **Eletromagnetic current for Bosons**

$$J^{\mu} = \Gamma^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2k - P' - P)^{\mu}}{(k^2 - m^2 + \imath\epsilon)((k - P)^2 - m^2 + \imath\epsilon)((k - P')^2 - m^2 + \imath\epsilon)} .$$

- **Plus Component of the electromaagnetic current:** J^+

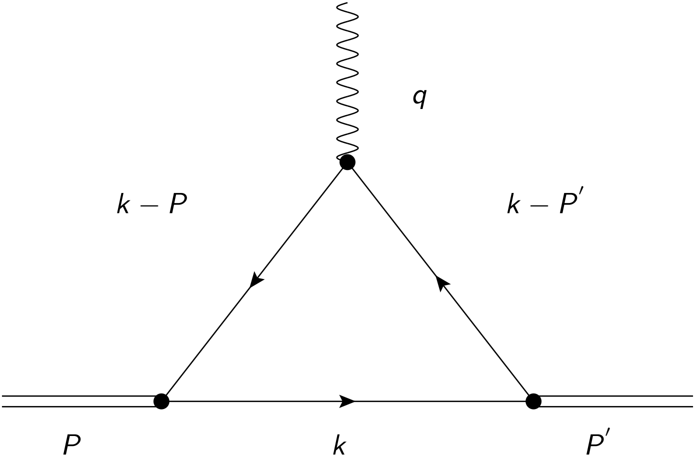
$$J^{+} = \Gamma^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2k - P' - P)^{+}}{(k^2 - m^2 + \imath\epsilon)((k - P)^2 - m^2 + \imath\epsilon)((k - P')^2 - m^2 + \imath\epsilon)}$$

- **And with LF coordinates:**

$$J^{+} = \Gamma^{2} \int \frac{d^{2}k_{\perp}dk^{+}dk^{-}}{2(2\pi)^{4}} \frac{(2k - P' - P)^{+}}{k^{+}(P^{+} - k^{+})(P'^{+} - k^{+})(k^{-} - \frac{f_1 - \imath\epsilon}{k^{+}})} \frac{1}{(P^{-} - k^{-} - \frac{f_2 - \imath\epsilon}{P^{+} - k^{+}})(P'^{-} - k^{-} - \frac{f_3 - \imath\epsilon}{P'^{+} - k^{+}})}$$

- **with the def.:**

$$f_1 = k_{\perp}^{2} + m^{2} \ , \ f_2 = (P - k)_{\perp}^{2} + m^{2} \ e \ f_3 = (P^{'} - k)_{\perp}^{2} + m^{2}$$


q
k − P
k − P′
P
k
P′

Poles position

$$
\begin{aligned}
k_1^- &= \frac{f_1 - \imath\epsilon}{k^+} \\
k_2^- &= P^- - \frac{f_2 - \imath\epsilon}{P^+ - k^+} \\
k_3^- &= P'^- - \frac{f_3 - \imath\epsilon}{P'^+ - k^+} \ .
\end{aligned}
$$

i) $k^+ < 0$

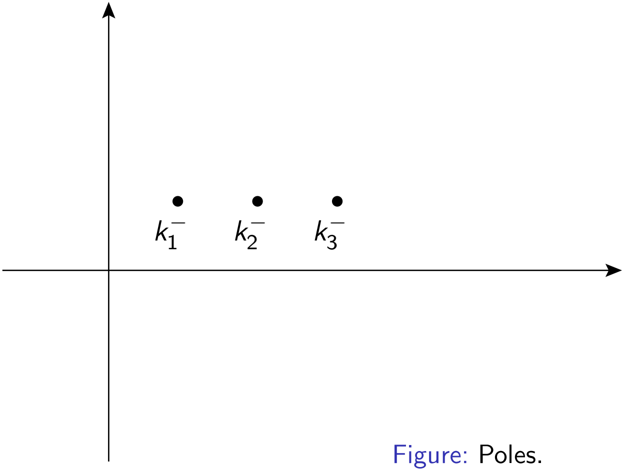


Figure: Poles.

