Conformal invariance of TMD rapidity evolution

Giovanni Antonio Chirilli

University of Regensburg

Light Cone 2019
Ecole Polytechnique, Palaiseau, France 16 - 20 September, 2019

Outline

- Motivation.

- Reminder of high-energy OPE for DIS
  - BK equation and $SL(2, \mathbb{C})$ Möbius group.

- Rapidity factorization for particle production in hadron-hadron collisions.

- Conformal invariance of TMD Operators.

- Conformal TMD Rapidity evolution equation in Sudakov region.

- Conclusions and outlook.
TMD evolution equations are analyzed by different methods at moderate $x_B$, CSS and SCET, and at small-$x_B$ resulting in different evolution equations.

At the future Electron Ion Collider, TMD will be probed from low to high $x_B$. It is then necessary to develop a formalism which is valid in both limits.

In the region of $x_B \sim 1$ TMD analysis is performed with a combination of UV and rapidity cutoff which gives two evolution equations in $\mu^2$ and $\zeta$ (related to rapidity). Such evolution equations are known at two and three-loop, but their relation to the conformal properties of TMD is not known.
I. Balitsky and A. Tarasov (2016): Evolution equation for gluon TMD valid for all $x_B$ and all $k_\perp$. Result is complicated and not unique. Conformal Invariance may help.

$$\frac{d}{d \ln \sigma} \tilde{F}_i^a(\beta_B, x_\perp) F_j^a(\beta_B, y_\perp) = -\alpha_s \text{Tr} \left\{ \int d^2k_\perp (x_\perp | \left\{ U^\dagger \frac{1}{\sigma \beta_B s + p_\perp^2} (Uk_k + p_k U) \frac{\sigma \beta_B s g_{\mu i} - 2k_\mu^\perp k_i}{\sigma \beta_B s + k_\perp^2} ight. \right.$$ 

$$\left. - 2k_\mu^\perp g_{ik} U^\dagger \frac{1}{\sigma \beta_B s + p_\perp^2} U - 2g_{\mu k} U^\dagger \frac{p_i}{\sigma \beta_B s + p_\perp^2} U + \frac{2k_\mu^\perp}{k_\perp^2} g_{ik} \} \tilde{F}^k(\beta_B + \frac{k_\perp^2}{\sigma s}) |k_\perp) \right.$$ 

$$\times (k_\perp | F^l(\beta_B + \frac{k_\perp^2}{\sigma s}) \{ \frac{\sigma \beta_B s \delta^\mu_j - 2k_\perp^\mu k_j}{\sigma \beta_B s + k_\perp^2} (k_l U^\dagger + U^\dagger p_l) \frac{1}{\sigma \beta_B s + p_\perp^2} U$$ 

$$- 2k_\perp^\mu g_{jl} U^\dagger \frac{1}{\sigma \beta_B s + p_\perp^2} U - 2\delta^\mu_l U^\dagger \frac{p_j}{\sigma \beta_B s + p_\perp^2} U + 2g_{jl} \frac{k_\mu^\perp}{k_\perp^2} \} |y_\perp)$$ 

$$+ 2 \tilde{F}_i(\beta_B, x_\perp) (y_\perp | \frac{p^m}{p_\perp^2} F_k(\beta_B) (i \partial_l + U_l) (2\delta^k_m \delta^l_j - g_{jm} g^{kl}) U^\dagger \frac{1}{\sigma \beta_B s - p_\perp^2 + i\epsilon} U$$ 

$$+ F_j(\beta_B) \frac{\sigma \beta_B s}{p_\perp^2 (\sigma \beta_B s - p_\perp^2 + i\epsilon)} |y_\perp)$$ 

$$+ 2 (x_\perp | - U^\dagger \frac{1}{\sigma \beta_B s - p_\perp^2 - i\epsilon} U (2\delta^k_m \delta^l_j - g_{jm} g^{kl}) (i \partial_k - U_k) \tilde{F}_l(\beta_B) \frac{p^m}{p_\perp^2}$$ 

$$+ \tilde{F}_i(\beta_B) \frac{\sigma \beta_B s}{p_\perp^2 (\sigma \beta_B s - p_\perp^2 - i\epsilon)} |x_\perp) F_j(\beta_B, y_\perp) \} + O(\alpha_s^2) \right\}.$$
High-Energy Operator Product Expansion

DIS amplitude is factorized in rapidity: $\eta$

$|B\rangle$ is the target state.

