Conformal invariance of TMD rapidity evolution

Giovanni Antonio Chirilli

University of Regensburg

Light Cone 2019
Ecole Polytechnique, Palaiseau, France 16 - 20 September, 2019

based on arXiv:1905.09144: published in PRD rapid communications

Outline

- Motivation.
- Reminder of high-energy OPE for DIS
  - BK equation and $SL(2, C)$ Möbius group.
- Rapidity factorization for particle production in hadron-hadron collisions.
- Conformal invariance of TMD Operators.
- Conformal TMD Rapidity evolution equation in Sudakov region.
- Conclusions and outlook.

Motivation

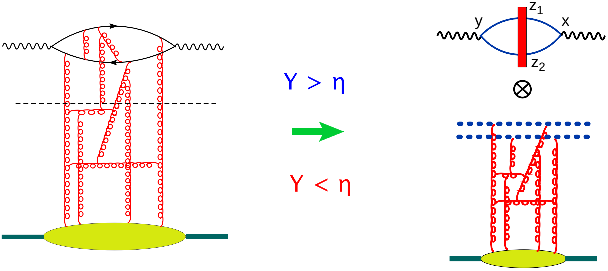
- TMD evolution equation are analyzed by different methods at moderate x_B, CSS and SCET, and at small-x_B resulting in different evolution equations.
- At the future Electron Ion Collider, TMD will be probed from low to high x_B. It is then necessary to develop a formalism which is valid in both limits.
- In the region of $x_B \sim 1$ TMD analysis is performed with a combination of UV and rapidity cutoff which gives two evolution equation in μ^2 and ζ (related to rapidity). Such evolution equations are known at two and three-loop, but their relation to the conformal properties of TMD is not known.

Motivation

- I. Balitsky and A. Tarasov (2016): Evolution equation for gluon TMD valid for all x_B and all $k_\perp$. Result is complicated and not unique. Conformal Invariance may help.

$$
\begin{aligned}
\frac{d}{d\ln\sigma}\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,y_\perp) \;=\; & -\alpha_s \mathrm{Tr}\Big\{\int d\!\!\!{}^-{}^2 k_\perp (x_\perp|\Big\{U^\dagger\frac{1}{\sigma\beta_B s+p_\perp^2}(Uk_k+p_kU)\frac{\sigma\beta_B s g_{\mu i}-2k_\mu^\perp k_i}{\sigma\beta_B s+k_\perp^2} \\
& -2k_\mu^\perp g_{ik}U^\dagger\frac{1}{\sigma\beta_B s+p_\perp^2}U-2g_{\mu k}U^\dagger\frac{p_i}{\sigma\beta_B s+p_\perp^2}U+\frac{2k_\mu^\perp}{k_\perp^2}g_{ik}\Big\}\tilde{\mathcal{F}}^k\big(\beta_B+\frac{k_\perp^2}{\sigma s}\big)|k_\perp) \\
& \times\,(k_\perp|\mathcal{F}^l\big(\beta_B+\frac{k_\perp^2}{\sigma s}\big)\Big\{\frac{\sigma\beta_B s\delta_j^\mu-2k_\perp^\mu k_j}{\sigma\beta_B s+k_\perp^2}(k_lU^\dagger+U^\dagger p_l)\frac{1}{\sigma\beta_B s+p_\perp^2}U \\
& -2k_\perp^\mu g_{jl}U^\dagger\frac{1}{\sigma\beta_B s+p_\perp^2}U-\;2\delta_l^\mu U^\dagger\frac{p_j}{\sigma\beta_B s+p_\perp^2}U+2g_{jl}\frac{k_\perp^\mu}{k_\perp^2}\Big\}|y_\perp) \\
& +2\tilde{\mathcal{F}}_i(\beta_B,x_\perp)(y_\perp|\frac{p^m}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l+U_l)(2\delta_m^k\delta_j^l-g_{jm}g^{kl})U^\dagger\frac{1}{\sigma\beta_B s-p_\perp^2+i\epsilon}U \\
& \qquad +\mathcal{F}_j(\beta_B)\frac{\sigma\beta_B s}{p_\perp^2(\sigma\beta_B s-p_\perp^2+i\epsilon)}|y_\perp) \\
& +2(x_\perp|-U^\dagger\frac{1}{\sigma\beta_B s-p_\perp^2-i\epsilon}U(2\delta_i^k\delta_m^l-g_{im}g^{kl})(i\partial_k-U_k)\tilde{\mathcal{F}}_l(\beta_B)\frac{p^m}{p_\perp^2} \\
& \qquad +\tilde{\mathcal{F}}_i(\beta_B)\frac{\sigma\beta_B s}{p_\perp^2(\sigma\beta_B s-p_\perp^2-i\epsilon)}|x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\Big\}\;+\;O(\alpha_s^2)
\end{aligned}
$$

