Monodromies and the structure of gauge and gravity amplitudes

Pierre Vanhove

ICHEP, July 22nd, 2010

based on the works

[0908.1923], [1004.1392], [0907.1425] and [1003.2403]

done in collaboration with

N.E.J. Bjerrum-Bohr, N. Berkovits, P. Damgaard, M.B. Green, J. Russo, T. Søndergaard

I just put 1.795372 and 2.204628 together.
And what does that mean?
Four!
(Doctor Who)

It is crucial for experimental and theoretical reasons to have efficient methods for evaluating amplitudes of physical processes in quantum field theory

- multilegs and multiloop amplitudes for LHC physics
- Quantum gravity: perturbative ultraviolet nature of $\mathcal{N} = 8$ supergravity

Unfortunately the number of individual Feynman graphs rises dramatically with the number of external legs or loop order, and tensor reduction methods increase the number of terms even more.

A huge number of cancellation are needed to get the result leading to

- instabilities due to large numerical cancellations in matrix elements
- obfuscation of the fundamental structure of the interactions: gauge invariance, ultraviolet divergences, infrared singularities, hidden symmetries

Explicit amplitude computations display rather unexpectedly simple structures

- One-loop multi-photon in QED and multi-graviton amplitudes in $\mathcal{N} = 8$ supergravity amplitudes share the *same* no-triangle property [Bjerrum-Bohr, Vanhove],[Badger,Bjerrum-Bohr,Vanhove]
- Simpler than expected subleading color contribution at one-loop for Higgs + 2 jets, $\bar{q}qggH$, process [Badger, Campbell, Ellis, Williams]
- Better UV behaviour of subleading color factor amplitudes at multi-loop order in $\mathcal{N} = 4$ SYM [Berkovits, Green, Russo, Vanhove]

All these simplifications hints on simple structures than the diagrammatics from Feynman rules suggest

Amplitudes from dressed φ^3 cubic vertices

[Bern, Carrasco, Johansson] ([Cf. Johansson's talk]) have proposed a parametrisation of gauge and gravity amplitude based

on n-points, L-loop skeleton graphs, $\gamma \in \Gamma_n^L$, with only cubic φ^3 vertices dressed by

- n_γ: lorentz factor build from external and loop momenta and polarisations
- c_γ: a color factor

- At tree level in gauge and gravity

$$\mathcal{A}_n^L = \sum_{\gamma \in \Gamma_n^L} \frac{n_\gamma c_\gamma}{\prod_{\alpha \in \gamma} p_\alpha^2}; \qquad \mathcal{M}_n^L = \sum_{\gamma \in \Gamma_n^L} \frac{n_\gamma \tilde{n}_\gamma}{\prod_{\alpha \in \gamma} p_\alpha^2}$$

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- c_γ: a color factor

- at loop order for $\mathcal{N} = 4$ super-Yang-Mills and $\mathcal{N} = 8$ supergravity

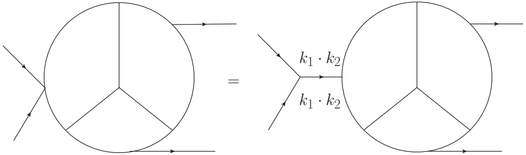
$$\mathcal{A}_n^L = \sum_{\gamma \in \Gamma_n^L} \int d^{3L-3}\ell_\alpha \frac{n_\gamma c_\gamma}{\prod_{\alpha\in\gamma} p_\alpha^2}; \qquad \mathcal{M}_n^L = \sum_{\gamma \in \Gamma_n^L} \int d^{3L-3}\ell_\alpha \frac{n_\gamma \tilde{n}_\gamma}{\prod_{\alpha\in\gamma} p_\alpha^2}$$

Amplitudes from dressed φ^3 cubic vertices

Gauge invariance, supersymmetry, crossing symmetry of the amplitudes requires *contact terms* given by higher point vertices

Such contact terms are resolved by inserting appropriate factors

$$1 = \frac{k_1 \cdot k_2}{k_1 \cdot k_2} :$$



the same manipulations for internal vertices as well lead to graphs build on φ^3 skeletons with dressing numerator factors characteristic of the theory one considers.

