Perturbative Quantum Gravity from Gauge Theory

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Zvi Bern, John Joseph Carrasco, HJ
Outline

- How to reorganize pure gauge theory amplitudes?
- Duality between color and kinematics (tree-level)!
  - Cubic vertices, Jacobi identity for kinematics
  - Novel relations between tree amplitudes
  - Duality automatically gives gravity amplitudes (~KLT)
- Does duality hold at quantum level?
  - Direct evidence of duality in loop amplitudes
- Is duality non-perturbative?
  - Lagrangian formulation
- Conclusion
Gauge theory and Gravity

Pure Yang-Mills

\[ \mathcal{L}_{YM} = -\frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu}, \quad A_\mu^a \]

\[ \sim 10 \text{ terms} \]

Pure Einstein gravity

\[ \mathcal{L} = \frac{2}{\kappa^2} \sqrt{g} R, \quad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \]

infinitely many vertices:

\[ \sim 10^2 \]

\[ \sim 10^3 \]

\[ \text{terms} \]

\[ \ldots \]

naively the two theories are very different!
**Amplitudes from standard techniques**

**Lagrangian, Feynman rules, etc.**

**Tree-level YM**
- difficult – but solved
- Berends-Giele recursion (‘88)

- factorial growth of # of diagrams
- gauge choice redundancy

**Loop-level YM**
- "impossible" beyond 4-5 points even at one loop

- additionally:
  - Faddeev-Popov ghosts
  - tensor integral reductions

**Tree-level Gravity**
- beyond control using Feynman rules

- additionally:
  - unlimited vertex expansion
  - extremely complicated vertex factors

**Loop-level Gravity**
- "insurmountable"

- additionally:
  - Batalin-Vilkovisky formalism
“Modern” amplitude calculations

Tree-level YM

easy with modern on-shell methods

Kawai-Lewellen-Tye (KLT)

Loop-level YM

(this talk)

Tree-level Gravity

Loop-level Gravity

modern methods dramatically simplifies calculations, but life is even better...
Ampl’s have hidden structures and beauty

Examples:

- **Park-Taylor** MHV formula (’86)

- **Witten**’s twistor string theory (tree-level)

- **Dual superconformal symmetry and Yangian** ($\mathcal{N}=4$ SYM)
  - Drummond, Henn, Korchemsky, Smirnov, Sokatchev, Plefka, etc.

- **Polygon Wilson loop duality** ($\mathcal{N}=4$ SYM)
  - Alday, Maldacena; Drummond, Henn, Korchemsky, Sokatchev; Brandhuber, Heslop, Spence, Travaglini, etc.

- **Grassmannian integrals** (’09) ($\mathcal{N}=4$ SYM)
  - Arkani-Hamed, Cachazo, Cheung, Kaplan; Mason, Skinner; Spradlin, Volovich; Korchemsky, Sokatchev etc.

Planar gauge theory - very clean and beautiful!

What about non-planar theories, and gravity?
Novel structure cleans up diagrams

Hidden duality: color ↔ kinematics

• Only cubic vertices:

\[ \ = f^{abc} V(k_1, k_2, k_3) \]

\[ f^{abc} \leftrightarrow V(k_1, k_2, k_3) \]

color ↔ kinematics

• Same algebraic properties:

\[ = - \]

\[ \text{relates planar and non-planar} \]

\[ \text{face } fedb = f bce feda - fabe febd \]
Gauge theory color decomposition

• Usual decomposition

\[ A_n^{\text{tree}}(1, 2, \ldots, n) = g^{n-2} \sum_{\mathcal{P}(2, \ldots, n)} \text{Tr}[T^{a_1} T^{a_2} \cdots T^{a_n}] A_n^{\text{tree}}(1, 2, \ldots, n) \]

• Alternative decomposition, 4pt example

\[ A_4^{\text{tree}}(1, 2, 3, 4) = g^2 \left( \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right) \]

gauge invariant

• Map

\[ \tilde{f}^{abc} \equiv i \sqrt{2} f^{abc} = \text{Tr}([T^a, T^b]T^c) \]

color factors

\[ c_u \equiv \tilde{f}^{a_4a_2b} \tilde{f}^{ba_3a_1} \]
\[ c_s \equiv \tilde{f}^{a_1a_2b} \tilde{f}^{ba_3a_4} \]
\[ c_t \equiv \tilde{f}^{a_2a_3b} \tilde{f}^{ba_4a_1} \]

color structures

\[ A_4^{\text{tree}}(1, 2, 3, 4) \equiv \frac{n_s}{s} + \frac{n_t}{t}, \]
kinematic numerators

\[ n_s, n_t, n_u \]

absorbs 4-pt contact terms

- but gauge dependent!

