THE FREE-ENERGY OF SPIN-TWO FIELDS

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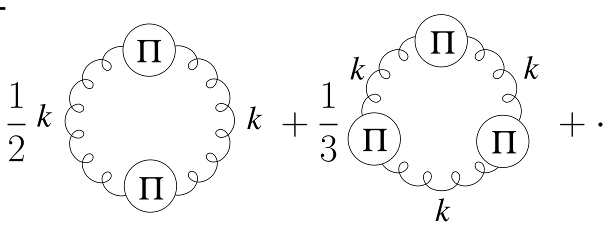
Abstract

We derive a closed form expression for the sum of all the infrared divergent contributions to the free-energy of a gas of gravitons. An important ingredient of our calculation is the use of a gauge fixing procedure such that the graviton propagator becomes both traceless and transverse. This has been shown to be possible, in a previous work, using a general gauge fixing procedure, in the context of the lowest order expansion of the Einstein-Hilbert action, describing non-interacting spin two fields. In order to encompass the problems involving thermal loops, such as the resummation of the free-energy, in the present work, we have extended this procedure to the situations when the interactions are taken into account.

The problem

How to obtain the sum of all the IR divergent (three or more loops) contributions to the free-energy of spin-two fields, in d-dimensions. This can be represented as [1, 2, 3].

$\Omega(T) = -\frac{1}{2}\left[\frac{1}{2}\right.$  $\left.+\cdots\right]$, (1)

where the curly line and the blob represent the tree-level graviton propagator and the static contribution to the graviton self-energy, respectively.

The method

The gauge theory of Spin-two fields is described by the Einstein-Hilbert action plus gauge fixing and ghosts, in the weak field approximation [4]

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}; \quad \kappa \equiv \sqrt{32\pi G}. \tag{2}$$

The *gauge freedom* is employed in order to obtain the most convenient expression for the tree-level graviton propagator, $D_{\mu\nu,\alpha\beta}$. More specifically, *if* there is a gauge fixing procedure such that the following conditions are fulfilled

$$\eta^{\mu\nu} D^{\rm TT}_{\mu\nu,\lambda\sigma}(k) = 0 \tag{3a}$$

$$k^{\mu} D^{\rm TT}_{\mu\nu,\lambda\sigma}(k) = 0, \tag{3b}$$

then the individual contributions in (1) can be simplified and the sum can be performed in a closed form expression. This gauge fixing is indeed possible as we have verified previously [5].

We also need the finite temperature result for the static graviton self-energy [6, 7]. Since this is a *gauge independent* quantity we do not need to compute it again. Nonetheless, we did compute it in order to test the gauge fixing procedure developed in [5] (the diagrams are shown in figures 1 and 2).

We employ the most convenient and physically illuminating tensor basis for both $D^{\rm TT}$ and the static graviton self-energy, which is given by the three linearly independent *transverse and traceless tensors*. With this choice, each term in the integrand of Eq. (1) can be reduced to

$$\frac{1}{(-|\vec{k}|^2)^n}\left(\frac{d(d-3)}{2}(\bar{\mathcal{C}}^A)^n + (d-2)(\bar{\mathcal{C}}^B)^n + (\bar{\mathcal{C}}^C)^n\right), \tag{4}$$

where the quantities C^I, $I = A, B, C$ are the transverse and traceless components of the self-energy.

Results

From Eq. (4) the sum of all the ring diagrams can be readily performed using $-\sum_{n=2}^{\infty}(-x)^n/n = \log(1+x) - x$, so that the free-energy can be written as

$$\Omega(T) = \frac{T}{2}\frac{1}{(2\pi)^{d-1}}\left[\frac{d(d-3)}{2}I(\bar{\mathcal{C}}^A) + (d-2)I(\bar{\mathcal{C}}^B) + I(\bar{\mathcal{C}}^C)\right], \tag{5}$$

where the integral

$$I(c) \equiv \int d^{d-1}k\left[\log\left(1+\frac{c}{|\vec{k}|^2}\right) - \frac{c}{|\vec{k}|^2}\right] \tag{6}$$

can be done in a closed form (the same kind of expression also arises in the context of scalar fields). This yields the following expression for the free-energy of gravitons in d-dimensions