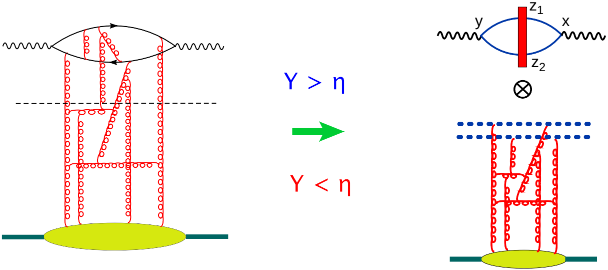
DIS amplitude is factorized in rapidity: η



$|B\rangle$ is the target state.

$$\langle B|T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|B\rangle = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle B|\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|B\rangle + \dots$$

High-Energy Operator Product Expansion



$$\langle B|T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|B\rangle = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle B|\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|B\rangle + \dots$$

Rapidity Regularization:

$$A^\eta_\mu(x) = \int đ^4k\,\theta(e^\eta - |k^+|)e^{-ik\cdot x}A_\mu(k)$$

$đ^n k \equiv \frac{d^n k}{(2\pi)^n}$

Non-linear evolution equation: Balitsky-Kovchegov equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

- LLA for DIS in pQCD $\Rightarrow$ BFKL
  - (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$): describes proliferation of gluons.

- LLA for DIS in semi-classical-QCD $\Rightarrow$ BK eqn
  - background field method: describes recombination process.

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;=\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+ \, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \; = \; \mathrm{Pexp}\left\{ ig\int_{-\infty}^{\infty} dx^+ \, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp / x_\perp^2$ and $x^+ \to x^+ / x_\perp^2$

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;=\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+\, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp/x_\perp^2$ and $x^+ \to x^+/x_\perp^2 \Rightarrow$

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;\to\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2}\, A_+\left(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2}\right)\right\} \;=\; [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;=\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+ \, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp / x_\perp^2$ and $x^+ \to x^+/x_\perp^2 \Rightarrow$

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;\to\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2} \, A_+(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2})\right\} \;=\; [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp / x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2 z}{\cancel{z^4}} \frac{(x-y)^2 \; \cancel{z^4}}{(x-z)^2 (z-y)^2} [\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}]$$

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z,\bar{z})] = z^2\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_+, \hat{U}(z,\bar{z})] = -\partial_z\hat{U}(z,\bar{z})$$

$z \equiv z^1 + iz^2, \quad \bar{z} \equiv z^1 - iz^2, \quad U(z_\perp) = U(z,\bar{z})$

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z,\bar{z})] = z^2\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_+, \hat{U}(z,\bar{z})] = -\partial_z\hat{U}(z,\bar{z})$$

$z \equiv z^1 + iz^2, \quad \bar{z} \equiv z^1 - iz^2, \quad U(z_\perp) = U(z,\bar{z})$

Conformal invariance of the evolution kernel

$$\frac{d}{d\eta}[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2}\int dz\, K(x,y,z)[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}\mathrm{Tr}\{U_x U_y^\dagger\}]$$

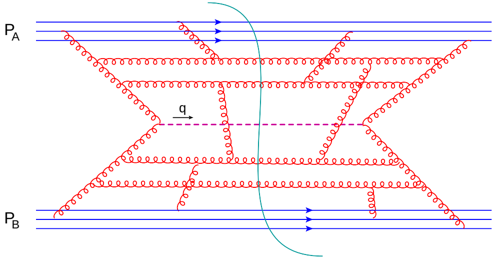
$$\Rightarrow \left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\right]K(x,y,z) = 0$$

Light cone vectors n, n' and Sudakov variables:

$$n^2 = n'^2 = 0\,, \quad n \cdot n' = 1\,, \qquad P_A = \sqrt{\frac{s}{2}}\,n + \frac{P_A^2}{\sqrt{2s}}n'\,, \quad P_B = \sqrt{\frac{s}{2}}\,n' + \frac{P_A^2}{\sqrt{2s}}n$$