- $\Longrightarrow$ **No contribuition to e.m. current!!**

ii) $k^+ > P^+$

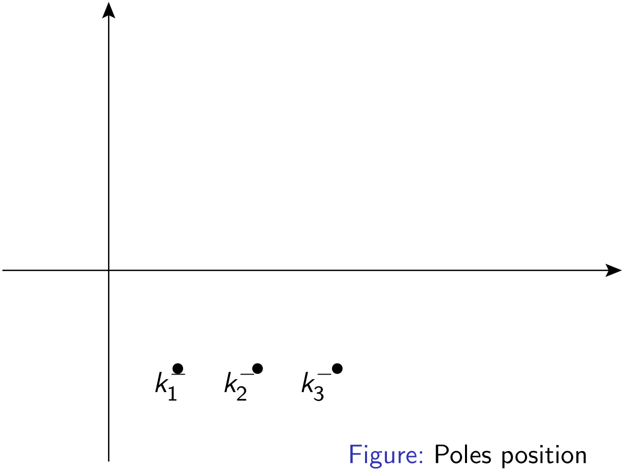


Figure: Poles position

- $\Longrightarrow$ **No contribuition to e.m. current!!**

iii) $0 < k^+ < P^+$

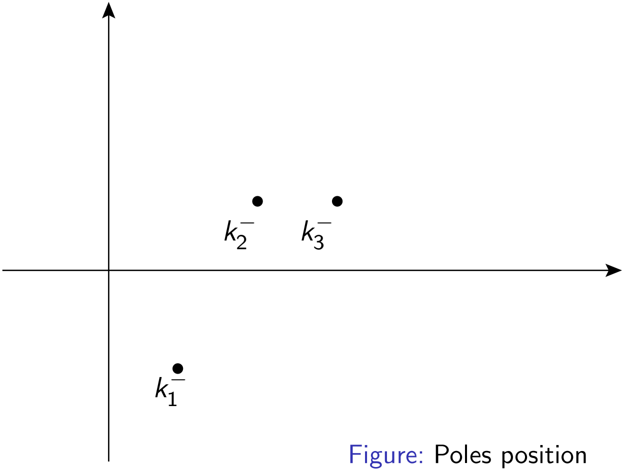


Figure: Poles position

$\star \implies$ **Contribuition to e.m. current!!**

- k^- **Integration**

$$J^+ = \Gamma^2 \int \frac{d^2k_\perp dk^+}{2(2\pi)^3} \frac{(2k-P'-P)^+\theta(P^+-k^+)\theta(k^+)}{k^+(P^+-k^+)(P'^+-k^+)}$$
$$\frac{1}{(P^- - \frac{f_1}{k^+} - \frac{f_2 - \imath\epsilon}{P^+-k^+})(P'^- - \frac{f1}{k^+} - \frac{f_3-\imath\epsilon}{P'^+-k^+})}$$

- $\Rightarrow \quad x = \frac{k^+}{P^+}$, **and** $0 < x < 1$
- $P^+P^- - P_\perp^2 = M_b^2 \iff M_b$ **Bound state mass (for two bosons)**

$$M_0^2 = \frac{f_1}{x} + \frac{f_2}{(1-x)} - P_\perp^2 \tag{2}$$
$$M_0'^2 = \frac{f_1}{x} + \frac{f_3}{(1-x)} - P_\perp'^2$$

$$J^+ \;=\; \Gamma^2 \int \frac{d^2k_\perp dx}{2(2\pi)^3} \frac{(2xP - 2P)^+}{x(1-x)^2} \frac{\theta(x)\theta(1-x)}{(M_b^2 - M_0^2)(M_b^2 - M_0'^2)}$$

- **Wave function**

$$\Phi(x, \vec{k_\perp}) = \frac{\Gamma}{(M_b^2 - M_0^2)}$$

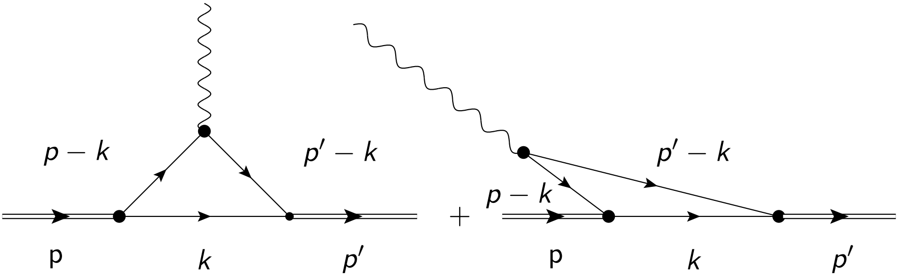
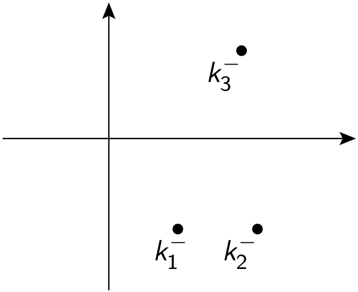


Figure: Feynman diagramms contribuitions to the elastic electromagnetic form factor, Left: The valence component contribuition from the electromagnetic current to the elelectromagnetic form factor. Right panel: Possible non-valence contribuition for the electromagnetic current with frames diferents from the Breit frame with the Drell-Yan condition.

