Adler Functions, DIS sum rules and Crewther Relations

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in collaboration to P. Baikov (MSU) and J. Kühn (KIT)

Phys.Rev.Lett.101:012002,2008; arXiv:0801.1821
Phys.Rev.Lett.104:132004,2010; arXiv:1001.3606v1
Nucl.Phys.B837:186-220,2010; arXiv:1004.1153
arXiv:1007.0478

35th ICHEP conference, Paris, 23.07.2010

- the current status of $R(s)/D(Q)$ and $\boldsymbol{\alpha_s}$ from $\boldsymbol{\tau}$ and Z at $\boldsymbol{\mathcal{O}(\alpha_s{}^4)}$

- $\boldsymbol{\zeta_3}$ in quenched QED piece $\Longrightarrow$ reliability problem?
  (two aspects: (A) master integrals and (B) algebra/reduction)

- (A): algebraic **and** numerical evaluation of **all masters**

- succesfull test of (B) with Bjorken sum rule for polarized DIS at $\boldsymbol{\mathcal{O}(\alpha_s{}^4)}$ and the generalized Crewther relation*

- results for the Gross-Llewellyn Smith sum rule in $\boldsymbol{\mathcal{O}(\alpha_s{}^4)}$ for a generic gauge group and constraints for the singlet part of R(s)/D(Q) (NEW!)

- Conclusions

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* Crewther (1972,1997); Broadhurst and Kataev (1993);
Braun, Korchemsky and Müller (2003), ...

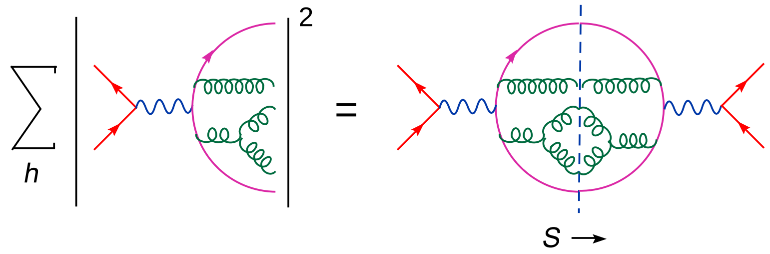
“gold plated” (Bjorken, 1979) QCD observables:

$$\boldsymbol{R_Z = \Gamma(Z_0 \to hadrons)/\sigma(Z_0 \to \mu^+\mu^-)}$$

$$\boldsymbol{R_\tau = \Gamma(\tau \to hadrons + \nu_\tau)/\Gamma(\tau \to l + \bar{\nu}_l + \nu_\tau)}$$

$$\boldsymbol{R(s) = \sigma_{tot}(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)}$$

(via unitarity) $R(s) \approx \Im\,\Pi(s - i\delta)$



$$\Pi(Q^2) \approx \int e^{iqx} \langle 0|T[\ j^v_\mu(x) j^v_\mu(0)\ ]|0\rangle dx$$

$R(s) \leftrightarrow D(Q) \quad \Longleftarrow$ Adler function $\equiv Q^2 \dfrac{d}{dQ^2}\Pi(q^2) = Q^2 \displaystyle\int \frac{R(s)}{(s+Q^2)^2}\, ds$

$$R(s) = 1 + \sum_{i\geq 1} r_i\, a_s(s)^i, \quad D = 1 + \sum_{i \geq 1} d_i\, a_s(Q)^i, \quad (a_s \equiv \alpha_s/\pi, \mu = Q, Q^2 \equiv -q^2)$$

- status of theory (in the massless limit):

$$R = 3 \sum_i Q_i^2 (\, 1 \; + \; \frac{\alpha_s}{\pi} \; + \; \# \left(\frac{\alpha_s}{\pi}\right)^2 \; + \; \# \left(\frac{\alpha_s}{\pi}\right)^3 \; + \; \# \left(\frac{\alpha_s}{\pi}\right)^4 + \cdots)$$