Amplitudes from dressed φ^3 cubic vertices

- What are the constraints on the set of graphs Γ_n^L?
- Constraints on the Lorentz numerators factors n_γ for $\gamma \in \Gamma_n^L$?
- Hints of a "moduli space" of graphs similar to the one of string theory

Part I

Tree-level amplitudes

Tree-level amplitudes in gauge theory

Tree-level gauge theory amplitudes are decomposed as in color ordered factors

$$\mathcal{A}_n^{\text{tree}} = g_{\text{YM}}^{n-2} \sum_{\sigma \in \mathfrak{S}_n / \mathbb{Z}_n} \text{Tr}(\lambda^{\sigma(1)} \cdots \lambda^{\sigma(n)})\, A_n(\sigma(1), \cdots, \sigma(n))$$

All the information are in the color ordered partial amplitude

$$A_n^{\sigma} := A_n(\sigma(1), \cdots, \sigma(n)); \qquad \sigma \in \mathfrak{S}_n$$

The color ordered amplitudes are not independent and satisfy relations kinematic relations

- Cyclicity property

$$A_n(1,\ldots,n) = A_n(2,n,\ldots,1)$$

Reduces the number of independent amplitudes to $(n-1)!$

- Reflection property

$$A_n(1,\ldots,n) = (-1)^n A_n(n,\ldots,1)$$

Reduces further to $\frac{1}{2}\,(n-1)!$ independent amplitudes

- Photon decoupling identity

$$\sum_{\sigma\in\mathfrak{S}_n} A_n(\sigma(1),\ldots,\sigma(n)) = 0$$

What is the minimal number of independent amplitude ?

Tree-level amplitudes in gauge theory

Instead of considering the sum of the mutilple field theory graph individualy we treat the field theory amplitudes and the infinite tension limit $\alpha' \to 0$ of the tree-level string amplitudes

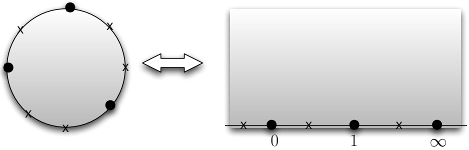
$$A_{\mathrm{SYM}}(\sigma(1),\ldots,\sigma(n)) = \lim_{\alpha'\to 0} \mathfrak{A}(\sigma(1),\ldots,\sigma(n))$$

$$\mathfrak{A}(\sigma(1),\ldots,\sigma(n)) = \left\langle U^{(1)}(z_1)U^{(n-1)}(z_{n-1})U^{(n)}(z_n) \prod_{i=2}^{n-2} \int_{\mathrm{ordered}} d^2 z_i V^{(i)} \right\rangle$$

where $U(z)$ and $V(z)$ are vertex operators and $\langle \cdots \rangle$ is the path integral over the world-sheet fields.

This can applied to any string theory formalism (Bosonic, RNS, Green-Schwarz, pure spinor, ...) in any spacetime dimensions

Open string tree-level amplitude on the disc



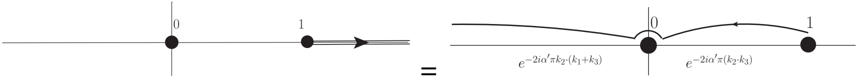
$PSL(2,\mathbb{R})$ invariance $z_1 = 0$, $z_{n-1} = 1$ and $z_n = +\infty$. (3 marked points)

$$\mathfrak{A}(1,\dots,n) = \int_{x_1<\cdots<x_n} \prod_{i=2}^{n-2} dx_i \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)^{2\alpha' k_i\cdot k_j} \sum_{(\zeta_j)\in\{0,1,x_i\}} L_k \prod_{i=2}^{n-2} \frac{1}{x_j - \zeta_j}$$

- The L_k factor encodes the information on the theory we consider: scalar or vector in the adjoint, etc

Monodromies from contour deformation

Contour deformation [Bjerrum-bohr, Damgaard, Vanhove]