Pole dislocation method*

- **Integral region:** $P^+ < k^+ < P'^+$, if $\underline{q^+ \neq 0}$

$\Rightarrow P'^+ = P^+ + q^+$, In the $\lim_{q^+\to 0_+} P'^+ = P^+$.



- **Integration in the pole k_3^-:**

$$J^+ = \Gamma^2 \int \frac{d^2k_\perp dk^+}{2(2\pi)^4} \frac{(2k-P'-P)^+\theta(P'^+-k^+)\theta(k^+-P^+)}{k^+(P^+-k^+)(P'^+-k^+)(P'^- - \frac{f_3}{P'^+-k^+} - \frac{f_1}{k^+})} \frac{1}{(\frac{f_3}{P'^+-k^+} - \frac{f_2}{P^+-k^+})}$$

- **The denominator:**

$$\left[\frac{f_3}{P'^+-k^+} - \frac{f_2}{P^+-k^+}\right]^{-1}$$

- **Three virtual particles propagation in $x^+ = 0$**
- **The denominator**

$$\left[P'^- - \frac{f_3}{P'^+-k^+} - \frac{f_1}{k^+}\right]^{-1}$$

- **Is the propagator for two final state particles**

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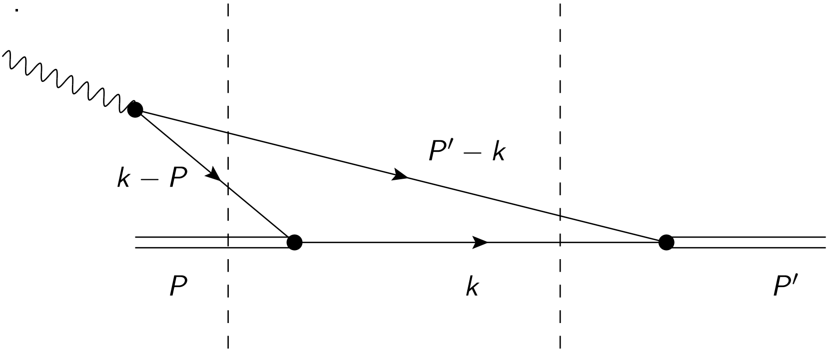

Figure: Pairs diagramm

•Ref. **H. Naus, J. de Melo and T. Frederico, Few Body Syst. 24, 99 (1998)**

• **J. de Melo, J. Sales, T. Frederico and P. U. Sauer, Nucl. Phys. A 631, 574C (1998)**

- **If use the transformation** $x = (k^+ - P^+)/q^+$

$$J^+ = \Gamma^2 \int \frac{d^2k_\perp dx}{2(2\pi)^3} \frac{q^{+2}(2x-1)\theta(1-x)\theta(x)}{(P^+ + q^+x)x(1-x)} \frac{1}{(\frac{f_3}{(1-x)} - q^+(\frac{f_1}{(P^+ + q^+x)} - P'^-))(\frac{f_3}{1-x} + \frac{f_2}{x}))}$$

- **Take the limit** $q^+ \to 0_+$, **we have** $J^+ \propto q^{+2}$

$$\boxed{\lim_{q^+\to 0_+} J^+ = 0}$$

Which means physically, for this component of the electromagnetic current the pair creation is suppressed

$J_\perp$ Component of the e.m. current

$$J_\perp = \Gamma^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2k - P' - P)_\perp}{(k^2 - m^2 + \imath\epsilon)((k-P)^2 - m^2 + \imath\epsilon)((k-P')^2 - m^2 + \imath\epsilon)}$$

i) $0 < k^+ < P^+ \implies k^{-(i)} = \frac{f_1 - \imath\epsilon}{k^+}$

$$J^i_\perp = \Gamma^2 \int \frac{d^2k_\perp dk^+}{2(2\pi)^3} \frac{(2k - P' - P)_\perp \theta(k^+)\theta(P^+ - k^+)}{k^+(P^+ - k^+)(P'^+ - k^+)} \frac{1}{(P^- - \frac{f_1}{k^+} - \frac{f_2 - \imath\epsilon}{P^+ - k^+})(P'^- - \frac{f_1}{k^+} - \frac{f_3 - \imath\epsilon}{P'^+ - k^+})}$$