$$\langle B| T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|B\rangle = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} \, I_{\mu\nu}^{LO}(x, y; z_1, z_2) \langle B| \text{tr}\{\hat{U}_\eta^{z_1} \hat{U}^{+\eta}_{z_2}\}|B\rangle + \ldots$$
\[ \langle B | T \{ \hat{j}_\mu(x) \hat{j}_\nu(y) \} | B \rangle = \int \frac{d^2 z_1 d^2 z_2}{z_{12}^4} \mathcal{I}^{LO}_{\mu\nu}(x, y; z_1, z_2) \langle B | \text{tr} \{ \hat{U}^\eta_{z_1} \hat{U}^{\dagger \eta}_{z_2} \} | B \rangle + \ldots \]

Rapidity Regularization:

\[ A^\eta_\mu(x) = \int \hat{d}^4 k \theta(e^\eta - |k^+|) e^{-i k \cdot x} A_\mu(k) \]

\[ \hat{d}^n k \equiv \frac{d^n k}{(2\pi)^n} \]
Non-linear evolution equation: Balitsky-Kovchegov equation

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\} \]

\[ \frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z (x - y)^2}{(x - z)^2(y - z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\} \]

- LLA for DIS in pQCD ⇒ BFKL
  - (LLA: \( \alpha_s \ll 1, \alpha_s \eta \sim 1 \)): describes proliferation of gluons.

- LLA for DIS in semi-classical-QCD ⇒ BK eqn
  - background field method: describes recombination process.
Conformal invariance of the BK equation

Formally, a light-like Wilson line

\[
[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} dx^+ A_+(x^+, x_\perp) \right\}
\]

is invariant under inversion (with respect to the point with \( x^- = 0 \)).
Formally, a light-like Wilson line

\[ \left[ \infty p_1 + x_\perp, -\infty p_1 + x_\perp \right] = \text{Pexp}\left\{ ig \int_{-\infty}^{\infty} dx^+ A_+ (x^+, x_\perp) \right\} \]

is invariant under inversion (with respect to the point with \( x^- = 0 \)).

Indeed,
\[(x^+, x_\perp)^2 = -x^2_\perp \Rightarrow \text{after the inversion } x_\perp \rightarrow x_\perp / x^2_\perp \text{ and } x^+ \rightarrow x^+ / x^2_\perp \]
Conformal invariance of the BK equation

Formally, a light-like Wilson line

\[ \left[ \infty p_1 + x_\perp, -\infty p_1 + x_\perp \right] = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} dx^+ A_+(x^+, x_\perp) \right\} \]

is invariant under inversion (with respect to the point with \( x^- = 0 \)).

Indeed,
\((x^+, x_\perp)^2 = -x^\perp_- \Rightarrow \text{after the inversion } x_\perp \rightarrow x_\perp / x^\perp_- \text{ and } x^+ \rightarrow x^+ / x^\perp_- \Rightarrow \)

\[ \left[ \infty p_1 + x_\perp, -\infty p_1 + x_\perp \right] \rightarrow \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} \frac{dx^+}{x^\perp_-} A_+ \left( \frac{x^+}{x^\perp_-}, \frac{x_\perp}{x^\perp_-} \right) \right\} = \left[ \infty p_1 + \frac{x_\perp}{x^\perp_-}, -\infty p_1 + \frac{x_\perp}{x^\perp_-} \right] \]
Conformal invariance of the BK equation

Formally, a light-like Wilson line

\[
[\infty p_1 + x_-, -\infty p_1 + x_-] = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} dx^+ A_+(x^+, x_-) \right\}
\]

is invariant under inversion (with respect to the point with \( x^- = 0 \)).

Indeed,

\[(x^+, x_-)^2 = -x_-^2 \Rightarrow \text{after the inversion } x_- \rightarrow x_-/x_-^2 \text{ and } x^+ \rightarrow x^+/x_-^2 \Rightarrow\]

\[
[\infty p_1 + x_-, -\infty p_1 + x_-] \rightarrow \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} \frac{dx^+}{x_-^2} A_+\left(\frac{x^+}{x_-^2}, \frac{x_-}{x_-^2}\right) \right\} = [\infty p_1 + \frac{x_-}{x_-^2}, -\infty p_1 + \frac{x_-}{x_-^2}]
\]