- The real and imaginary part of the monodromy relations lead to a set of linear system of equations

$$\mathfrak{A}_n(\beta_1,\dots,\beta_r,1,\alpha_1,\dots,\alpha_s,n) = (-1)^r \times$$

$$\times \Re\mathrm{e}\Big[\prod_{1\leqslant i<j\leqslant r} e^{(\beta_i\cdot\beta_j)} \sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} \prod_{i=1}^{r}\prod_{j=1}^{s} e^{(\alpha_i,\beta_j)}\mathfrak{A}_n(1,\{\sigma\},n)\Big]$$

$$0 = \Im\mathrm{m}\Big[\prod_{1\leqslant i<j\leqslant r} e^{(\beta_i\cdot\beta_j)} \sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} \prod_{i=1}^{r}\prod_{j=1}^{s} e^{(\alpha_i,\beta_j)}\mathfrak{A}_n(1,\{\sigma\},n)\Big]$$

$\exp(\alpha,\beta) = \exp(2i\pi\alpha'\, k_\alpha\cdot k_\beta)$ if $\Re\mathrm{e}(z_\beta - z_\alpha) > 0$ or 1 otherwise

Minimal Basis for tree-level amplitudes

- This leads to a linear system of rank $(n-3)!$ in the amplitudes
- All ordered amplitudes can be expanded in the *minimal* basis B_n

$$(B_n)^{\sigma} := A(1, \underbrace{\sigma(2), \ldots, \sigma(n-2)}_{\textit{permutation}}, n-1, n); \qquad \sigma \in \mathfrak{S}_{n-3}$$

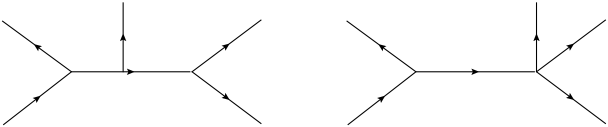
$$A_n^{\sigma} = \sum_{\sigma' \in \mathfrak{S}_{n-3}} M^{\sigma}_{\sigma'} \, (B_n)^{\sigma'}$$

- These monodromy relations apply for *all* matter content

The five-point case

- We consider the for color ordered gauge amplitudes

$$A_5(\sigma(1),\ldots,\sigma(5)) = \sum_{i=1}^{5} \frac{n_{r_i}}{p_{1,i}^2 p_{2,i}^2}$$



The five-point case

- We consider the for color ordered gauge amplitudes

$$A_5(\sigma(1),\ldots,\sigma(5)) = \sum_{i=1}^{5} \frac{n_{r_i}}{p_{1,i}^2 p_{2,i}^2}$$

- The numerator factors are *not* gauge invariant
- The monodromy relations between the color ordered amplitudes imply

$$0 = (s_{13}+s_{23})A_5(1,2,3,4,5) - s_{35}A_5(1,2,4,3,5) + s_{13}A_5(1,3,2,4,5)$$

The five-point case

- We consider the for color ordered gauge amplitudes

$$A_5(\sigma(1),\dots,\sigma(5)) = \sum_{i=1}^{5} \frac{n_{r_i}}{p_{1,i}^2 p_{2,i}^2}$$

- The system is solved by the generalized dual Jacobi relations

$$X_{ijk} = n_i - n_j + n_k = P_n(s_{ij}); \qquad c_i - c_j + c_k = 0$$

- The n_i are not uniquely defined by the pairing $n_i c_i$ summed over the graph gives the gauge invariant amplitudes
- $P_n = 0$ is the [Bern, Carrasco, Johansson] solution
- $P_n \neq 0$ is from the freedom in resolving the higher point vertices [Tye, Zhang], [Bjerrum-Bohr, Damgaard, Søndergaard, Vanhove]