- Final result

$$J^i_\perp = \Gamma^2 \int \frac{d^2k_\perp dx}{2(2\pi)^3} \frac{(2k - P' - P)_\perp \theta(x)\theta(1-x)}{x(1-x)^2(M_b^2 - M_0^2)(M_b^2 - M_0'^2)} \neq 0$$

ii) $P^+ < k^+ < P'^+ \implies k_3^- = P'^- - \frac{f_3 - \imath\epsilon}{P'^+ - k^+}$

- Use $x = (k^+ - P^+)/q^+$, then

$$J_\perp^{ii} = \Gamma^2 \int \frac{d^2 k_\perp dk^+}{2(2\pi)^3} \frac{q^+ \; (2k - P' - P)_\perp \theta(1-x)\theta(x)}{(P^+ + q^+ x)x(1-x)}$$
$$\frac{1}{(-\frac{f_3}{1-x} - q^+(\frac{f_1}{P^+ + q^+ x} - P'^-))(\frac{f_3}{1-x} + \frac{f_2}{x})} \propto q^+$$

$$\boxed{\lim_{q^+ \to 0_+} J^\perp = 0}$$

Again, the pair creation is suppressed!!

J^- Component of the e.m. current

$$J^- = \Gamma^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2k - P' - P)^-}{(k^2 - m^2 + \imath\epsilon)((k-P)^2 - m^2 + \imath\epsilon)((k-P')^2 - m^2 + \imath\epsilon)}$$

- **Integral intervals**

i) $0 < k^+ < P^+ \implies k^{-(i)} = \frac{f_1 - \imath\epsilon}{k^+}$

ii) $P^+ < k^+ < P'^+ \implies k_3^- = P'^- - \frac{f_3 - \imath\epsilon}{P'^+ - k^+}$

- Results

$$J^{-i} = -\imath\Gamma^2 \int \frac{d^2k_\perp dk^+}{2(2\pi)^3} \frac{[2f_1 - k^+(P'^- + P^-)]\ \theta(P^+ - k^+)\theta(k^+)}{k^{+2}(P^+ - k^+)(P'^+ - k^+)(P^- - \frac{f_1}{k^+} - \frac{f_2}{P^+ - k^+})} \frac{1}{(P'^- - \frac{f_1}{k^+} - \frac{f_3}{P'^+ - k^+})} \neq 0$$

$$J^{-ii} = -\imath\Gamma^2 \int \frac{d^2k_\perp dk^+}{2(2\pi)^3} \frac{[-2\frac{f_3}{P'^+ - k^+} + P'^- - P^-]}{k^+(P^+ - k^+)(P'^+ - k^+)} \frac{\theta(P'^+ - k^+)\theta(k^+ - P^+)}{(P'^- - \frac{f_3}{P'^+ - k^+} - \frac{f_1}{k^+})(\frac{f_3}{P'^+ - k^+} - \frac{f_2}{P^+ - k^+})}$$

- Final results for interval "(ii)" with $x = \frac{(k^+ - P^+)}{q^+}$

$$J^{-ii} = -\imath\Gamma^2 \int \frac{d^2 k_\perp dx}{2(2\pi)^3} \frac{-2f_3 + q^+(1-x)(P'^- - P^-)\theta(x)\theta(1-x)}{(P^+ + q^+ x)x(1-x)^2}$$
$$\frac{1}{(-\frac{f_3}{1-x} - q^+(\frac{f_1}{P^+ + q^+ x} - P'^-))(\frac{f_3}{1-x} + \frac{f_2}{x})}$$

- In the limit $q^+ \to 0_+$

$$J^{-ii} = -\imath\frac{\Gamma^2}{P^+}\int_0^1 \frac{d^2 k_\perp dx}{(2\pi)^3} \frac{1}{xf_3 + (1-x)f_2}$$
$$= \boxed{\imath\frac{\Gamma^2}{P^+}\int \frac{d^2 k_\perp}{(2\pi)^3} \frac{\ln(f_2) - \ln(f_3)}{f_3 - f_2} \neq 0}$$