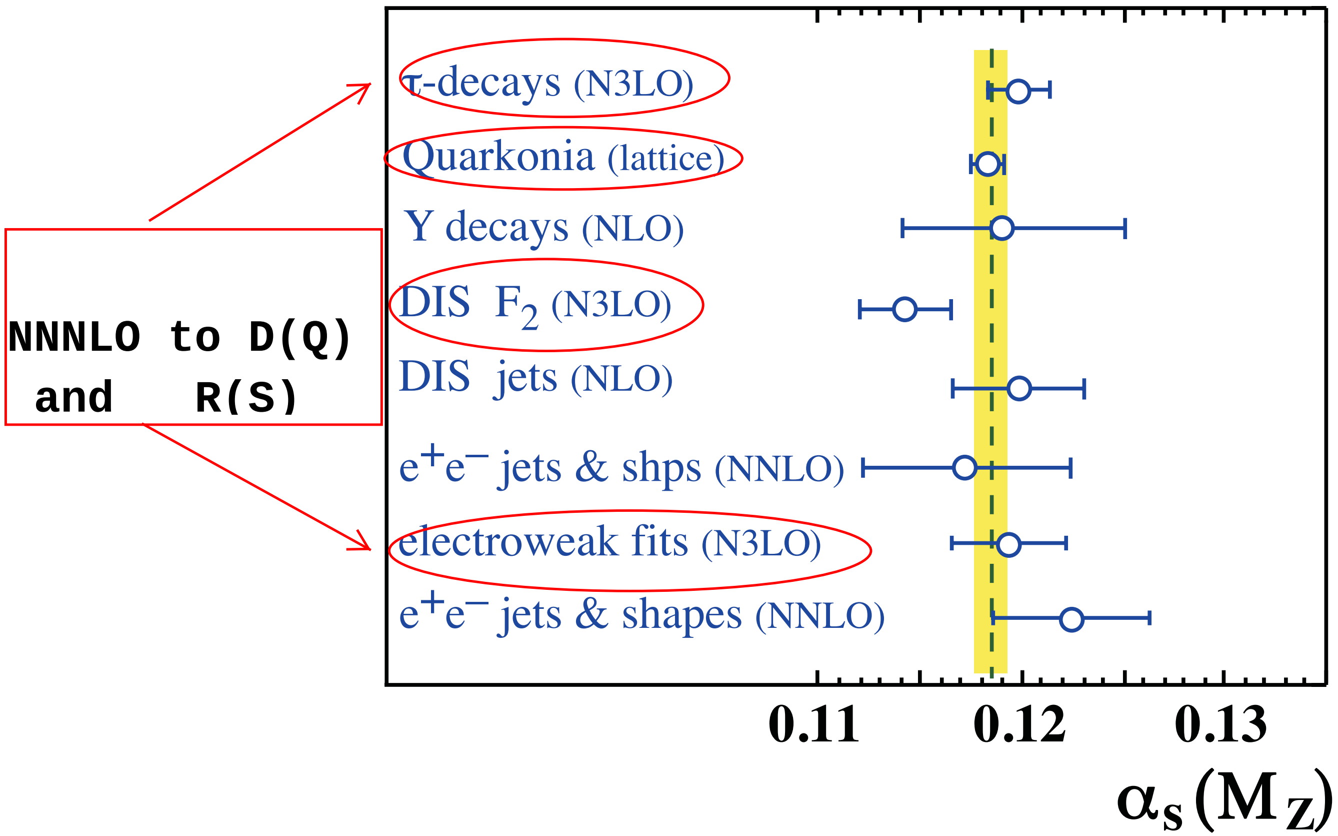
parton model	QED Källen+ Sabry 1955	K. Ch., Kataev, Tkachov; Dine, Sapirstein; Celmaster 1979	Gorishny, Kataev, Larin; Surguladze, Samuel 1991 K. Ch. /gen. gauge/ 1996	Baikov, K. Ch. , Kühn 2008 (Feynman gauge and SU(3)-colour group only)

Tool-box for R(s) at α_s^4:

- reduction to Masters: "direct and automatic" construction of CF's through $\boldsymbol{1/D}$ expansion within the Baikov's representation for Feynman integrals (Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Suppl.116:378-381,2003)

- computing time and required resources: could be huge; we have been using parallel FORM /Vermaseren, Retey, Fliegner, Tentyukov, ...(2000 – . . .) **and** HP XC4000 supercomputer of the Karlsruhe University

World Summary of α_s 2009:



–> $\alpha_s(M_Z) = 0.1184 \pm 0.0007$

(Bethke, arXiv:0908.1135)