$$\Omega(T) = -\frac{2\Gamma\left(\frac{5-d}{2}\right)}{(d-1)(d-3)(2\pi)^{d-1}}\left[\frac{2^{(5-d)}\pi^{\frac{5-d}{2}}\Gamma(d+1)\zeta(d)}{\Gamma\left(\frac{d-1}{2}\right)}\right]^{\frac{d-1}{2}}\left(GT^{d-2}\right)^{\frac{d-1}{2}}T^d \times\left\{\frac{d(d-3)}{2}\left(\frac{d-3}{d-1}\right)^{\frac{d-1}{2}} + (d-2)\left(\frac{d-3}{d-1}\right)^{\frac{d-1}{2}} + \left[\left(1-\frac{(d-2)d(d+2)}{2(d-1)}\right)\left(\frac{d-3}{d-1}\right)\right]^{\frac{d-1}{2}}\right\}. \tag{7}$$

This result exhibts some interesting features. First, for odd space-time dimensions it is a real and singular function; for even space-time dimensions, it is a finite and non-analytic function of GT^{d-2} as one would expect for a *non-perturbative quantity*. However, in this case it acquires an imaginary part. For instance, for $d = 4$ the third term inside the curly brackets, which can be traced back to the T^C component of the self-energy, is equal to $(-7/3)^{3/2}$. As a result, one would conclude that the gravitational C-mode is unstable, since the imaginary part of the free energy is connected with the decay rate of the quantum vacuum [8]. However, a detailed investigation shows that the graviton self-energy, which is proportional to GT^4, is of the same order as the solution of the Einstein equation for the curvature tensor, when the thermal energy momentum tensor is taken into account. Therefore, by consistency, one should also take into account the curvature corrections in the analysis of instabilities of gravity at finite temperature. These corrections [6, 7] have the effect of adding some extra contributions to the self-energy in such a way that the C-mode contribution to $\Omega(T)$ would change the third term of the curly bracket of Eq. (7) to $(-7/3+5/27)^{3/2}$, which is still imaginary. This term may be related to an imaginary value of a thermal Jeans mass [6, 7], which reflects the instability of the system due to the universal attractive nature of gravity.