\(\Rightarrow\) The dipole kernel is invariant under the inversion \( V(x_-) = U(x_-/x_-^2) \)

\[
\frac{d}{d\eta} \text{Tr} \{ V_x V_y^\dagger \} = \frac{\alpha_s}{2\pi^2} \int \frac{d^2 z}{z^A} \frac{(x - y)^2 z^A}{(x - z)^2 (z - y)^2} \left[ \text{Tr} \{ V_x V_z^\dagger \} \text{Tr} \{ V_z V_y^\dagger \} - N_c \text{Tr} \{ V_x V_y^\dagger \} \right]
\]
Conformal invariance of the BK equation

**SL(2,C) for Wilson lines**

\[ \hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2) \]

\[ [\hat{S}_0, \hat{S}_{\pm}] = \pm \hat{S}_{\pm}, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0, \]

\[ [\hat{S}_-, \hat{U}(z, \bar{z})] = z^2 \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_0, \hat{U}(z, \bar{z})] = z \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_+, \hat{U}(z, \bar{z})] = -\partial_z \hat{U}(z, \bar{z}) \]

\[ z \equiv z^1 + iz^2, \quad \bar{z} \equiv z^1 - iz^2, \quad U(z_{\perp}) = U(z, \bar{z}) \]
Conformal invariance of the BK equation

**SL(2,C) for Wilson lines**

$$\hat{S}_- \equiv \frac{i}{2} (K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2} (D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2} (P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z, \bar{z})] = z^2 \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_0, \hat{U}(z, \bar{z})] = z \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_+, \hat{U}(z, \bar{z})] = -\partial_z \hat{U}(z, \bar{z})$$

$$z \equiv z^1 + iz^2, \quad \bar{z} \equiv z^1 - iz^2, \quad U(z_\perp) = U(z, \bar{z})$$

**Conformal invariance of the evolution kernel**

$$\frac{d}{d\eta} [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2} \int dz \ K(x, y, z) [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\}] \text{Tr}\{U_x U_y^\dagger\}]$$

$$\Rightarrow \left[ x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z} \right] K(x, y, z) = 0$$
Light cone vectors $n, n'$ and Sudakov variables:

\[ n^2 = n'^2 = 0, \quad n \cdot n' = 1, \quad P_A = \sqrt{\frac{s}{2}} n + \frac{P_A^2}{\sqrt{2s}} n', \quad P_B = \sqrt{\frac{s}{2}} n' + \frac{P_A^2}{\sqrt{2s}} n \]

\[ x^+ = \sqrt{\frac{2}{s}} p_2 \cdot x, \quad x^- = \sqrt{\frac{2}{s}} p_1 \cdot x, \quad \frac{x^0 \pm x^+}{\sqrt{2}} \]

\[ k^\mu = \alpha p_1^\mu + \beta p_2^\mu + k_\perp = \sqrt{\frac{2}{s}} k^+ p_1^\mu + \sqrt{\frac{2}{s}} k^- p_2^\mu + k_\perp \]

e.g. production of Higgs particle
- **typical TMD region**: \( s \sim q^2 = m_X^2 \gg q^2_{\perp} \sim 1\text{Gev} \)

- **Sudakov region**: \( s \sim q^2 = m_X^2 \gg q^2_{\perp} \gg 1\text{Gev} \)

- **small-x region**: \( s \gg q^2 \sim q^2_{\perp} \gg m_N^2 \)
\[
\frac{d\sigma}{d\eta d^2 q_\perp} = \sum_f \int d^2 b_\perp e^{i(q,b)_\perp} \mathcal{D}_{f/A}(x_A, b_\perp, \eta) \mathcal{D}_{f/B}(x_B, b_\perp, \eta) \sigma(ff \to H) \\
+ \text{power corrections} + Y - \text{terms}
\]

See Collins’ book for references
\[
\frac{d\sigma}{d\eta d^2q_\perp} = \sum_f \int d^2b_\perp e^{i(q,b)_\perp} D_{f/A}(x_A, b_\perp, \eta) D_{f/B}(x_B, b_\perp, \eta) \sigma(ff \rightarrow H) \\
+ \text{power corrections} + Y - \text{terms}
\]