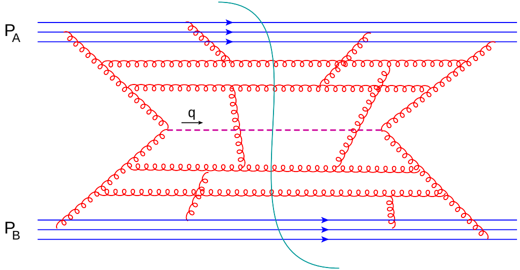
$$x^+ = \sqrt{\frac{2}{s}}\,p_2 \cdot x\,, \quad x^- = \sqrt{\frac{2}{s}}\,p_1 \cdot x\,, \qquad \frac{x^0 \pm x^+}{\sqrt{2}}$$

$$k^\mu = \alpha p_1^\mu + \beta p_2^\mu + k_\perp = \sqrt{\frac{2}{s}}\,k^+ p_1^\mu + \sqrt{\frac{2}{s}}\,k^- p_2^\mu + k_\perp^\mu$$



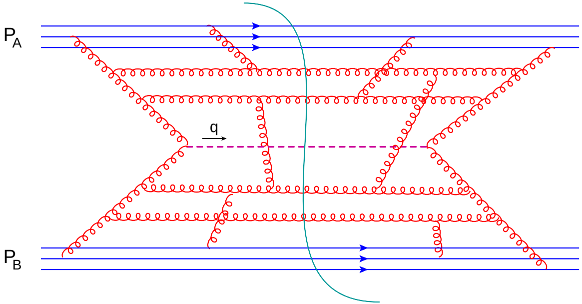
e.g. production of Higgs particle

Particle production in hadron-hadron collisions



- typical TMD region: $s \sim q^2 = m_X^2 \gg q_\perp^2 \sim 1$Gev
- Sudakov region: $s \sim q^2 = m_X^2 \gg q_\perp^2 \gg 1$Gev
- small-x region : $s \gg q^2 \sim q_\perp^2 \gg m_N^2$

TMD factorization



$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_f \int d^2b_\perp e^{i(q,b)_\perp}\mathcal{D}_{f/A}(x_A, b_\perp, \eta)\mathcal{D}_{f/B}(x_B, b_\perp, \eta)\sigma(ff \to H)$$

$$+ \text{ power corrections } + \text{ Y} - \text{terms}$$

See Collins' book for references

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_f \int d^2b_\perp e^{i(q,b)_\perp} \mathcal{D}_{f/A}(x_A, b_\perp, \eta)\mathcal{D}_{f/B}(x_B, b_\perp, \eta)\sigma(ff \to H)$$
$$+ \text{ power corrections } + \text{Y}-\text{terms}$$

- rapidity: $\eta = \frac{1}{2}\ln\frac{q^+}{q^-}$.
- $\mathcal{D}_{f/A}$: TMD density of parton f in hadron A.
- $\sigma(ff \to H)$: cross section of production of particle H of invariant mass $m_H^2 = Q^2$ from two partons scattering.
- Power corrections: $\frac{q_\perp^2}{Q^2}$.
- Y-terms: for $q_\perp^2 \sim q^2$ allow transition to collinear factorization formula.

TMD factorization in coordinate space

$$\langle p_A, p_B | g^2 F^a_{\mu\nu} F^{a\mu\nu}(z_1) g^2 F^b_{\lambda\rho} F^{b\lambda\rho}(z_2) | p_A, p_B \rangle$$
$$= \frac{1}{N_c^2 - 1} \langle p_A | \tilde{\mathcal{O}}_{ij}(z_1^-, z_{1\perp}; z_2^-, z_{2\perp}) | p_A \rangle^{\sigma_A} \langle p_B | \mathcal{O}^{ij}(z_1^+, z_{1\perp}; z_2^+, z_{2\perp}) | p_B \rangle^{\sigma_B} + \ldots$$

Gluon TMD operator

$$\mathcal{O}_{ij}(z_1^+, z_{1\perp}; z_2^+, z_{2\perp}) = \mathcal{F}^a_i(z_1)[z_1 - \infty n, z_2 - \infty n]^{ab} \mathcal{F}^b_j(z_2)\Big|_{z_1^- = z_2^- = 0},$$

$$\tilde{\mathcal{O}}_{ij}(z_1^-, z_{1\perp}; z_2^-, z_{2\perp}) = \mathcal{F}^a_i(z_1)[z_1 - \infty n', z_2 - \infty n']^{ab} \mathcal{F}^b_j(z_2)\Big|_{z_1^+ = z_2^+ = 0},$$

$$\left(\mathcal{F}^{i,a}(z_\perp, z^+)\right)^\sigma \equiv g F^{-i,m}(z) \left[\mathrm{P} e^{ig \int_{-\infty}^{z^+} dz^+ A^{-,\sigma}(up_1 + x_\perp)} \right]^{ma},$$