Our result for $R(s)$ in numerical form is ($a_s \equiv \alpha_s(s)$)

$$R = 1 + a_s + (1.9857 - 0.1152\, n_f)\, a_s^2$$
$$(-6.63694 - 1.20013 n_f - 0.00518 n_f^2)\, a_s^3$$
$$\boldsymbol{+(-156.61 + 18.77\, n_f - 0.7974\, n_f^2 + 0.0215\, n_f^3)\, a_s^4}$$

Impact on α_s from $Z-$decays:

$$\boldsymbol{\alpha_s(M_Z)^{NNLO} = 0.1185 \pm 0.0026^{\rm exp} + \pm 0.002^{\rm th}}$$

Including the α_s^4 term leads to a shift of $\boldsymbol{\delta\alpha_s(M_Z) = 0.0005}$ and to *four-fold* decrease of the theory error!

$$\boldsymbol{\alpha_s(M_Z)^{NNNLO} = 0.1190 \pm 0.0026^{\rm exp} + \pm 0.0005^{\rm th}}$$

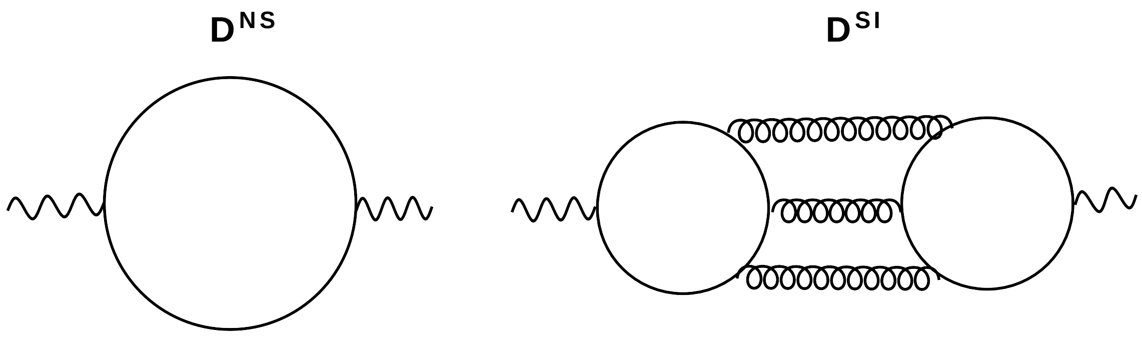
First point of concern:

1. What about impact of the singlet α_s^4 diagrams (contributing to $e^+e^- \to hadrons$ and $\Gamma(Z_0 \to hadrons)$ **but NOT** to R_τ)

$$R^{NS} = 3\sum_i Q_i^2 \left(1 + \frac{\alpha_s}{\pi} + \# \left(\frac{\alpha_s}{\pi}\right)^2 + \# \left(\frac{\alpha_s}{\pi}\right)^3 + \# \left(\frac{\alpha_s}{\pi}\right)^4 + \cdots \right)$$

parton model	QED Källen+ Sabry 1955	K. Ch., Kataev, Tkachov; Dine, Sapirstein; Celmaster 1979	Gorishny, Kataev, Larin; Surguladze, Samuel 1991 K. Ch. /gen. gauge/ 1996	Baikov, K. Ch. , Kühn 2008 (Feynman Gauge only)

$$R^{SI} = \left(\sum_i Q_i\right)^2 \left(\# \left(\frac{\alpha_s}{\pi}\right)^3 + ?? \left(\frac{\alpha_s}{\pi}\right)^4 + \cdots \right)$$



Second (and most severe) point of concern:

2. Doubts on the overall reliability of the result (Kataev, 2008) due to the "problem of quenched QED" at five loops:

i. β^{qQED} is esentially equal $R(s)$ with only terms of order $(\alpha_s C_F)^i$ retained and $\alpha_s C_F$ set to $A \equiv \frac{\alpha}{4\pi}$

ii. many simple people (including us!) and even real experts in formal aspects of QFT believed that β^{qQED} should be rational in all orders ...

ii. β^{qQED} at 5 loops (computed by us as warming-up exercise at 2006) have disobeyed to the expert's view (**!**):

$$\boldsymbol{\beta^{\mathrm{qQED}} = \frac{4}{3} A + 4 A^2 - 2 A^3 - 46 A^4 + \left(\frac{4157}{6} + \boxed{128\,\zeta_3}\right) A^5}$$

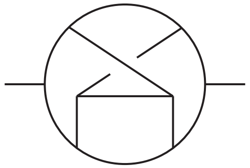
TWO things must be checked:

A. the masters

B. reduction to masters

recently both A and B have been SUCCESSFULLY tested!!!:

A. All 13 non-trivial masters have independently computed analytically [1] **and** numerically [2] with **completely agreeing** results, for instance:



$M_{62},\ \varepsilon^0$

$$= -\frac{\mathbf{10\zeta_5}}{\boldsymbol{\varepsilon}} + \mathbf{130\zeta_5} - \mathbf{10\,\zeta_3^2} - \mathbf{25\zeta_6} - \mathbf{70\,\zeta_7} + \mathcal{O}(\varepsilon)$$

$$= -\frac{\mathbf{10.369277551433697}}{\boldsymbol{\epsilon}} + \mathbf{24.333174217955914} \quad \text{(exact)}$$

$$= -\frac{\mathbf{10.36933 \pm 0.00006}}{\boldsymbol{\varepsilon}} + \mathbf{24.3355 \pm 0.0013} \quad \text{(numerical)}$$

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[1] P. Baikov, J. Kühn, K. Ch., Nucl. Phys.B837:186-220,2010.

[2] A. Smirnov, Tentyukov, Nucl .Phys. .B837:40-49,2010.

To check reduction to masters we have made 3 new calculations in order $\alpha_s{}^4$, all for *general gauge group!*:

1. (non-singlet) Adler function $D^{NS}(a_s)$
2. CF $C^{Bjp}(a_s)$ in the Bjorken sum rule:

$$Bjp(Q^2) = \int_0^1 [g_1^{ep}(x, Q^2) - g_1^{en}(x, Q^2)]dx = \frac{1}{6}|\frac{g_A}{g_V}| C^{Bjp}(a_s)$$

3. CF $C_{GLS}(a_s)$ in the the Gross-Llewellyn Smith sum rule:

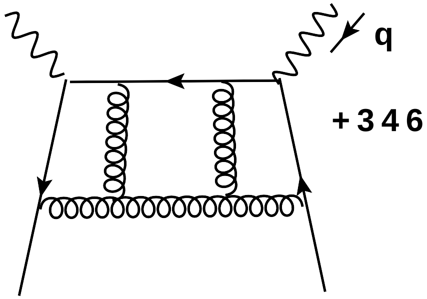
$$GLS(Q^2) = \frac{1}{2}\int_0^1 F_3^{\nu p + \overline{\nu} p}(x, Q^2)dx = 3\, C_{GLS}(a_s)$$

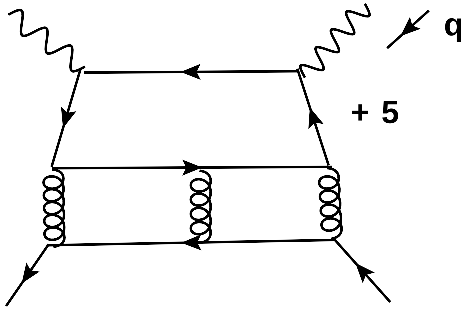
Note that both sum rules are unambiguous QCD predictions /modulo higher twists!/ confrontable with data; they ideally suited to study interface between perturbative and non-perturbarive contributions to the running of *effective* strong coupling constant at intermediate scales⋆

⋆ see, e.g. A. Deur et al., PLN B650 (2997) 244;
R. Pasechnik, D. Shirkov and O. Teryaev, Phys.Rev.D78:071902,2008.

Typical diagrams at ${\alpha_s}^3$ (computed in early nineties /Larin & Vermaseren/),

Bjp *and* GLS $\qquad\qquad$ GLS *only*





Our results for DIS sum rules in numerical form:

$$
\begin{aligned}
C^{NS}_{GLS} \equiv C_{BJp} = 1 \quad & - \quad a_s + a_s^2 \left[-4.583 + 0.3333\, n_f\right] \\
& + \quad a_s^3 \left[-41.44 + 7.607\, n_f - 0.1775\, n_f^2\right] \\
& + \quad a_s^4 \left[-479.4 + 123.4\, n_f - 7.697\, n_f^2 + 0.1037\, n_f^3\right]
\end{aligned}
$$

$$
C^{SI}_{GLS} = 0.4132\, n_f\, a_s^3 + a_s^4\, n_f \left(5.80157 - 0.233185\, n_f\right)
$$

Note: $C^{SI}_{GLS} \ll C^{NS}_{GLS}$ as expected ($\overline{\text{MS}}$-scheme):

