The b-quark mass and the heavy-light decay constant from lattice HQET

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To constraint the standard model and see a signal of new physics the theoretical uncertainties should be decreased.

Two examples:

- Theoretical uncertainty on the inclusive determination on $|V_{ub}|$ dominated by the one of the b-quark mass $\delta V_{ub}/V_{ub} \sim 4 \frac{\delta m_b}{m_b}$
  
  Now $\delta m_b = 40\text{ MeV} \Rightarrow \delta V_{ub}/V_{ub} = 3.5\%$ [Hitlin et al. 09]

- $\mathcal{B}_R(B_s \rightarrow \mu^+ \mu^-) = F^2_{B_s}(C_{SM} + \tan^6 \beta_{MSSM})$

⇒ In the B sector, high precision results are needed

⇒ Lattice HQET is a natural candidate to study heavy-light mesons
   It is theoretically sound and can give precise results
Effective theories for heavy quark

Momentum of a heavy quark (inside a hadron) \( p = m_Q v + k \)

Interaction with light dof \( k \sim \Lambda_{\text{QCD}} \ll m_Q \)

Separate the higher and lower components of the heavy quark, and find an effective Lagrangian (see eg [Grozin '02])

\[
\mathcal{L}_{\text{eff}}^{\text{heavy}} = \bar{\psi}_h(x) \left[ i v \cdot D + \frac{(iD_\perp)^2}{2m_Q} + \frac{g \sigma \cdot G}{4m_Q} + ... \right] \psi_h(x)
\]

Different choices of lattice implementation

- Expansion in \( \Lambda_{\text{QCD}}/m_Q \): HQET \( \rightarrow \) This talk
- Expansion in \( v \) and \( 1/am_Q \): NRQCD
- Fermilab Method [El-Khadra et al '96]
- Relativistic heavy quarks [Aoki et al '01, Christ et al, Lin et al '06 ]

(Note a recent proposal by ETMC for B physics [ETMC '10 ])

See talks by E. Gamiz and M Della Morte in the lattice session
Should you like (lattice) HQET?

- **pros**
  - Theoretically well defined, (continuum limit, renormalization)
  - Can be implemented non-perturbatively
  - The static propagator is numerically cheap
  - In many cases the $1/m$ terms are doable
  - Convergence expected to be fast

- **cons**
  - Effective theory, not QCD
  - Linear divergence in the static energy [Eichten & Hill '90]

$$E_{\text{stat}} \simeq \frac{19.95}{12\pi^2} \times \frac{g_0^2}{a} + \ldots$$

- Ratio **Noise/signal** $\rightarrow \exp{(E_{\text{stat}}x_0)}$
  $\Rightarrow$ Can one get a signal?
“Recent” improvements in HQET

- Conceptual improvement:
  Non Perturbative matching with HQET [Heitger & Sommer 03]
  \( \Rightarrow \) Subtractions of the divergences

- Technical improvement:
  1. Reduction of the Ratio Noise/Signal
     [Della Morte, Dürr, Heitger, Molke, Rolf, Shindler, Sommer '03]

  2. Application on variational techniques and all to all propagators
     [Blossier, Della Morte, von Hippel, Mendes, Sommer '09]
The static part is given by the Eichten-Hill action \cite{Eichten & Hill 90}

\[ S_{\text{stat}} = a^4 \sum_x \overline{\psi}_h(x) D_0 \psi_h(x) \]

with \( P_+ \psi_h = \psi_h \), \( \overline{\psi}_h P_+ = \overline{\psi}_h \), \( P_+ = \frac{1}{2}(1 + \gamma_0) \)

The static energy contains a linear divergence (\( \propto 1/a \)) which is absorbed by \( m_{\text{bare}} \)

\[ m_B = E_{\text{stat}} + m_{\text{bare}} \]

The \( 1/m \) corrections are the kinetic and chromomagnetic terms

\[ O_{\text{kin}} = -\overline{\psi}_h(D^2)\psi_h \quad O_{\text{spin}} = -\overline{\psi}_h(\sigma \cdot \mathbf{B})\psi_h \]

with coefficient \( \omega_{\text{kin}}, \omega_{\text{spin}} \) \( \Rightarrow \) Classically \( \omega_{\text{kin}} = \omega_{\text{spin}} = 1/(2m) \)