TMD factorization in coordinate space

Gluon TMD operator

$$\mathcal{O}_{ij}(z_1^+, z_{1\perp}; z_2^+, z_{2\perp}) = \mathcal{F}_i^a(z_1)[z_1 - \infty n, z_2 - \infty n]^{ab} \mathcal{F}_j^b(z_2)\Big|_{z_1^- = z_2^- = 0},$$

$$\tilde{\mathcal{O}}_{ij}(z_1^-, z_{1\perp}; z_2^-, z_{2\perp}) = \mathcal{F}_i^a(z_1)[z_1 - \infty n', z_2 - \infty n']^{ab} \mathcal{F}_j^b(z_2)\Big|_{z_1^+ = z_2^+ = 0},$$

$$\left(\mathcal{F}^{i,a}(z_\perp, z^+)\right)^\sigma \equiv gF^{-i,m}(z)\left[\mathrm{P}e^{ig\int_{-\infty}^{z^+} dz^+ A^{-,\sigma}(up_1 + x_\perp)}\right]^{ma},$$

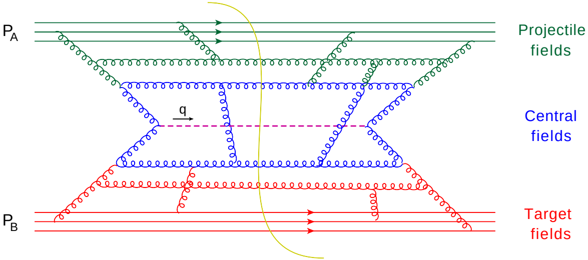
$$A_\mu^\sigma(x) = \int \frac{d^4k}{16\pi^4}\, \theta\Big(\frac{\sigma\sqrt{2}}{z_{12_\perp}} - |k^+|\Big) e^{-ik\cdot x} A_\mu(k)$$

$\frac{\sigma\sqrt{2}}{z_{12_\perp}}$ cutoff preserving conformal invariance

$$[x, y] \equiv \mathrm{P}e^{ig\int du (x-y)^\mu A_\mu(ux + (1-u)y)}$$

Rapidity Factorization

I. Balitsky and G.A.C. (2008),
I. Balitsky and A. Tarasov (2015)



- Projectile fields: $k^- < \sigma_a \quad \Rightarrow \quad$ Projectile TMD
- Central fields: coefficient functions
- Target fields: $k^+ < \sigma_b \quad \Rightarrow \quad$ Target TMD

Conformal group has 15 generators: Poincare + Dilatation + Special conformal trasformation (inversion + shift + inversion)

$$
\begin{aligned}
&i[M_{\mu\nu}, M_{\alpha\beta}] = g_{\mu\alpha}M_{\nu\beta} + g_{\nu\beta}M_{\mu\alpha} - g_{\mu\beta}M_{\nu\alpha} - g_{\nu\alpha}M_{\mu\beta} \\
&i[M_{\alpha\beta}, P_{\mu}] = g_{\alpha\mu}P_{\beta} - g_{\beta\mu}P_{\alpha} \\
&i[M_{\alpha\beta}, K_{\mu}] = g_{\alpha\mu}K_{\beta} - g_{\beta\mu}K_{\alpha} \\
&i[D, P_{\mu}] = P_{\mu}, \quad i[D, K_{\mu}] = -K_{\mu}, \quad i[K_{\mu}, P_{\nu}] = 2(g_{\mu\nu}D + M_{\mu\nu})
\end{aligned}
$$

Action on scalar field of canonical dimension Δ;

$$
\begin{aligned}
&i[D, \Phi(x)] = (x^{\alpha}\partial_{\alpha} + \Delta)\Phi(x) \\
&i[K^{\mu}, \Phi(x)] = (2x^{\mu}x^{\alpha}\partial_{\alpha} - x^2\partial^{\mu} + 2\Delta x^{\mu})\Phi(x)
\end{aligned}
$$

quantum correction: $\Delta \to \Delta + \text{anomalous}$

Conformal invariance of TMD operator

Conformal $SO(2,4)$ group has 15 generators

TMD operator transform covariantly under 11 generators:

TMD Conformal group

$$P^i, \;\; P^-, \;\; M^{12}, \;\; M^{-i}, \;\; D, \;\; K^i, \;\; K^-, \;\; M^{-+}$$

Generators P^+, K^+, M^{+i} do not preserve the form of $\mathcal{F}^{-j}$.