- And

$$\boxed{\mathbf{J}^+ = \mathbf{J}^- = \mathbf{J}^{-(\mathbf{i})} + \mathbf{J}^{-(\mathbf{ii})}}$$

- **Two bosons bound state**

$$q_\mu . J^\mu = \Sigma(P^{'}) - \Sigma(P)$$

- **Electromagnetic current for two bosons bound state**

$$q_\mu . J^\mu = \int \frac{d^4k}{(2\pi)^4} \frac{q_\mu .(2k - P' - P)^\mu}{(k^2 - m^2 + \imath\epsilon)((k-P)^2 - m^2 + \imath\epsilon)((k-P^{'})^2 - m^2 + \imath\epsilon)}$$

- $\Rightarrow$ $\mathbf{q}^\mu = (\mathbf{P}^{'\mu} - \mathbf{P}^\mu - \mathbf{k}^\mu + \mathbf{k}^\mu)$

Then,

$$\begin{aligned} -(P^{'\mu} - k^\mu - P^{'\mu} + k^\mu).(P_\mu - k_\mu + P^{'}_\mu - k_\mu) \\ = -(P^{'\mu} - k^\mu)^2 + (P^\mu - k^\mu)^2 - m^2 + m^2 \end{aligned}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{-(P'^{\mu}-k^{\mu})^2+m^2+(P^{\mu}-k^{\mu})^2-m^2}{(k^2-m^2+\imath\epsilon)((k-P)^2-m^2+\imath\epsilon)((P'-k)^2-m^2+\imath\epsilon)} =$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2+\imath\epsilon)((P'-k)^2-m^2+\imath\epsilon)} -$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2+\imath\epsilon)((P-k)^2-m^2+\imath\epsilon)}$$

Is the Ward-Takahashi again:

$$q_{\mu}.J^{\mu} = \Sigma(P^{'}) - \Sigma(P)$$

- Self-energy

$$\Sigma(P) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2+\imath\epsilon)((k-P)^2-m^2+\imath\epsilon)}$$

- **Integration with the interval:** $P'^{+} > P^{+} > 0$

$$\Sigma(P) = \int \frac{d^2k_\perp dk^+ dk^-}{2(2\pi)^4} \frac{1}{k^+(P^+ - k^+)(k^- - \frac{f_1 - \imath\epsilon}{k^+})(P^- - k^- - \frac{f_2 - \imath\epsilon}{P^+ - k^+})}$$

$$\begin{aligned}\Sigma(P) &= 2\pi\imath \int_0^{P^+} \frac{d^2k_\perp dk^+}{2(2\pi)^4} \frac{1}{k^+(P^+ - k^+)(P^- - \frac{f_1}{k^+} - \frac{f_2}{P^+ - k^+})} \\ &= \pi\imath \int_0^1 \frac{d^2k_\perp dx}{(2\pi)^4} \frac{1}{x(1-x)(P^2 - M_0^2 + \imath\epsilon)}\end{aligned}$$

- $x = \frac{k^+}{P^+}$, M_0^2 and $M_0'^2$,

$$\begin{aligned}M_0^2 &= \frac{f_1}{x} + \frac{f_2}{(1-x)} - P_\perp^2 \\ M_0'^2 &= \frac{f_1}{x} + \frac{f_3}{(1-x)} - P_\perp'^2\end{aligned}$$

$$\Sigma(P^{'}) - \Sigma(P) = \pi\imath \int_0^1 \frac{d^2k_\perp dx}{(2\pi)^4} \frac{-P'^2 + P^2}{x(1-x)(P'^2 - M_0'^2)(P^2 - M_0^2)}$$

- Or writing as,

$$\Sigma(P^{'}) - \Sigma(P) = (P_\mu + P^{'}_\mu)(P^\mu - P'^\mu)\ I_\Sigma$$

- I_Σ is,

$$I_\Sigma = \pi\imath \int d^2k_\perp \int_0^1 \frac{dx}{(2\pi)^4} \frac{1}{x(1-x)(P'^2 - M_0'^2)(P^2 - M_0^2)}$$