HQET coefficients \( m_{\text{bare}}, \omega_{\text{kin}}, \omega_{\text{spin}} \) are determined non-perturbatively \( \Rightarrow \) renormalizability
We want to compute hadronic quantities at the $1/m$ order of HQET, for example

$$m_B = m_{\text{bare}} + E^{\text{stat}} + \omega_{\text{kin}} E^{\text{spin}} + \omega_{\text{spin}} E^{\text{spin}}$$

$$\langle 0 | A_0^{\text{HQET}} | B \rangle = Z_A^{\text{HQET}} \left( \langle 0 | A_0^{\text{stat}} | B \rangle + \omega_{\text{kin}} \langle 0 | A_0^{\text{kin}} | B \rangle + \omega_{\text{spin}} \langle 0 | A_0^{\text{spin}} | B \rangle \right)$$

⇒ To achieve such a computation, one needs:

- large volume matrix element and energies $E^{\text{stat}}, E^{\text{kin}}, \langle 0 | A_0^{\text{stat}} | B \rangle, \ldots$
  → use variational techniques on top of all-to-all propagators

- HQET parameters $m_{\text{bare}}, \omega_{\text{kin}}, Z_A^{\text{HQET}}, \ldots$
  → non perturbative matching
Strategy

[Heitger & Sommer 03]

\[ L_1 \sim 0.5 \text{ fm} \]

\[ \Phi^{QCD}(L_1, am_b) \]

\[ L_2 = 2L_1 \]

\[ \Phi(L_{\text{inf}}, am_b) \]

\[ \Phi^{HQET}(L_{\text{inf}} - \Phi^{HQET}(L_1) \]

\[ \Phi^{HQET}(L_2) - \Phi^{HQET}(L_1) \]

\[ \Phi^{HQET}(L_{\text{inf}}) - \Phi^{HQET}(L_2) \]

Experiment

Nicolas Garron (University of Edinburgh)  b-quark and decay constant in HQET  July 21, 2010
Simulate QCD in small volume $L_1 \sim 0.5$ fm with $a m_b \ll 1$. Compute a set of observables and take the continuum limit $\Phi(L_1, m_q)$.
- Simulate QCD in small volume $L_1 \sim 0.5 \text{ fm}$ with $am_b \ll 1$. Compute a set of observables and take the continuum limit $\Phi(L_1, m_q)$.

- Compute the corresponding quantities at a given order of the effective theory for various lattice spacing $a$. Impose the matching $\Rightarrow$ HQET parameters for these values of the lattice spacings.

  e.g. static meson mass $\Gamma^{QCD}(L_1, m_q) = m_{\text{bare}}(m_q, a) + \Gamma^{\text{stat}}(L_1, a)$.
Simulate QCD in small volume $L_1 \sim 0.5 \text{ fm}$ with $am_b \ll 1$. Compute a set of observables and take the continuum limit $\Phi(L_1, m_q)$.

Compute the corresponding quantities at a given order of the effective theory for various lattice spacing $a$. Impose the matching $\Rightarrow$ HQET parameters for these values of the lattice spacings.

For example, static meson mass

$$\Gamma^{\text{QCD}}(L_1, m_q) = m_{\text{bare}}(m_q, a) + \Gamma^{\text{stat}}(L_1, a)$$

Perform another simulation of HQET, with the same $a$'s but in a larger volume, for example $L_2 = 2L_1$.

Use the HQET parameters computed in the previous step, to obtain the observables in the volume $L_2$, and take their continuum limit $\Phi(L_2, m_q)$ (cancellation of the divergences).

Static meson mass:

$$\Gamma(L_2, m_q) = \lim_{a \to 0} \left( \Gamma^{\text{stat}}(L_2, a) - \Gamma^{\text{stat}}(L_1, a) \right) + \Gamma^{\text{QCD}}(L_1, m_q)$$

$$= L_1 \sigma^m(\bar{g}^2(L_1)) + \Gamma^{\text{QCD}}(L_1, m_q)$$
Simulate QCD in small volume \( L_1 \sim 0.5 \text{ fm} \) with \( a m_b \ll 1 \).
Compute a set of observables and take the continuum limit \( \Phi(L_1, m_q) \)

Compute the corresponding quantities at a given order of the effective theory for various lattice spacing \( a \).
Impose the matching \( \Rightarrow \) HQET parameters for these values of the lattice spacings.
e.g. static meson mass \( \Gamma^{QCD}(L_1, m_q) = m_{\text{bare}}(m_q, a) + \Gamma^{\text{stat}}(L_1, a) \)

Perform another simulation of HQET, with the same \( a \)'s but in a larger volume, for example \( L_2 = 2L_1 \).
Use the HQET parameters computed in the previous step, to obtain the observables in the volume \( L_2 \), and take their continuum limit \( \Phi(L_2, m_q) \) (cancelation of the divergences).