Action of conformal generators on TMD operators

$$-iP^i\mathcal{F}^{-j}(x^+,x_\perp) = \partial^i\mathcal{F}^{-j}(x^+,x_\perp)\,, \qquad -iP^-\mathcal{F}^{-j}(x^+,x_\perp) = \frac{\partial}{\partial x^+}\mathcal{F}^{-j}(x^+,x_\perp)\,,$$

$$-iM^{-i}\mathcal{F}^{-j}(x^+,x_\perp) = -x^i\frac{\partial}{\partial x^+}\mathcal{F}^{-j}(x^+,x_\perp)\,,$$

$$-iD\mathcal{F}^{-i}(x^+,x_\perp) = (x^+\frac{\partial}{\partial x^+} + x^k\frac{\partial}{\partial x^k} + 2)\mathcal{F}^{-i}(x^+,x_\perp)\,,$$

$$-iM^{ij}\mathcal{F}^{-k}(x^+,x_\perp) = (x^i\partial^j - x^j\partial^i)\mathcal{F}^{-k}(x^+,x_\perp) + g^{ik}\mathcal{F}^{-j}(x^+,x_\perp) - g^{jk}\mathcal{F}^{-i}(x^+,x_\perp)\,,$$

$$-K^i\mathcal{F}^{-j}(x^+,x_\perp) = 2x^i(x^+\frac{\partial}{\partial x^+} + x^k\frac{\partial}{\partial x^k} + 2)\mathcal{F}^{-j}(x^+,x_\perp) + x_\perp^2\frac{\partial}{\partial x^i}\mathcal{F}^{-j}(x^+,x_\perp)$$
$$-2x^j\mathcal{F}^{-i}(x^+,x_\perp) + 2g^{ij}x_i\mathcal{F}^{-i}(x^+,x_\perp)\,,$$

$$-iK^-\mathcal{F}^{-j}(x^+,x_\perp) = x_\perp^2\frac{\partial}{\partial x^+}\mathcal{F}^{-j}(x^+,x_\perp)\,,$$

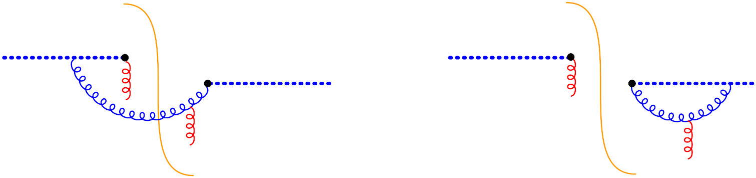
$$-iM^{+-}\mathcal{F}^{-i}(x^+,x_\perp) = (x^+\frac{\partial}{\partial x^+} + 1)\mathcal{F}^{-i}(x^+,x_\perp)$$

Rapidity evolution at leading in Sudakov region

Sudakov region: $s \sim q^2 = m_X^2 \gg q_\perp^2 \gg 1$Gev

Sudakov region in coord. space: $z_{12\parallel}^2 \equiv z_{12}^- z_{12}^+ \gg z_{12\perp}^2$

Typical LO diagrams



The approximation to calculate these diagrams is: $k^+ \gg \frac{z_{12}^+}{z_{12\perp}^2}$

$$\mathcal{O}^{\sigma_2}(z_1^+, z_2^+) \;=\; \frac{\alpha_s N_c}{2\pi}\int_{\frac{\sigma_1\sqrt{2}}{z_{12\perp}}}^{\frac{\sigma_2\sqrt{2}}{z_{12\perp}}}\frac{dk^+}{k^+}\, K\mathcal{O}^{\sigma_1}(z_1^+, z_2^+)$$

where the kernel K is given by

$$K\mathcal{O}(z_1^+, z_2^+) \;=\; \mathcal{O}(z_1^+, z_2^+)\int_{-\infty}^{z_1^+}\frac{dz^+}{z_2^+ - z^+}e^{-i\frac{z_{12\perp}\sigma}{\sqrt{2}(z_2-z)^+}} + \mathcal{O}(z_1^+, z_2^+)\int_{-\infty}^{z_2^+}\frac{dz^+}{z_1^+ - z^+}e^{i\frac{z_{12\perp}\sigma}{\sqrt{2}(z_1-z)^+}}$$
$$-\int_{-\infty}^{z_1^+}dz^+\frac{\mathcal{O}(z_1^+, z_2^+) - \mathcal{O}(z^+, z_2^+)}{z_1^+ - z^+} - \int_{-\infty}^{z_2^+}dz^+\frac{\mathcal{O}(z_1^+, z_2^+) - \mathcal{O}(z_1^+, z^+)}{z_2^+ - z^+}$$