Part II

Loop amplitudes

Higher-loop extensions

[Bern, Carrasco, Johansson] have extended this parametrisation to higher-loop amplitudes

$$\mathfrak{A}_{4,L}^{(D)} = g_{\rm YM}^{2L+2} \sum_{\gamma\in\Gamma_4^L} \int \prod_{i=1}^{L} \frac{d^D\ell_i}{(2\pi)^D} \frac{1}{S_j} \frac{n_\gamma c_\gamma}{\prod_{r=1}^{3L+1} p_r^2},$$

for gravity

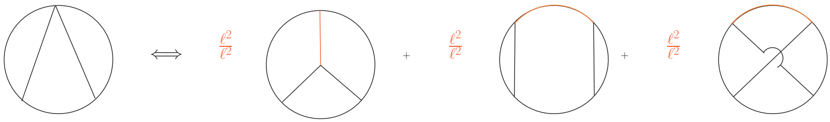
$$\mathfrak{M}_{4,L}^{(D)} = \kappa_{(D)}^{2L+2} \sum_{\gamma\in\Gamma_4^L} \int \prod_{i=1}^{L} \frac{d^D\ell_i}{(2\pi)^D} \frac{1}{S_j} \frac{n_\gamma \tilde n_\gamma}{\prod_{r=1}^{3L+1} p_r^2}.$$

- We show how this parametrisation allows to reproduce the ultraviolet behaviour of the 4-point amplitude in $\mathcal{N}=4$ SYM
- Apply it to the ultraviolet behaviour of $\mathcal{N}=8$ supergravity

Parametrisation of the $\mathcal{N} = 4$ SYM amplitudes

From 3-loop order the parametrisation of the four-point amplitude needs 4-point contact terms [Berkovits, Green, Russo,Vanhove]

The contact terms are resolved into cubic vertices as follows



- The contact terms are needs for the total amplitude to be gauge invariant
- Important for the counting of the fermionic zero modes and the ultraviolet behaviour of the amplitude in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA

  [Berkovits, Green, Russo,Vanhove], [Green, Bjornsson], [Vanhove]

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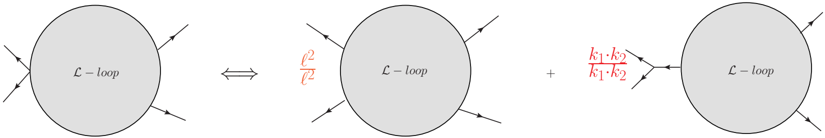

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  [Berkovits, Green, Russo,Vanhove], [Green, Bjornsson], [Vanhove]

Parametrisation of the $\mathcal{N}=4$ SYM amplitudes

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It was shown in [Berkovits, Green, Russo, Vanhove] that the *sub-leading* color contribution of the four-point L-loop amplitude in $\mathcal{N}=4$ SYM has a *better* ultraviolet behaviour than the leading (planar) color factor contribution

- 1-particle irreducible graphs

$$n^{1PI} \sim s\, F^4\, k_i k_j\, t_1^{1PI}(\ell, k_i)$$

- 1-particle irreducible graphs with 1 and 2 inverse derivative factors ($L \geqslant 4$ loops)

$$n_2^{1PR} \sim s^3\, F^4\, k_i k_j\, t_3^{1PR}(\ell, k_i)$$

Parametrisation of the $\mathcal{N}=4$ SYM amplitudes

$$A^{(D)}_{4,L} = g_{\rm YM}^{2L+2} \sum_{\gamma\in\Gamma_4^L} \int \prod_{i=1}^{L} \frac{d^D\ell_i}{(2\pi)^D} \frac{1}{S_j} \frac{n_\gamma c_\gamma}{\prod_{r=1}^{3L+1} p_r^2},$$

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- 1-particle irreducible graphs: large loop momentum behaviour $\ell\sim\Lambda$

$$\lim_{\ell\sim\Lambda} n^{1PI} \sim s^2\, F^4 \;\times \Lambda^{2L-6}$$

- 1-particle irreducible graphs 2 inverse derivative factors ($L\geqslant 4$ loops)

$$\lim_{\ell\sim\Lambda} n_2^{1PR} \sim s^3\,{\rm tr}F^4 \;\times \Lambda^{2L-4} + s^4\,({\rm tr}F^2)^2 \;\times \Lambda^{2L-6}$$