kinematic structures

\[ A_4^{\text{tree}}(1, 3, 4, 2) \equiv -\frac{n_u}{u} - \frac{n_s}{s}, \]

\[ A_4^{\text{tree}}(1, 4, 2, 3) \equiv -\frac{n_t}{t} + \frac{n_u}{u}, \]
A Jacobi-like 4pt identity

\[ A_{4}^{\text{tree}}(1, 2, 3, 4) = g^2 \left( \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right) \]

• Jacobi identity for color and for kinematics

\[ C_u = C_s - C_t \quad \Leftrightarrow \quad n_u = n_s - n_t \]

• Duality between color and kinematics

• Kinematic numerators gauge dependent - but 4pt identity is gauge invariant

\[-n'_s + n'_t + n'_u = -n_s + n_t + n_u + \Delta(k_j, \varepsilon_j)(s + t + u) = 0\]

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Similar duality at higher points

- Decomposing 5pt amplitude in terms of 15 cubic diagrams

\[ A_5^{\text{tree}} = g^3 \left( \frac{n_1 c_1}{s_{12}s_{45}} + \frac{n_2 c_2}{s_{23}s_{51}} + \frac{n_3 c_3}{s_{34}s_{12}} + \frac{n_4 c_4}{s_{45}s_{23}} + \frac{n_5 c_5}{s_{51}s_{34}} + \frac{n_6 c_6}{s_{14}s_{25}} \right) 
+ \frac{n_7 c_7}{s_{32}s_{14}} + \frac{n_8 c_8}{s_{25}s_{43}} + \frac{n_9 c_9}{s_{13}s_{25}} + \frac{n_{10} c_{10}}{s_{42}s_{13}} + \frac{n_{11} c_{11}}{s_{51}s_{42}} + \frac{n_{12} c_{12}}{s_{12}s_{35}} \right), \]

- Equivalent to partial amplitudes

\[ A_5^{\text{tree}} (1, 2, 3, 4, 5) \equiv \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}} \]

- Duality between color and kinematics hold

\[ n_3 - n_5 + n_8 = 0 \quad \iff \quad c_3 - c_5 + c_8 = 0 \]

\[ n_3 - c_3 = n_5 - c_5 = n_8 - c_8 = 0 \]

\[ s_{ij} = (k_i + k_j)^2 \]

but is no longer gauge invariant...
Generalized gauge transformation

Define “generalized gauge transformation” on amplitude as

\[ A_n^{\text{tree}} = \sum_i \frac{c_i n_i}{\prod_\alpha p_\alpha^2} \]

such that

\[ \sum_i \frac{c_i \Delta_i}{\prod_\alpha p_\alpha^2} = 0 \]

Amplitudes invariant under this transformation, but not duality

\[ n_i + n_j + n_k \neq 0 \quad \iff \quad c_i + c_j + c_k = 0 \]

Conjecture: transformation always exists such we can make \( n_i \) satisfy the Jacobi identity – making duality manifest.
Amplitude relations

- Assuming the duality can be made manifest at 5pts:
  - 15 different $n_i$
  - 9 Jacobi identities
  - fix 2 $n_i$ using two partial amplitudes
  - remaining 4 $n_i \Leftrightarrow$ residual gauge freedom

Gives curious amplitude relations:

$$A_{5}^{\text{tree}}(1, 3, 4, 2, 5) = \frac{-s_{12}s_{45}A_{5}^{\text{tree}}(1, 2, 3, 4, 5) + s_{14}(s_{24} + s_{25})A_{5}^{\text{tree}}(1, 4, 3, 2, 5)}{s_{13}s_{24}}$$

$$A_{5}^{\text{tree}}(1, 2, 4, 3, 5) = \frac{-s_{14}s_{25}A_{5}^{\text{tree}}(1, 4, 3, 2, 5) + s_{45}(s_{12} + s_{24})A_{5}^{\text{tree}}(1, 2, 3, 4, 5)}{s_{24}s_{35}}$$

- Any 5pt tree is a linear combination of two basis amplitudes

$$A_{5}(\ldots ..) = \alpha A_{5}(1,2,3,4,5) + \beta A_{5}(1,4,3,2,5)$$
Amplitude relations for any number of legs