The (generalized) Crewther relation

$$C^{Bjp}(a_s)\, D^{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s}\Big[K^{NS} = K_1\, a_s + K_2\, a_s^2 + K_3\, a_s^3 + \ldots\Big] (\star)$$

with $\frac{\beta(\alpha_s)}{\alpha_s} \equiv -\beta_0 a_s + \ldots, \quad \beta_0 = \frac{11}{12} C_A - \frac{T_f\, n_f}{3}$

$(\star)$ **implies 6 constraints on 12 color strtuctures**

$$C_F^4\,,\; C_F^3 C_A\,,\; C_F^2 C_A^2\,,\; C_F C_A^3\,,\; C_F^3 T_F n_f\,,\; C_F^2 C_A T_F n_f\,,$$
$$C_F C_A^2 T_F n_f\,,\; C_F^2 T_F^2 n_f^2\,,\; C_F C_A T_F^2 n_f^2\,,\; C_F T_F^3 n_f^3\,,\; d_F^{abcd} d_A^{abcd}\,,\; n_f d_F^{abcd} d_F^{abcd}$$

appearing at $\mathcal{O}(\alpha_s^4)$ in the difference

$$D^{NS} - 1/C^{Bjp}$$

3 of them are very simple: the above difference cannot contain color structures

$$C_F^4,\;\; d_F^{abcd} d_A^{abcd} \quad n_f d_F^{abcd} d_F^{abcd}$$

remaining three are a bit more complicated

All 6 constaints are indeed fulfilled!

	d_4	$(1/C^{Bjp})_4$
C_F^4	$\frac{4157}{2048} + \frac{3}{8}\zeta_3$	$\frac{4157}{2048} + \frac{3}{8}\zeta_3$
$n_f \frac{d_F^{abcd} d_F^{abcd}}{d_R}$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5$
$\frac{d_F^{abcd} d_A^{abcd}}{d_R}$	$\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5$	$\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5$
$C_F T_f^3$	$-\frac{6131}{972} + \frac{203}{54}\zeta_3 + \frac{5}{3}\zeta_5$	$-\frac{605}{972}$
$C_F^2 T_f^2$	$\frac{5713}{1728} - \frac{581}{24}\zeta_3 + \frac{125}{6}\zeta_5 + 3\zeta_3^2$	$\frac{869}{576} - \frac{29}{24}\zeta_3$
$C_F T_f^2 C_A$	$\frac{340843}{5184} - \frac{10453}{288}\zeta_3 - \frac{170}{9}\zeta_5 - \frac{1}{2}\zeta_3^2$	$\frac{165283}{20736} + \frac{43}{144}\zeta_3 - \frac{5}{12}\zeta_5 + \frac{1}{6}\zeta_3^2$
$C_F^3 T_f$	$\frac{1001}{384} + \frac{99}{32}\zeta_3 - \frac{125}{4}\zeta_5 + \frac{105}{4}\zeta_7$	$-\frac{473}{2304} - \frac{391}{96}\zeta_3 + \frac{145}{24}\zeta_5$
$C_F^2 T_f C_A$	$\frac{32357}{13824} + \frac{10661}{96}\zeta_3 - \frac{5155}{48}\zeta_5 - \frac{33}{4}\zeta_3^2 - \frac{105}{8}\zeta_7$	$-\frac{17309}{13824} + \frac{1127}{144}\zeta_3 - \frac{95}{144}\zeta_5 - \frac{35}{4}\zeta_7$
$C_F T_f C_A^2$	$-\frac{(\cdots)}{(\cdots)} + \frac{8609}{72}\zeta_3 + \frac{18805}{288}\zeta_5 - \frac{11}{2}\zeta_3^2 + \frac{35}{16}\zeta_7$	$-\frac{(\cdots)}{(\cdots)} - \frac{59}{64}\zeta_3 + \frac{1855}{288}\zeta_5 - \frac{11}{12}\zeta_3^2 + \frac{35}{16}$
$C_F^3 C_A$	$-\frac{253}{32} - \frac{139}{128}\zeta_3 + \frac{2255}{32}\zeta_5 - \frac{1155}{16}\zeta_7$	$-\frac{8701}{4608} + \frac{1103}{96}\zeta_3 - \frac{1045}{48}\zeta_5$
$C_F^2 C_A^2$	$-\frac{592141}{18432} - \frac{43925}{384}\zeta_3 + \frac{6505}{48}\zeta_5 + \frac{1155}{32}\zeta_7$	$-\frac{435425}{55296} - \frac{1591}{144}\zeta_3 + \frac{55}{9}\zeta_5 + \frac{385}{16}\zeta_7$
$C_F C_A^3$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\zeta_3 - \frac{77995}{1152}\zeta_5 + \frac{605}{32}\zeta_3^2 - \frac{385}{64}\zeta_7$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\zeta_3 - \frac{12545}{1152}\zeta_5 + \frac{121}{96}\zeta_3^2 - $