Static meson mass:

\[
\Gamma(L_2, m_q) = \lim_{a \to 0} \left( \Gamma^{\text{stat}}(L_2, a) - \Gamma^{\text{stat}}(L_1, a) \right) + \Gamma^{QCD}(L_1, m_q)
\]

\[
= L_1 \sigma^m(\bar{g}^2(L_1)) + \Gamma^{QCD}(L_1, m_q)
\]

Restart from step 1, with \( \Phi^{QCD}(L_1, m_q) \rightarrow \Phi(L_2, m_q) \) until the volume is large enough to compute hadronic quantities.
For the meson mass at the static order, we obtain for various quark masses $m_q$:

$$
\Gamma(L_\infty, m_q) = \lim_{a \to 0} (\Gamma_{\text{stat}}(L_\infty, a) - \Gamma_{\text{stat}}(L_2, a)) + \lim_{a \to 0} (\Gamma_{\text{stat}}(L_2, a) - \Gamma_{\text{stat}}(L_1, a)) + \Gamma_{\text{QCD}}(L_1, m_q)
$$

And we interpolate at the $B(s)$-meson mass to obtain the $b$-quark mass.

In the case of the heavy-light decay constant we use the parameters interpolated at the obtained value of the $b$-quark mass, and obtain $F_{B(s)}$. 
Implementation: Schrödinger functional of size $T \times L^3$

- Dirichlet boundary conditions in time (at $x_0 = 0$ and $x_0 = T$)
- Periodic boundary conditions in space, up to a phase $\Psi(x + \hat{k}L) = e^{i\theta} \Psi(x)$.

Transition amplitude for $C(x_0 = 0) \rightarrow C'(x_0 = T)$

$$Z[C', C] = \langle C' | e^{-\hat{H}T} \mathbb{P} | C \rangle$$

$$= \sum_{n=0}^{\infty} e^{-E_n T} \psi_n[C'] \psi_n[C]^*$$
Implementation: 2-point functions in QCD

Boundary to current correlators

\[ f_A(x_0) = -\frac{a^6}{2} \sum_{y,z} \left\langle (A_{I0}(x) (\bar{\zeta}_b(y) \gamma_5 \zeta_1(z)) \right\rangle \]

and boundary to boundary correlator

\[ f_1 = -\frac{a^{12}}{2L^6} \sum_{y,z,y',z'} \left\langle (\bar{\zeta}'_b(y') \gamma_5 \zeta'_1(z')) (\bar{\zeta}_b(y) \gamma_5 \zeta_1(z)) \right\rangle \]

\[ k_1 = -\frac{a^{12}}{2L^6} \sum_{y,z,y',z'} \left\langle (\bar{\zeta}'_b(y') \gamma_k \zeta'_1(z')) (\bar{\zeta}_b(y) \gamma_k \zeta_1(z)) \right\rangle \]
Implementation: 2-point functions in the static theory

Boundary to current correlators

\[ f_{A}^{\text{stat}}(x_{0}) = -\frac{a^{6}}{2} \sum_{y,z} \left\langle (A_{I}^{\text{stat}})_{0}(x) (\bar{\zeta}_{h}(y) \gamma_{5} \zeta_{l}(z)) \right\rangle \]

and boundary to boundary correlator

\[ f_{1}^{\text{stat}} = -\frac{a^{12}}{2L^{6}} \sum_{y,z,y',z'} \left\langle (\bar{\zeta}'_{h}(y') \gamma_{5} \zeta'_{l}(z')) (\bar{\zeta}_{h}(y) \gamma_{5} \zeta_{l}(z)) \right\rangle \]
Implementation: 2-point functions at the $1/m$ order

Boundary to current correlators

$$f_{A}^{\text{kin}}(x_0) = -\frac{a^6}{2} \sum_{y,z,u} \left\langle A_0^{\text{stat}}(x) O^{\text{kin}}(u) (\bar{\zeta}_h(y)\gamma_5\zeta_1(z)) \right\rangle$$

Boundary to boundary correlator

$$f_{1}^{\text{kin}} = -\frac{a^{12}}{2L^6} \sum_{y,z,y',z',u} \left\langle (\bar{\zeta}'_h(y')\gamma_5\zeta'_1(z')) O^{\text{kin}}(u) (\bar{\zeta}_h(y)\gamma_5\zeta_1(z)) \right\rangle$$

And the same for $f_{A}^{\text{spin}}, f_{1}^{\text{spin}}$. 
\( F_B \), including \( 1/m \) corrections

From the current-to-boundary \( f_A \) and the boundary-to-boundary \( f_1 \) correlators

Build an observables related to the decay constant:

\[
\phi^{QCD}_2 = \ln \left( \frac{-f_A(x_0)}{\sqrt{f_1}} \right) \quad \overset{L \gg 1}{\longrightarrow} \quad \ln \left( \frac{1}{2} F_B \sqrt{m_B L^3} \right)
\]