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• General relations for gauge theory partial amplitudes

\[ A_n^{\text{tree}}(1, 2, \{\alpha\}, 3, \{\beta\}) = \sum_{\{\sigma\}_j \in \text{POP}(\{\alpha\}, \{\beta\})} A_n^{\text{tree}}(1, 2, 3, \{\sigma\}_j) \prod_{k=4}^{m} \frac{\mathcal{F}(3, \{\sigma\}_j, 1|k)}{g_{2,4,...,k}} \]

where

\[ \{\alpha\} \equiv \{4, 5, \ldots, m-1, m\}, \quad \{\beta\} \equiv \{m+1, m+2, \ldots, n-1, n\} \]

and

\[ \mathcal{F}(3, \sigma_1, \sigma_2, \ldots, \sigma_{n-3}, 1|k) \equiv \mathcal{F}(\{\rho\}|k) = \begin{cases} \sum_{l=t_k}^{n-1} g(k, \rho_l) & \text{if } t_{k-1} < t_k \\ -\sum_{l=1}^{t_k} g(k, \rho_l) & \text{if } t_{k-1} > t_k \\ 0 & \text{else} \end{cases} + \begin{cases} g_{2,4,...,k} & \text{if } t_{k-1} < t_k < t_{k+1} \\ -g_{2,4,...,k} & \text{if } t_{k-1} > t_k > t_{k+1} \end{cases} \]

and \( t_k \) is the position of leg \( k \) in the set \( \{\rho\} \)

\[ A_n(\sigma_1, \sigma_2, \ldots, \sigma_n) = \alpha_1 A_n(1, 2, \ldots, n) + \alpha_2 A_n(2, 1, \ldots, n) + \ldots + \alpha_{(n-3)!} A_n(3, 2, \ldots, n) \]

**Basis size:** \((n-3)!\)  
**Compare to Kleiss-Kuijf relations** \((n-2)!\)

Recent proofs: Bjerrum-Bohr, Damgaard, Vanhove; Feng, Huang, Jia
String worldsheet monodromy

Monodromy relations on the open string worldsheet is shown to capture both the Kleiss-Kuijf relations and the relations implied by the duality

Bjerrum-Bohr, Damgaard, Vanhove (2009)

\[ A(1, 3, 2, 4) = -\text{Re} \left[ e^{-2i\alpha' \pi k_2 \cdot k_3} A(1, 2, 3, 4) + e^{-2i\alpha' k_2 \cdot (k_1 + k_3)} A(2, 1, 3, 4) \right] \quad \text{“Kleiss-Kuijf”} \]

\[ 0 = \text{Im} \left[ e^{-2i\alpha' \pi k_2 \cdot k_3} A(1, 2, 3, 4) + e^{-2i\alpha' k_2 \cdot (k_1 + k_3)} A(2, 1, 3, 4) \right] \quad \text{new relations} \]

Original relations recovered in the field theory limit:

\[ A(1, 3, 2, 4) = \frac{\sin(2i\alpha' \pi k_1 \cdot k_4)}{\sin(2i\alpha' \pi k_2 \cdot k_4)} A(1, 2, 3, 4) \]

\[ \alpha' \to 0 : A(1, 3, 2, 4) = \frac{k_1 \cdot k_4}{k_2 \cdot k_4} A(1, 2, 3, 4) \]

Provides partial proof of duality conjecture
Gravity as a bonus
KLT Relations

\[ \mathcal{L} = \frac{2}{\kappa^2} \sqrt{g} R, \]

Feynman rules:

Kawai-Lewellen-Tye relations

Originally string theory tree level identity:

Field theory limit \( \Rightarrow \) gravity theory \( \sim \) (gauge theory) \( \times \) (gauge theory)

\[
M_4^{\text{tree}}(1, 2, 3, 4) = -i s_{12} A_4^{\text{tree}}(1, 2, 3, 4) \bar{A}_4^{\text{tree}}(1, 2, 4, 3)
\]

\[
M_5^{\text{tree}}(1, 2, 3, 4, 5) = i s_{12} s_{34} A_5^{\text{tree}}(1, 2, 3, 4, 5) \bar{A}_5^{\text{tree}}(2, 1, 4, 3, 5)
\]

\[
+ i s_{13} s_{24} A_5^{\text{tree}}(1, 3, 2, 4, 5) \bar{A}_5^{\text{tree}}(3, 1, 4, 2, 5)
\]