Comments:

The Crewther test is highly non-trivial:

- **four-loop box-type diagrams (in propagator kinematics) versus five loop propagators**

- **No IR-trickery is neccessary in calculation of $C_{Bjp}(a_s)$**

- **As a result we he have been able to check that $C_{Bjp}(a_s)$ is indeed gauge-independent (the Adler finction was computed in the simplest, Feynman gauge only!)**

- **in the course of our calculations we have had to extend the Larin treatment of Hooft-Veltman γ_5 at 4-loop level**

Crewther relation between $D = D^{NS} + D^{SI}$ and C_{GLS}

$$\left(D^{NS} + d_3^{SI} a_s^3 + d_4^{SI} a_s^4\right)\left(C_{GLS}^{NS} + c_3^{SI} a_s^3 + c_4^{SI} a_s^4\right) =$$
$$1 + \frac{\beta(\alpha_s)}{\alpha_s}\left[K^{NS} + a_s^3 K_3^{SI} n_f \frac{d_F^{abc} d_F^{abc}}{d_R}\right] \quad \star$$

with $\frac{\beta(\alpha_s)}{\alpha_s} \equiv -\beta_0 a_s + \ldots, \quad \beta_0 = \frac{11}{12} C_A - \frac{T_f}{3}$

$$d_3^{SI} = n_f \frac{d_F^{abc} d_F^{abc}}{d_R} d_{3,1}^{SI}, \quad d_4^{SI} = n_f \frac{d_F^{abc} d_F^{abc}}{d_R}\left(C_F d_{4,1}^{SI} + C_A d_{4,2}^{SI} + T_F d_{4,3}^{SI}\right)$$

$$c_3^{SI} = n_f \frac{d_F^{abc} d_F^{abc}}{d_R} c_{3,1}^{SI}, \quad c_4^{SI} = n_f \frac{d_F^{abc} d_F^{abc}}{d_R}\left(C_F c_{4,1}^{SI} + C_A c_{4,2}^{SI} + T_F c_{4,3}^{SI}\right)$$

rhs of $\star$ depends on only 1 unknown parameter, K_3^{SI}, thus we have 3-1 =2 constraints on 3 unknown coefficients in d_4^{SI}

Obvious solution of these constraints reads:

$$d^{SI}_{4,1} = -\frac{3}{2}c^{SI}_{3,1} - c^{SI}_{4,1} = -\frac{13}{64} - \frac{\zeta_3}{4} + \frac{5\zeta_5}{8}$$

$$d^{SI}_{4,2} = -c^{SI}_{4,2} + \frac{11}{12}\,K^{SI}_{3,1}$$

$$d^{SI}_{4,3} = -c^{SI}_{4,3} + \frac{1}{3}\,K^{SI}_{3,1}$$

CONCLUSION

- **The Adler $D^{NS}(a_s)$-function and the CF $C^{Bjp}(a_s)$ of the Bjorken sum rule for the polarized DIS have been both analytically evaluated for a generic gauge group at $\mathcal{O}({\alpha_s}^4)$**

- **The generalized Crewther relation puts as many as 6 highly non-tivial constraints on the difference $d_4 - (C^{Bjp})_4$ which are all fulfilled!**

- **CF $C^{GLS}(a_s)$ of the Gross-Llewellyn Smith sum rule has been analytically evaluated for a generic gauge group at $\mathcal{O}({\alpha_s}^4)$**

- **At order $\mathcal{O}({\alpha_s}^4)$ the singlet Adler function D^{SI} is contrubuted by exactly three color structures. All three are fixed by the corresponding Crewther relation up to only one still unknown constant**

- **The full calculation of D^{SI} at order $\mathcal{O}({\alpha_s}^4)$ is under way and should be finished soon (another non-trivial check of the reduction and the Crewther relation!)**