Reproduces the UV behaviour of the 4-point $\mathcal{N}=4$ SYM amplitudes

Parametrisation of the $\mathcal{N} = 8$ SUGRA amplitudes

- We recycle the previous parametrisation in the $\mathcal{N} = 8$ amplitudes

$$M_{4,L}^{(D)} = \kappa_{(D)}^{2L+2} \sum_{\gamma\in\Gamma_4^L} \int \prod_{i=1}^{L} \frac{d^D\ell_i}{(2\pi)^D} \frac{1}{S_j} \frac{n_\gamma \tilde{n}_\gamma}{\prod_{r=1}^{3L+1} p_r^2} .$$

- We use left/right symmetric expression

$$n_i\tilde{n}_i \sim |\partial^4 F^4\, t(\ell,k)|^2 \sim \partial^8\, R^4\, t(\ell,k)\tilde{t}(\ell,k)$$

- The 1PI graphs have the UV behaviour

$$[\mathfrak{M}_{4,L}^{(D)}]^{1PI} \sim \Lambda^{(D-2)L-14}\, \partial^8 R^4$$

- The 1PR cancellation of the contributions $1/(k^2)^2$ the UV behaviour for $L \geqslant 4$

$$[\mathfrak{M}_{4,L}^{(D)}]^{1PR} \sim \Lambda^{(D-2)L-14} \partial^8 R^4$$

- The total crossing symmetry of the amplitudes implies that the behaviour of the 1PR graph is the same as the 1PI graph contributions

The ultraviolet behaviour of $\mathcal{N}=8$ supergravity

- For $L \leqslant 4$ the UV behaviour is

$$[\mathfrak{M}^{(D)}_{4;L}] \sim \Lambda^{(D-4)L-6}\,\partial^{2L}R^4 \qquad 2 \leqslant L \leqslant 4$$

- identical to the one of $\mathcal{N}=4$ SYM

- After 4-loop only $\partial^8 R^4$ is factorized $\beta_L = 4$ for $L \geqslant 4$

$$[\mathfrak{M}^{(D)}_{4;L}] \sim \Lambda^{(D-2)L-14}\,\partial^{8}R^4 \qquad L \geqslant 4$$

- worse than the one of $\mathcal{N}=4$ SYM

- At five-loop order the 4-point amplitude in
  - $\mathcal{N}=4$ SYM divergences for $5 < 26/5 \leqslant D$
  - $\mathcal{N}=8$ SUGRA divergences for $24/5 \leqslant D$
- Would imply a *seven-loop* divergence in $D=4$ with counter-term $\partial^8 R^4$

- Yes: explicitly check by string/duality or field theory computation
- Black 'yes': not explicitly checked but implied by power counting in superspace
- Red: coefficient has not been evaluated but supersymmetry argument does not rule out this counter-terms

	R^4	$\partial^4 R^4$	$\partial^6 R^4$	$\partial^8 R^4$	$\partial^{10} R^4$	$\partial^{12} R^4$
D=10	– –	– –	– –	– –	L=2 yes	– –
D=9	– –	– –	– –	L=2 yes	– –	– –
D=8	L=1 yes	– –	L=2 yes	– –	– –	L=3 yes
D=7	– –	L=2 yes	– –	– –	– –	– –
D=6	– –	– –	L=3 yes	– –	L=4 yes	– –
D=5	L=2 no	– –	L=4 no	– –	– –	L=6 yes
D=4	L=3 no	L=5 no	L=6 no	L=7 !	L=8 ?	L=9 yes

Outlook

We have analyzed the parametrisation of amplitudes in gauge and gravity based on a dressing of φ^3 types of diagrams

- The Lorentz and color factors need to satisfy some constraints for consistence
- At tree-level the monodromy relations between the color order amplitude give model independent relations
  - Best possible reorganization of the tree-level amplitude in gravity and gauge theory [Bjerrum-Bohr, Damgaard, Vanhove]
- At loop orders contact terms are needed to match the structure of the multiloop amplitudes gauge and gravity
- Structure of the space of the various graphs in Γ_n^L