Gravity states are products of gauge theory states:

\[ |1\rangle_{\text{grav}} = |1\rangle_{\text{gauge}} \otimes |1\rangle_{\text{gauge}} \]
Jacobi identity + KLT

KLT:

\[ M^\text{tree}_4 = s_1 A^\text{tree}_4(1, 2, 3, 4) \tilde{A}^\text{tree}_4(1, 2, 4, 3) = \frac{n_s \tilde{n}_s}{s} + \frac{n_t \tilde{n}_t}{t} + \frac{n_u \tilde{n}_u}{u} \]

follows after using: \[ n_u = n_s - n_t \]

\[ A^\text{tree}_4 = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \]

\[ M^\text{tree}_4 = \frac{n_s \tilde{n}_s}{s} + \frac{n_t \tilde{n}_t}{t} + \frac{n_u \tilde{n}_u}{u} \]

Unlike KLT this gravity formula is for local objects \( n_i \) and is manifestly crossing (Bose) symmetric

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\[
\begin{align*}
\text{gauge theory} & \quad \times \text{color} \\
\text{gravity} & \quad = \left( \begin{array}{c}
\text{} \\
\end{array} \right)^2
\end{align*}
\]
Gravity = double copy of YM

- At 5 points

\[ A_{\text{tree}}^5 = g^3 \left( \frac{n_1 c_1}{s_{12} s_{45}} + \frac{n_2 c_2}{s_{23} s_{51}} + \frac{n_3 c_3}{s_{34} s_{12}} + \frac{n_4 c_4}{s_{45} s_{23}} + \frac{n_5 c_5}{s_{51} s_{34}} + \frac{n_6 c_6}{s_{14} s_{25}} \right) + \frac{n_7 c_7}{s_{13} c_{13}} + \frac{n_8 c_8}{s_{25} s_{13}} + \frac{n_9 c_9}{s_{25} s_{13}} + \frac{n_{10} c_{10}}{s_{13} s_{25}} + \frac{n_{11} c_{11}}{s_{42} s_{13}} + \frac{n_{12} c_{12}}{s_{13} s_{45}} \]

\[ M_{\text{tree}}^5 = i \left( \frac{\kappa}{2} \right)^3 \left( \frac{n_1 \tilde{n}_1}{s_{12} s_{45}} + \frac{n_2 \tilde{n}_2}{s_{23} s_{51}} + \frac{n_3 \tilde{n}_3}{s_{34} s_{12}} + \frac{n_4 \tilde{n}_4}{s_{45} s_{23}} + \frac{n_5 \tilde{n}_5}{s_{51} s_{34}} + \frac{n_6 \tilde{n}_6}{s_{14} s_{25}} \right) + \frac{n_7 \tilde{n}_7}{s_{13} \tilde{n}_{13}} + \frac{n_8 \tilde{n}_8}{s_{25} s_{43}} + \frac{n_9 \tilde{n}_9}{s_{13} s_{25}} + \frac{n_{10} \tilde{n}_{10}}{s_{42} s_{13}} + \frac{n_{11} \tilde{n}_{11}}{s_{51} s_{42}} + \frac{n_{12} \tilde{n}_{12}}{s_{12} s_{35}} \]

Remarkably only one family of numerators (either \( n_i \) or \( \tilde{n}_i \)) need to satisfy the Jacobi identities.
Tree-level gravity to all orders

- Conjecture to all orders (checked through 8 points)  
  \[ A^\text{tree}_n = \sum_i \frac{n^i c_i}{\prod_\alpha p^2_\alpha} \quad \Leftrightarrow \quad M^\text{tree}_n = \sum_i \frac{n^i \tilde{n}^i}{\prod_\alpha p^2_\alpha} \]

  \[ \begin{array}{c}
  \begin{array}{c}
  \text{Left sector} \\
  n_{L,i}
  \end{array}
  \end{array} \quad \Leftrightarrow \quad \text{modes in spacetime} \quad R^{(1,D-1)} \]

  \[ \begin{array}{c}
  \begin{array}{c}
  \text{Right sector} \\
  \tilde{n}_{R,i}
  \end{array}
  \end{array} \quad \Leftrightarrow \quad \text{modes in spacetime} \quad R^{(1,D-1)} \times T^{N_c} \]

Connection to Heterotic string by Tye and Zhang

- Double copy of YM

  Proof: Bern, Dennen, Huang, Kiermaier
Classical → Quantum
Duality present in $\mathcal{N} = 4$ SYM 4-pt ampl.