Solution of the evolution equation

To solve the evolution equation, perform Fourier transform

$$Ke^{-ik^- z_1^+ + ik'^- z_2^+} = \Big[- 2\ln\sigma z_{12_\perp} - \ln(ik^-)$$
$$- \ln(-ik'^-) + \ln 2 - 4\gamma_E + O\Big(\frac{z_{12}^+}{z_{12_\perp}\sigma}\Big)\Big] e^{-ik^- z_1^+ + ik'^- z_2^+}$$

Result ($\bar\alpha_s = \frac{\alpha_s N_c}{4\pi}$)

I. Balitsky and G.A.C (2019)

$$\mathcal{O}^{\sigma_2}(z_1^+, z_2^+) = e^{-2\bar\alpha_s \ln\frac{\sigma_2}{\sigma_1}\left[\ln\sigma_1\sigma_2 + 4\gamma_E - \ln 2\right]} \int dz_1'^+ dz_2'^+ \, \mathcal{O}^{\sigma_1}(z_1'^+, z_2'^+)\, z_{12_\perp}^{-2\bar\alpha_s \ln\frac{\sigma_2}{\sigma_1}}$$
$$\times \frac{1}{4\pi^2}\left[\frac{i\Gamma\big(1 - 2\bar\alpha_s\ln\frac{\sigma_2}{\sigma_1}\big)}{(z_1^+ - z_1'^+ + i\epsilon)^{1-2\bar\alpha_s\ln\frac{\sigma_2}{\sigma_1}}} + c.c.\right]\left[\frac{i\Gamma\big(1 - 2\bar\alpha_s\ln\frac{\sigma_2}{\sigma_1}\big)}{(z_2^+ - z_2'^+ + i\epsilon)^{1-2\bar\alpha_s\ln\frac{\sigma_2}{\sigma_1}}} + c.c.\right]$$

Result transform covariantly under the TMD-conformal group generators except the Lorentz boost M^{+-} which is the generator of the evolution equation: The Lorentz boost in z direction changes the cutoffs for the evolution.

Conformal invariance of the TMD matrix element

Sudakov-region result is applicable in the region between:

$$\sigma_1 = \sigma_B = \frac{z_{12\perp}}{z_{12}^{-}\sqrt{2}} \quad \text{and} \quad \sigma_1 = \frac{z_{12}^{+}\sqrt{2}}{z_{12\perp}}$$

Lorentz boost: $z^+ \to \lambda z^+, \qquad z^- \to \frac{1}{\lambda} z^-$

- $\langle p_A|\mathcal{O}|p_B\rangle \to \langle p_A|\mathcal{O}|p_B\rangle \exp\{4\lambda\bar{\alpha}_s \ln \frac{z^2_{12\parallel}}{z^2_{12_\perp}}\}$
- $\langle p_A|\tilde{\mathcal{O}}|p_B\rangle \to \langle p_A|\tilde{\mathcal{O}}|p_B\rangle \exp\{-4\lambda\bar{\alpha}_s \ln \frac{z^2_{12\parallel}}{z^2_{12_\perp}}\}$

So,

$$\langle p_A, p_B|g^2 F^a_{\mu\nu}F^{a\mu\nu}(z_1) g^2 F^b_{\lambda\rho}F^{b\lambda\rho}(z_2)|p_A, p_B\rangle$$
$$= \frac{1}{N_c^2 - 1}\langle p_A|\tilde{\mathcal{O}}_{ij}(z_1^-, z_{1_\perp}; z_2^-, z_{2_\perp})|p_A\rangle^{\sigma_A}\langle p_B|\mathcal{O}^{ij}(z_1^+, z_{1_\perp}; z_2^+, z_{2_\perp})|p_B\rangle^{\sigma_B}$$

is invariant

- The conformal TMD group has been obtained: it is made out of 11 generators of the full conformal group.
- We obtained conformal evolution of gluon and quark ($N_c \to C_F$) TMDs in the Sudakov region.

Outlook

- Conformal properties of TMD evolution in the small-x region.
- Conformal evolution for all x_B.
- The plan is to perform the calculation for 6 point function similarly to the one done for the 4 point function in $\mathcal{N}$=4 and in in QCD.