- **rapidity:** \( \eta = \frac{1}{2} \ln \frac{q^+}{q^-} \).
- **\( D_{f/A} \):** TMD density of parton \( f \) in hadron \( A \).
- **\( \sigma(ff \rightarrow H) \):** cross section of production of particle \( H \) of invariant mass \( m_H^2 = Q^2 \) from two partons scattering.
- **Power corrections:** \( \frac{q_\perp^2}{Q^2} \).
- **Y-terms:** for \( q_\perp^2 \sim q^2 \) allow transition to collinear factorization formula.
\begin{align*}
\langle p_A, p_B | g^2 F^a_{\mu\nu} F^{a\mu\nu}(z_1) g^2 F^b_{\lambda\rho} F^{b\lambda\rho}(z_2) | p_A, p_B \rangle \\
= \frac{1}{N_c^2 - 1} \langle p_A | \tilde{O}_{ij}(z_1^-, z_1_\perp; z_2^-, z_2_\perp) | p_A \rangle^{\sigma_A} \langle p_B | O^{ij}(z_1^+, z_1_\perp; z_2^+, z_2_\perp) | p_B \rangle^{\sigma_B} + \ldots
\end{align*}

Gluon TMD operator

\begin{align*}
O_{ij}(z_1^+, z_1_\perp; z_2^+, z_2_\perp) &= F_i^a(z_1) [z_1 - \infty n, z_2 - \infty n]^{ab} F_j^b(z_2) \bigg|_{z_1^- = z_2^- = 0}, \\
\tilde{O}_{ij}(z_1^-, z_1_\perp; z_2^-, z_2_\perp) &= F_i^a(z_1) [z_1 - \infty n', z_2 - \infty n']^{ab} F_j^b(z_2) \bigg|_{z_1^+ = z_2^+ = 0},
\end{align*}

\begin{equation}
(F^{i,a}(z_\perp, z^+))^{\sigma} \equiv gF^{-i,m}(z) \left[ Pe^{ig \int_{-\infty}^{z^+} dz^+ A^{-,\sigma}(u p_1 + x_\perp)} \right]^{ma},
\end{equation}
Gluon TMD operator

\[ \mathcal{O}_{ij}(z_1^+, z_1^\perp; z_2^+, z_2^\perp) = \mathcal{F}_i^a(z_1)[z_1 - \infty n, z_2 - \infty n]^{ab} \mathcal{F}_j^b(z_2) \bigg|_{z_1^- = z_2^- = 0}, \]

\[ \tilde{\mathcal{O}}_{ij}(z_1^-, z_1^\perp; z_2^-, z_2^\perp) = \mathcal{F}_i^a(z_1)[z_1 - \infty n', z_2 - \infty n']^{ab} \mathcal{F}_j^b(z_2) \bigg|_{z_1^+ = z_2^+ = 0}, \]

\[ (\mathcal{F}^{i,a}(z_\perp, z^+))^\sigma \equiv gF^{-i,m}(z) \left[ P e^{ig \int_{-\infty}^{z^+} dz^+ A^{-, \sigma}(u p_1 + x_\perp)} \right]^{ma}, \]

\[ A_\mu^\sigma(x) = \int \frac{d^4k}{16\pi^4} \theta \left( \frac{\sigma \sqrt{2}}{z_{12}\perp} - |k^+| \right) e^{-ik \cdot x} A_\mu(k) \]

\[ \frac{\sigma \sqrt{2}}{z_{12\perp}} \text{ cutoff preserving conformal invariance} \]

\[ [x, y] \equiv P e^{ig \int du(x-y)^\mu A_\mu(ux+(1-u)y)} \]
Rapidity Factorization

I. Balitsky and G.A.C. (2008),
I. Balitsky and A. Tarasov (2015)

- **Projectile fields:** $k^- < \sigma_a \Rightarrow$ Projectile TMD
- **Central fields:** coefficient functions
- **Target fields:** $k^+ < \sigma_b \Rightarrow$ Target TMD
Conformal SO(2,4) group

Conformal group has 15 generators: Poincare + Dilatation + Special conformal transformation (inversion + shift + inversion)

\[ i[M_{\mu\nu}, M_{\alpha\beta}] = g_{\mu\alpha}M_{\nu\beta} + g_{\nu\beta}M_{\mu\alpha} - g_{\mu\beta}M_{\nu\alpha} - g_{\nu\alpha}M_{\mu\beta} \]
\[ i[M_{\alpha\beta}, P_\mu] = g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha \]
\[ i[M_{\alpha\beta}, K_\mu] = g_{\alpha\mu}K_\beta - g_{\beta\mu}K_\alpha \]
\[ i[D, P_\mu] = P_\mu, \quad i[D, K_\mu] = -K_\mu, \quad i[K_\mu, P_\nu] = 2(g_{\mu\nu}D + M_{\mu\nu}) \]