For particularly simple loop amplitudes one can show that the quantum duality follows from the tree-level one.

1-loop: $K^1\left\{ \begin{array}{c} 2 \ \ \ \ 3 \\ 1 \ \ \ \ 4 \end{array} + \begin{array}{c} 3 \ \ \ \ 4 \\ 1 \ \ \ \ 2 \end{array} + \begin{array}{c} 4 \ \ \ \ 2 \\ 1 \ \ \ \ 3 \end{array} \right\}$

2-loop: $K^1\left\{ \begin{array}{c} 2 \ \ \ \ 3 \\ 1 \ \ \ \ 4 \end{array} s^1 + \begin{array}{c} 3 \\ 1 \ \ \ \ 2 \end{array} s^1 + \text{perms} \right\}$

Prefactor contains helicity structure: $K = stA^\text{tree}_4$

Duality: $\mathcal{N} = 8$ sugra is obtained if $1 \rightarrow 2$ “numerator squaring”
New nontrivial evidence

3-loop $\mathcal{N}=4$ SYM admits manifest realization of duality – and $\mathcal{N}=8$ SUGRA is simply the square

$\tau_{ij} = 2k_i \cdot l_j$

<table>
<thead>
<tr>
<th>Integral $I^{(g)}$</th>
<th>$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8}$ supergravity) numerator</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)-(d)</td>
<td>$s^2$</td>
</tr>
<tr>
<td>(e)-(g)</td>
<td>$\left( s(-\tau_{35}+\tau_{45}+t)-t(\tau_{25}+\tau_{45})+u(\tau_{25}+\tau_{35})-s^2 \right)/3$</td>
</tr>
<tr>
<td>(h)</td>
<td>$\left( s(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u) +t(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17})+s^2 \right)/3$</td>
</tr>
<tr>
<td>(i)</td>
<td>$\left( s(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t) +t(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46})+u\tau_{25}+s^2 \right)/3$</td>
</tr>
<tr>
<td>(j)-(l)</td>
<td>$s(t-u)/3$</td>
</tr>
</tbody>
</table>
Works for non-susy theories

All-plus helicity QCD amplitude:

\[ + + \quad + + \quad + + \quad + + \quad + + = n_{DB} + n_{BT} + \ldots \]

All-plus helicity Einstein gravity amplitude:

\[ ++ \quad ++ \quad ++ \quad ++ \quad ++ = n^2_{DB} + n^2_{BT} + \ldots \]

(with dilation and axions in loops)

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Lagrangian formulation

- Lagrangian formulation with manifest duality

YM Lagrangian receives corrections at 5 points and higher

\[ \mathcal{L}_{YM} = \mathcal{L} + \mathcal{L}_5' + \mathcal{L}_6' + \ldots \]

corrections proportional to the Jacobi identity (thus equal to zero)

\[ \mathcal{L}_5' \sim \text{Tr} [A^\nu, A^\rho] \frac{1}{\Box} ([[\partial_\mu A_\nu, A_\rho], A^\mu] + [[A_\rho, A^\mu], \partial_\mu A_\nu] + [[A^\mu, \partial_\mu A_\nu], A_\rho]) \]

Introduction of auxiliary “dynamical” fields gives local cubic Lagrangian

\[ \mathcal{L}_{YM} = \frac{1}{2} A^{a\mu} \Box A^a_\mu - B^{a\mu\nu\rho} \Box B^{a}_{\mu\nu\rho} - g f^{abc} (\partial_\mu A^a_\nu + \partial^\rho B^{a}_{\rho\mu\nu}) A^{b\mu} A^{c\nu} + \ldots \]

“squaring” gives gravity Lagrangian.

→ non-perturbative insight?
Conclusion

Pure gauge theories have a new hidden structure - duality between color and kinematics at tree level.

The duality gives partial amplitudes relations, and (local) relations between gravity and gauge theory, clarifying KLT (and more).

Nontrivial checks at two and tree loops hints that duality survives at the quantum level – natural extension of conjecture.

Lagrangian formulation, connection to string theory, give hints of future potential. May be a key tool for non-planar gauge theory. May be exploited/extended towards non-perturbative physics.

What is the “physics” of the duality?
- Is there an underlying “Lie group” that controls the kinematics?
- What is the physical interpretation of gravity as a double copy of gauge theory? Compositeness?

Detailed physical understanding awaits us!