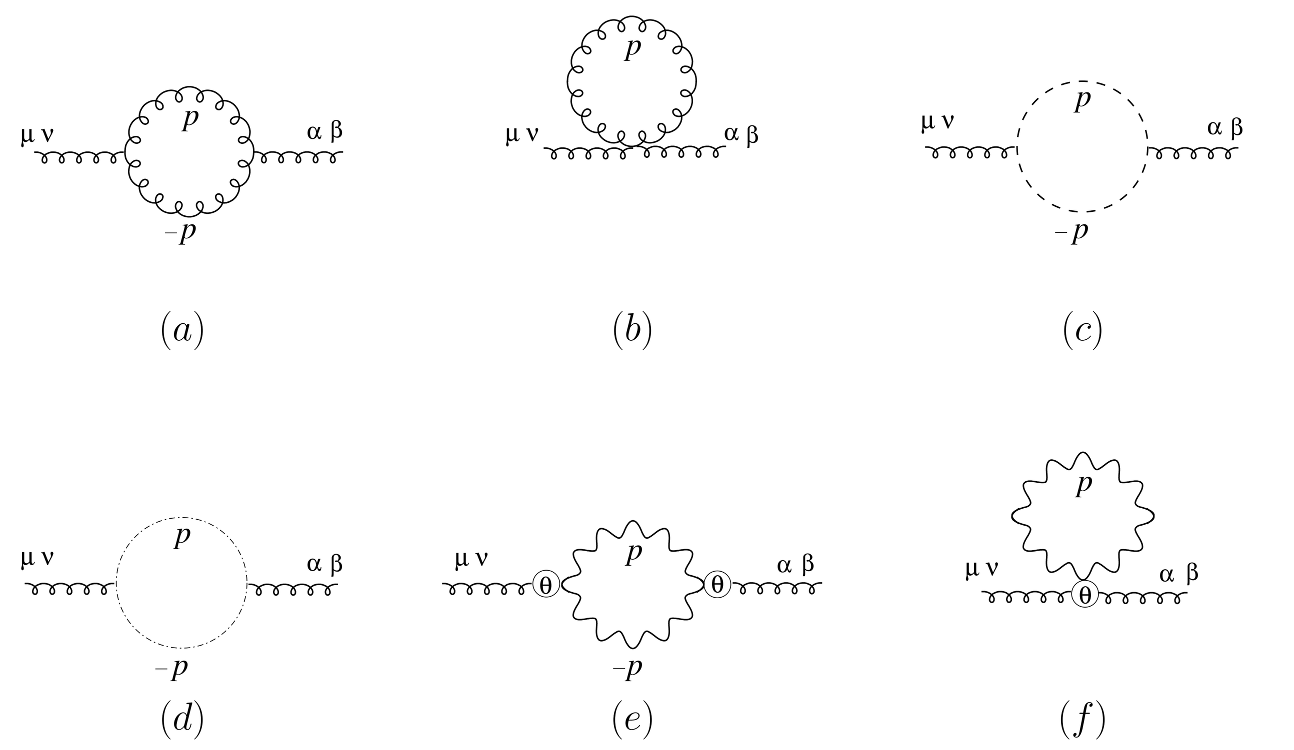


Figure 1: Diagrams which contribute the static limit of the graviton self-energy. The curly and wave lines represent respectively gravitons and the θ ghost. The dashed and dot-dashed lines represent the two types Fermionic ghosts. Graphs (a), (b), (e) and (f) have a symmetry factor $1/2$. A factor of (-1) is associated with the Fermionic ghost loops in figures (c) and (d).

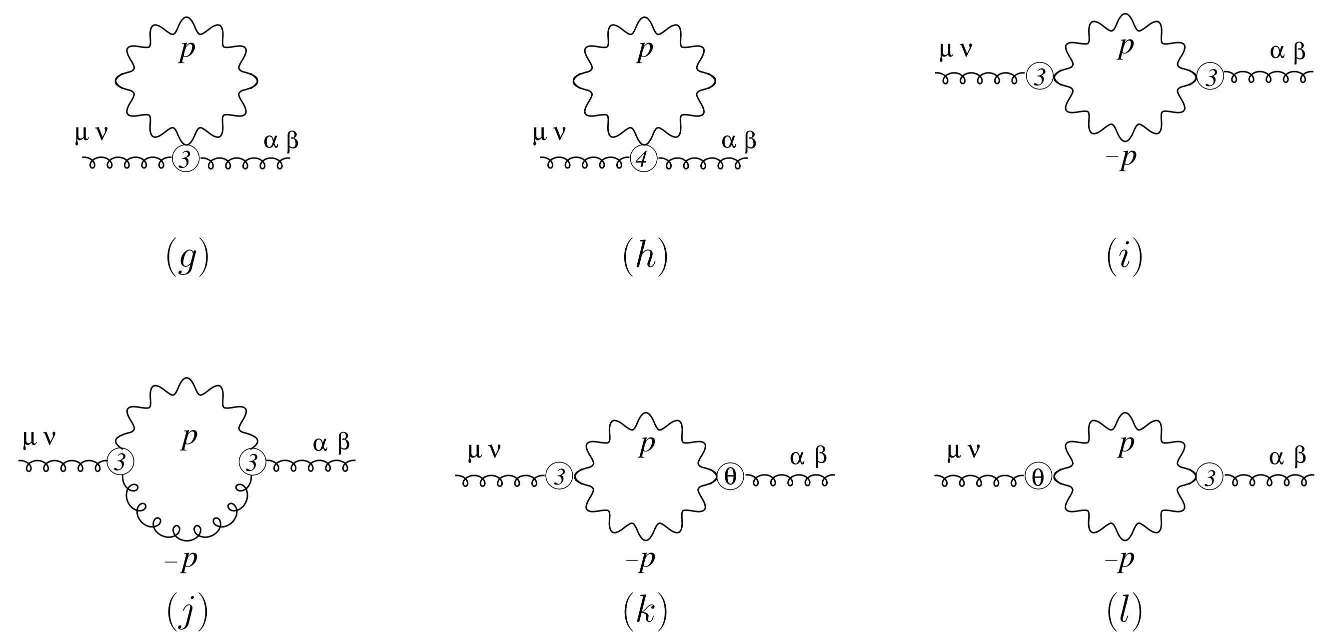


Figure 2: Contributions involving of the θ ghosts (wavy lines) to the static limit of the graviton self-energy. Except for graph (j), all graphs have a symmetry factor $1/2$.

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