Action on scalar field of canonical dimension \( \Delta \):

\[ i[D, \Phi(x)] = (x^\alpha \partial_\alpha + \Delta)\Phi(x) \]
\[ i[K^\mu, \Phi(x)] = (2x^\mu x^\alpha \partial_\alpha - x^2 \partial^\mu + 2\Delta x^\mu)\Phi(x) \]

quantum correction: \( \Delta \rightarrow \Delta + \text{anomalous} \)
Conformal invariance of TMD operator

Conformal $SO(2, 4)$ group has 15 generators

TMD operator transform covariantly under 11 generators:

<table>
<thead>
<tr>
<th>TMD Conformal group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^i$, $P^-$, $M^{12}$, $M^{-i}$, $D$, $K^i$, $K^-$, $M^{-+}$</td>
</tr>
</tbody>
</table>

Generators $P^+, K^+, M^{+i}$ do not preserve the form of $\mathcal{F}^{-j}$. 
Conformal invariance of TMD operator

Action of conformal generators on TMD operators

\[-iP^i \mathcal{F}^{-j}(x^+, x_\perp) = \partial^i \mathcal{F}^{-j}(x^+, x_\perp), \quad -iP^- \mathcal{F}^{-j}(x^+, x_\perp) = \frac{\partial}{\partial x^+} \mathcal{F}^{-j}(x^+, x_\perp),\]

\[-iM^{-i} \mathcal{F}^{-j}(x^+, x_\perp) = -x^i \frac{\partial}{\partial x^+} \mathcal{F}^{-j}(x^+, x_\perp),\]

\[-iD \mathcal{F}^{-i}(x^+, x_\perp) = (x^+ \frac{\partial}{\partial x^+} + x^k \frac{\partial}{\partial x^k} + 2) \mathcal{F}^{-i}(x^+, x_\perp),\]

\[-iM^{ij} \mathcal{F}^{-k}(x^+, x_\perp) = (x^i \partial^j - x^j \partial^i) \mathcal{F}^{-k}(x^+, x_\perp) + g^{jk} \mathcal{F}^{-j}(x^+, x_\perp) - g^{jk} \mathcal{F}^{-i}(x^+, x_\perp),\]

\[-K^i \mathcal{F}^{-j}(x^+, x_\perp) = 2x^i(x^+ \frac{\partial}{\partial x^+} + x^k \frac{\partial}{\partial x^k} + 2) \mathcal{F}^{-j}(x^+, x_\perp) + x^1 \frac{\partial}{\partial x^i} \mathcal{F}^{-j}(x^+, x_\perp)
  - 2x^i \mathcal{F}^{-i}(x^+, x_\perp) + 2g^{ij} x_i \mathcal{F}^{-i}(x^+, x_\perp),\]

\[-iK^- \mathcal{F}^{-j}(x^+, x_\perp) = x_\perp^2 \frac{\partial}{\partial x^+} \mathcal{F}^{-j}(x^+, x_\perp),\]

\[-iM^{+-} \mathcal{F}^{-i}(x^+, x_\perp) = (x^+ \frac{\partial}{\partial x^+} + 1) \mathcal{F}^{-i}(x^+, x_\perp)\]
Rapidity evolution at leading in Sudakov region

Sudakov region: \( s \sim q^2 = m^2_x \gg q^2 \gg 1\text{Gev} \)

Sudakov region in coord. space: \( z_{12\parallel}^2 \equiv z_{12}^+ z_{12}^- \gg z_{12\perp}^2 \)

Typical LO diagrams

The approximation to calculate these diagrams is: \( k^+ \gg \frac{z_{12}^+}{z_{12\perp}} \)
Rapidity evolution at leading in Sudakov region

\[ \mathcal{O}^{\sigma_2}(z_1^+, z_2^+) = \frac{\alpha_s N_c}{2\pi} \frac{\sigma_2 \sqrt{2}}{z_{12}^\perp} \int \frac{dk^+}{k^+} K \mathcal{O}^{\sigma_1}(z_1^+, z_2^+) \]

where the kernel \( K \) is given by

\[ K \mathcal{O}(z_1^+, z_2^+) = \mathcal{O}(z_1^+, z_2^+) \int_{-\infty}^{z_1^+} \frac{dz^+}{z_2^+ - z^+} e^{-i \frac{z_{12}^\perp}{\sqrt{2}(z_2^- - z^-)}} + \mathcal{O}(z_1^+, z_2^+) \int_{-\infty}^{z_2^+} \frac{dz^+}{z_1^+ - z^+} e^{i \frac{z_{12}^\perp}{\sqrt{2}(z_1^- - z^-)}} \]

\[ -\int_{-\infty}^{z_1^+} dz^+ \frac{\mathcal{O}(z_1^+, z_2^+) - \mathcal{O}(z_1^+, z_2^+)}{z_1^+ - z^+} - \int_{-\infty}^{z_2^+} dz^+ \frac{\mathcal{O}(z_1^+, z_2^+) - \mathcal{O}(z_1^+, z_2^+)}{z_2^+ - z^+} \]
Solution of the evolution equation