At the \( 1/m \) order of HQET

\[
\phi^{HQET}_2 = \ln Z^{HQET}_A + \ln \left( \frac{-f_{\text{stat}}}{\sqrt{f_{\text{stat}}}} \right) \\
+ c^{HQET}_A \frac{f_{\text{stat}}}{f_{\text{stat}}^A} \delta A + \omega_{\text{kin}} \left( \frac{f_{\text{kin}}}{f_{\text{stat}}^A} - \frac{1}{2} \frac{f_{\text{kin}}}{f_{\text{stat}}^1} \right) + \omega_{\text{spin}} \left( \frac{f_{\text{spin}}}{f_{\text{stat}}^A} - \frac{1}{2} \frac{f_{\text{spin}}}{f_{\text{stat}}^1} \right)
\]
We define the 5 dimensional vectors $\Phi$, $\eta$, $\omega$ and a 5 by 5 matrix $\phi$

$$\Phi(L, m_q) = \lim_{a \to 0} \left[ \phi(L, a) \omega(m_q, a) + \eta(L, a) \right]$$
Results

Continuum extrapolation of the QCD observables

The meson mass

The heavy-light decay constant

The RGI quark masses $M$ are such that $z = L_1 M \in (4, 6, 7, 9, 11, 13, 15, 18, 21)$

$L_1/r_0 = 0.9 \Rightarrow L_1 \sim 0.45$ fm

$L_1/a = 40, 32, 24(20)$

$\beta = 6.638, 6.4574, 6.2483$
Results

Continuum extrapolation of the static observables in $L_2$

The meson mass

$\Phi_1$

$\Phi_2$

The heavy-light decay constant

$(a/L_2)^2 x 10^{-3}$

The RGI quark masses $M$ are such that $z = L_1 M \in (4, 6, 7, 9, 11, 13, 15, 18, 21)$

$L_1/r_0 = 0.9 \Rightarrow L_1 \sim 0.45 \text{ fm}$

$L_1/a = 12, 10, 8 \quad L_1/a = 24, 20, 16$

$\beta = 5.758, 5.619, 5.4689$ \quad Point with $L_2/a = 32, L_1/a = 16$ will be added in the near future
Continuum extrapolation of the $1/m$ observables in $L_2$

The RGI quark masses $M$ are such that $z = L_1 M \in (4, 6, 7, 9, 11, 13, 15, 18, 21)$

$L_1/r_0 = 0.9 \Rightarrow L_1 \sim 0.45$ fm

$L_1/a = 12, 10, 8 \quad L_1/a = 24, 20, 16$

$\beta = \ldots, \ldots \quad$ Point with $L_2/a = 32, L_1/a = 16$ will be added in the near future
Results

Example of static parameter: $m_{\text{stat bare}}$

The RGI quark masses $M$ are such that $z = L_1 M \in (4, 6, 7, 9, 11, 13, 15, 18, 21)\$

$L_1/r_0 = 0.9 \Rightarrow L_1 \sim 0.45$ fm

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$L_1/a = 12, 10, 8 \quad L_1/a = 24, 20, 16$

$\beta = \ldots, \ldots \quad$ Point with $L_2/a = 32, L_1/a = 16$ will be added in the near future
Results

Interpolation at the physical mass, in the static approximation for $n_f = 2$

The RGI quark masses $M$ are such that $z = L_1 M \in (13, 15, 18)$
Continuum limit of $F_{B_s}$ for $n_f = 0$

Only static (upper curve) and static + $1/m$
Conclusion - Status of the project

- $n_f = 0$
  - b-quark mass  [Alpha '06]
    \[
    m_b(m_b) = 4.350(64) \text{ GeV} - 0.049(29) \text{ GeV} + O(\Lambda^3/m_b^2) + O(\Lambda^2/m_b) \]

- I. HQET parameters  [Alpha '10]
- II. Spectoscopy  [Alpha '10]
- III. Decay constant (submitted in June)
  \[
  F_{B_s}^{\text{stat}} = 229 \pm 6 \text{ MeV} \quad F_{B_s}^{\text{stat+1/m}} = 219 \pm 8 \text{ MeV} \]

- $n_f = 2$
  - HQET parameter: almost finished
  - Large volume part: preliminary results (1 lattice spacing)
    See Talk by Della Morte

**VERY PRELIMINARY**
\[
  m_b(m_b)^{\text{stat}} = 4.255(25)(50)(??) \quad m_b(m_b)^{\text{HQET}} = 4.276(25)(50)(??)
\]
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