- **Electromagnetic current in the Light-front**

$$q_\mu . J^\mu = \frac{1}{2}(\ q^+ J^- + q^- J^+)\ - \vec{q_\perp} . \vec{J_\perp}$$

$$J^- = \int \frac{d^4 k}{(2\pi)^4} \frac{(2k - P' - P)^-}{(k^2 - m^2 + \imath\epsilon)((k-P)^2 - m^2 + \imath\epsilon)((k-P')^2 - m^2 + \imath\epsilon)} \ ,$$

$$J^+ = \int \frac{d^4 k}{(2\pi)^4} \frac{(2k - P' - P)^+}{(k^2 - m^2 + \imath\epsilon)((k-P)^2 - m^2 + \imath\epsilon)((k-P')^2 - m^2 + \imath\epsilon)} \ ,$$

$$J_\perp = \int \frac{d^4 k}{(2\pi)^4} \frac{(2k - P' - P)_\perp}{(k^2 - m^2 + \imath\epsilon)((k-P)^2 - m^2 + \imath\epsilon)((k-P')^2 - m^2 + \imath\epsilon)}$$

Breit frame off-shell: $P^- \ \neq \ P^{'-}$, $P^+ \ \neq \ P^{'+}$.

$$J^{-} = -\pi\imath\int\frac{d^2k_\perp}{(2\pi)^4}\int_0^{P^+}dk^+\frac{(2\frac{f_1}{k^+}-P'^- - P^-)(P'^- - f_1 - \frac{f_3}{P'^+-k^+})^{-1}}{k^+(P^+-k^+)(P'^+-k^+)(P^- - f_1 - \frac{f_2}{P^+-k^+})}$$

$$\pi\imath\int\frac{d^2k_\perp}{(2\pi)^4}\int_{P^+}^{P'^+}dk^+\frac{(-2\frac{f_3}{P'^+-k^+}+P'^- - P^-)(P^- - P'^- + \frac{f_3}{P'^+-k^+} - \frac{f_2}{P^+-k^+})^{-1}}{k^+(P^+-k^+)(P'^+-k^+)(P'^- - \frac{f3}{P'^+-k^+} - \frac{f_1}{k^+})}$$

$$J^{+} = -\pi\imath\int\frac{d^2k_\perp}{(2\pi)^4}\int_0^{P^+}dk^+\frac{(2k^+ - P'^+ - P^+)(P'^- - \frac{f_1}{k^+} - \frac{f_3}{P'^+-k^+})^{-1}}{k^+(P^+-k^+)(P'^+-k^+)(P^- - f_1 - \frac{f_2}{P^+-k^+})} \quad -$$

$$\pi\imath\int\frac{d^2k_\perp}{(2\pi)^4}\int_{P^+}^{P'^+}dk^+\frac{(2k^+ + P'^+ - P^+)(P^- - P'^- + \frac{f_3}{P'^+-k^+} - \frac{f_2}{P^+-k^+})^{-1}}{k^+(P^+-k^+)(P'^+-k^+)(P'^- - \frac{f_3}{P'^+-k^+} - \frac{f_1}{k^+})}\,.$$

$$J_{\perp} = -\pi\imath\int\frac{d^2k_\perp}{(2\pi)^4}\int_0^{P^+}dk^+\frac{(2k_\perp - P'_\perp - P_\perp)(P'^- - \frac{f_1}{k^+} - \frac{f_3}{P'^+-k^+})^{-1}}{k^+(P^+-k^+)(P'^+-k^+)(P^- - f_1 - \frac{f_2}{P^+-k^+})} \quad -$$

$$\pi\imath\int\frac{d^2k_\perp}{(2\pi)^4}\int_{P^+}^{P'^+}dk^+\frac{(2k_\perp - P'_\perp - P_\perp)(P^- - P'^- + \frac{f_3}{P'^+-k^+} - \frac{f_2}{P^+-k^+})^{-1}}{k^+(P^+-k^+)(P'^+-k^+)(P'^- - \frac{f3}{P'^+-k^+} - \frac{f_1}{k^+})}$$

- q^μ as, $q^\mu = P'^\mu - P^\mu + \bar{k}^\mu - \bar{k}^\mu$

and in the interval (i) $\bar{k}^- = \frac{f_1}{k^+}$, $\bar{k}^+$ e $\bar{k}_\perp = k_\perp$.

$$[(P-\bar{k})^2 - m^2] - [(P' - \bar{k})^2 - m^2]$$
$$= (P^+ - k^+)[P^- - \frac{f_1}{k^+} - \frac{f_2}{P^+ - k^+}] - (P'^+ - k^+)[P'^- - \frac{f_1}{k^+} - \frac{f_3}{P'^+ - k^+}]$$