To solve the evolution equation, perform Fourier transform

\[
Ke^{-ik^- z_1^+ + ik' ^- z_2^+} = \left[ -2 \ln \sigma z_{12 \perp} - \ln(ik^-) \right.
\]

\[
- \ln(-ik'^-) + \ln 2 - 4\gamma_E + O(\frac{z_{12}^+}{z_{12 \perp} \sigma}) e^{-ik^- z_1^+ + ik'^- z_2^+}
\]

Result \((\bar{\alpha}_s = \frac{\alpha_s N_c}{4\pi})\)


\[
O^{\sigma_2}(z_1^+, z_2^+ ) = e^{-2\bar{\alpha}_s \ln \frac{\sigma_2}{\sigma_1}} \left[ \ln \sigma_1 \sigma_2 + 4\gamma_E - \ln 2 \right] \int dz_1^+ dz_2^+ \ O^{\sigma_1}(z_1'^+, z_2'^+ ) \ z_{12 \perp}^{-2\bar{\alpha}_s \ln \frac{\sigma_2}{\sigma_1}} \times \frac{1}{4\pi^2} \left[ \frac{i\Gamma(1 - 2\bar{\alpha}_s \ln \frac{\sigma_2}{\sigma_1})}{(z_1^+ - z_1'^+ + i\epsilon)^{1 - 2\bar{\alpha}_s \ln \frac{\sigma_2}{\sigma_1}}} + c.c. \right] \left[ \frac{i\Gamma(1 - 2\bar{\alpha}_s \ln \frac{\sigma_2}{\sigma_1})}{(z_2^+ - z_2'^+ + i\epsilon)^{1 - 2\bar{\alpha}_s \ln \frac{\sigma_2}{\sigma_1}}} + c.c. \right]
\]

Result transform covariantly under the TMD-conformal group generators except the Lorentz boost \(M^{+-}\) which is the generator of the evolution equation: The Lorentz boost in \(z\) direction changes the cutoffs for the evolution.
Conformal invariance of the TMD matrix element

Sudakov-region result is applicable in the region between:

$$\sigma_1 = \sigma_B = \frac{z_{12\perp}}{z_{12\perp} \sqrt{2}} \quad \text{and} \quad \sigma_1 = \frac{z_{12\perp}^+ \sqrt{2}}{z_{12\perp}}$$

Lorentz boost: $$z^+ \rightarrow \lambda z^+, \quad z^- \rightarrow \frac{1}{\lambda} z^-$$

- $$\langle p_A | O | p_B \rangle \rightarrow \langle p_A | O | p_B \rangle \exp\{4 \lambda \bar{\alpha}_s \ln \frac{z_{12\parallel}^2}{z_{12\perp}^2}\}$$
- $$\langle p_A | \tilde{O} | p_B \rangle \rightarrow \langle p_A | \tilde{O} | p_B \rangle \exp\{-4 \lambda \bar{\alpha}_s \ln \frac{z_{12\parallel}^2}{z_{12\perp}^2}\}$$

So,

$$\langle p_A, p_B | g^2 F_{\mu\nu}^a F^{a\mu\nu}(z_1) g^2 F_{\lambda\rho}^{b\lambda\rho}(z_2) | p_A, p_B \rangle = \frac{1}{N_c^2 - 1} \langle p_A | \tilde{O}_{ij}(z_1^-; z_1\perp; z_2^-; z_2\perp) | p_A \rangle^{\sigma_A} \langle p_B | O^{ij}(z_1^+; z_1\perp; z_2^+; z_2\perp) | p_B \rangle^{\sigma_B}$$

is invariant
The conformal TMD group has been obtained: it is made out of 11 generators of the full conformal group.

We obtained conformal evolution of gluon and quark \((N_c \rightarrow C_F)\) TMDs in the Sudakov region.

**Outlook**

- Conformal properties of TMD evolution in the small-\(x\) region.
- Conformal evolution for all \(x_B\).
- The plan is to perform the calculation for 6 point function similarly to the one done for the 4 point function in \(\mathcal{N}=4\) and in QCD.