- **Final for interval (i)**

$$I_i = -\pi\imath \int \frac{d^2k_\perp}{(2\pi)^4} \int_0^{P^+} dk^+ \frac{1}{k^+(P'^+ - k^+)(P'^- - \frac{f_1}{k^+} - \frac{f_3}{P'^+ - k^+})} +$$
$$\pi\imath \int \frac{d^2k_\perp}{(2\pi)^4)} \int_0^{P^+} dk^+ \frac{1}{k^+(P^+ - k^+)(P^- - \frac{f_1}{k^+} - \frac{f_2}{P^+ - k^+})}$$

- **Interval (ii)** $\bar{k}'^{-} = P'^{-} - \frac{f_3}{P'^{+}-k^{+}}$, $\bar{k}'^{+} = k^{+}$ and $\bar{k}'_{\perp} = k_{\perp}$

$$q^{\mu} = (P'^{\mu} - P^{\mu}) = (P' - \bar{k}')^{\mu} - (P - \bar{k}')^{\mu}$$

- And with on-shell condition

$$(P'^{\mu} - \bar{k}'^{\mu})^2 - m^2 = 0$$

- We have below,

$$\begin{aligned} q^{\mu}.(2\bar{k'}^{\mu} - P^{\mu} - P'^{\mu}) &= \\ -(P'^{\mu} - \bar{k'}^{\mu} - P^{\mu} + \bar{k'}^{\mu})(P'^{\mu} - \bar{k'}^{\mu} + P^{\mu} - \bar{k'}^{\mu}) &= \\ -((P' - \bar{k}')^2 - m^2) + ((P - \bar{k}')^2 - m^2) &= \\ -(P - \bar{k}')^2 + m^2 \end{aligned}$$

- With Ligh-front coordinates,

$$(P - \bar{k}')^2 - m^2 = (P^{+} - k^{+})[P^{-} - \frac{f_2}{P^{+} - k^{+}} - P'^{-} + \frac{f_3}{P'^{+} - k^{+}}]$$

- **Final results to interval (ii)**

$$I_{ii} = -\pi \imath \int \frac{d^2k_\perp}{(2\pi)^4} \int_{P^+}^{P'^+} dk^+ \frac{1}{k^+(P'^+ - k^+)(P'^- - \frac{f_1}{k^+} - \frac{f_3}{(P'^+ - k^+)})}$$

- **The sum (i) +(ii) is**

$$I = I_i + I_{ii} = -\pi \imath \int \frac{d^2k_\perp}{(2\pi)^4} \int_{0}^{P'^+} dk^+ \frac{1}{k^+(P'^+ - k^+)(P'^- - \frac{f_1}{k^+} - \frac{f_3}{P'^+ - k^+})}$$
$$+\pi \imath \int \frac{d^2k_\perp}{(2\pi)^4} \int_{0}^{P^+} dk^+ \frac{1}{k^+(P^+ - k^+)(P^- - \frac{f_1}{k^+} - \frac{f_3}{P^+ - k^+})}$$
$$-\pi \imath \int \frac{d^2k_\perp}{(2\pi)^4} \int_{P^+}^{P'^+} dk^+ \frac{1}{k^+(P'^+ - k^+)(P'^- - \frac{f_1}{k^+} - \frac{f_3}{P'^+ - k^+})}$$

- Working more, we have the final Ward-Takahashi ,

$$
\begin{aligned}
I_i + I_{ii} = I = \quad & -\pi \imath \int \frac{d^2 k_\perp}{(2\pi)^4} \int_0^1 dx \frac{1}{x(1-x)} \frac{1}{(P'^2 - M_0'^2)} + \\
& \pi \imath \int \frac{d^2 k_\perp}{(2\pi)^4} \int_0^1 dx \frac{1}{x(1-x)} \frac{1}{(P^2 - M_0^2)} \\
& = \pi \imath \int \frac{d^2 k_\perp}{(2\pi)^4} \int_0^1 dx \frac{1}{x(1-x)} \frac{(P^2 - P'^2)}{(P'^2 - M_0'^2)(P^2 - M_0^2)}
\end{aligned}
$$

With the conditions, $P'^2 = P^2 = M_b^2$.

If zero-modes, or non-valence contribuitions, is not included with Light-front calculations, WT is no satisfied!!

- **Some works**

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