Scattering amplitudes in maximally supersymmetric Yang-Mills theory

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**Why scattering amplitudes?**

*Hep-ph motivation:* Search for the Higgs boson at Large Hadron Collider

- Lots of produced particles in the final state leading to large background
- Identification of Higgs boson requires detailed understanding of scattering amplitudes
- Theory should provide solid basis for a successful physics program at the LHC

*Hep-th motivation:* Scattering amplitudes have a remarkable structure in planar $\mathcal{N} = 4$ SYM

- Simpler than Standard Model amplitudes but share many of the same properties
- All-order conjectures [Bern, Dixon, Smirnov], proposal for strong coupling via AdS/CFT [Alday, Maldacena]
- Hints for new symmetry – dual superconformal invariance [Drummond, Henn, GK, Sokatchev]
Maximally supersymmetric Yang-Mills theory

✔ Most (super)symmetric theory possible (without gravity)

✔ Uniquely specified by local internal symmetry group - e.g. number of colors $N_c$ for $SU(N_c)$

✔ Exactly scale-invariant field theory for any coupling $g^2$ (Green functions are powers of distances)

✔ *Gluon tree amplitudes* are the same in all gauge theories

Particle content:

- Massless spin-1 gluon ( = the same as in QCD)
- 4 massless spin-1/2 gluinos ( = cousin of the quarks)
- 6 massless spin-0 scalars

Interaction between particles:

All proportional to same dimensionless coupling $g$ and related to each other by supersymmetry
Gluon amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory

- On-shell matrix elements of $S$–matrix:

  $A_n = \text{tr} \left[ T^{a_1} T^{a_2} \ldots T^{a_n} \right] A_{n}^{h_1, h_2, \ldots, h_n}(p_1, p_2, \ldots, p_n) + \text{[Bose symmetry]}$

- Quantum numbers of scattered gluons:
  - Color: $a_i = 1, \ldots, N_c^2 - 1$
  - Light-like momenta: $(p_i^\mu)^2 = 0$
  - Polarization state (helicity): $h_i = \pm 1$

- Color-ordered planar gluon amplitudes:

- Supersymmetry all-loop relations:

  $A_n^{++\ldots+} = A_n^{-+\ldots+} = 0$

- Classification of amplitudes according to the total helicity $h = \sum_1^n h_i$

  $\text{MHV} = \{ A_n^{--+\ldots+}, A_n^{--+\ldots+}, \ldots \}$, \hspace{1cm} $\text{NMHV} = \{ A_n^{---+\ldots+}, A_n^{--+\ldots+}, \ldots \}$

  MHV = Maximal Helicity Violating amplitudes, NMHV = next-to-MHV, ...
Hints for integrability

Gluon amplitudes at tree level:

\[ S = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} \delta^{(4)}(\sum_{i} p_i) \]

Where does this simplicity come from ... Look for symmetries
Conformal symmetry of the amplitude

$\mathcal{N} = 4$ SYM has (super)conformal $SU(2, 2|4)$ symmetry to all loops:

✔ Consequences of conformal symmetry for the correlation functions

$$\langle O(x)O(0) \rangle \sim (x^2)^{-\Delta}$$

$$\langle O_1(x_1)O_2(x_2)O_3(0) \rangle \sim (x_{12}^2)^{-\Delta_3 - \Delta_1 - \Delta_2}(x_2^2)^{-\Delta_1 - \Delta_2 - \Delta_3}(x_1^2)^{-\Delta_2 - \Delta_3 - \Delta_1}$$

✔ Conformal symmetry acts locally in $x -$ space but non-locally in $p -$ space

✔ Realization of conformal symmetry for the amplitudes

$$k_{\alpha \dot{\alpha}} = \sum_i \frac{\partial^2}{\partial \lambda_i^\alpha \partial \bar{\lambda}_i^{\dot{\alpha}}} \implies k_{\alpha \dot{\alpha}} A_n^{MHV} = 0$$

Can be extended to the full $SU(2, 2|4)$ superconformal invariance

$$g \cdot A_n^{MHV} = 0, \quad g = \{p, m, k, q, \bar{q}, s, \bar{s}, \ldots\} \in SU(2, 2|4)$$

Much less trivial to verify for NMHV, $N^2$MHV,... amplitudes

✔ Conformal symmetry alone is not powerful enough to fix the tree amplitudes

[ICHEP 2010, 22 July 2010 - p. 6/13]
Dual $\mathcal{N} = 4$ (super)conformal symmetry

The $\mathcal{N} = 4$ amplitudes have a much bigger, dual conformal symmetry

[Drummond, Henn, GK, Sokatchev]

✔ Examine absolute value of the amplitude:

$$\left| \hat{A}^\text{MHV}_n \right|^2 = \frac{(S_{12})^4}{S_{12}S_{23}\ldots S_{n1}}, \quad \text{(with } S_{ij} = (p_i + p_j)^2)$$

✔ Introduce dual variables (not a Fourrier transform!)

$\times \quad p_i = x_i - x_{i+1}, \quad x_{n+1} \equiv x_1$

$\times \quad p_i^2 = 0 \implies (x_i - x_{i+1})^2 = 0$

$\times \quad S_{i,i+1} = (x_i - x_{i+2})^2$

✔ The MHV amplitude in the dual space

$$\left| \hat{A}^\text{MHV}_n \right|^2 = \frac{[(x_1 - x_3)^2]^3}{(x_2 - x_4)^2(x_3 - x_5)^2\ldots(x_n - x_1)^2}$$

Looks like $n$–point correlation function in $x$–space, BUT $x$’s are the momenta!
Conformal inversions in dual $x-$space

\[ x_i^\mu \rightarrow \frac{x_i^\mu}{x_i^2} \quad \Rightarrow \quad S_{i,i+1} \rightarrow (x_i^2 x_{i+2}^2)^{-1} S_{i,i+1} \]

Acts locally on the momenta $\Rightarrow$ is not related to conformal symmetry of $\mathcal{N} = 4$ SYM

The tree-level MHV amplitude is covariant under dual conformal inversions

\[ I \left[ \mathcal{A}_{n}^{\text{MHV}} \right] = (x_1^2 x_2^2 \ldots x_n^2) \times \mathcal{A}_{n}^{\text{MHV}} \]

Dual conformal symmetry can be extended to dual superconformal $\widetilde{SU}(2,2|4)$ symmetry

\[ G \cdot \mathcal{A}_{n}^{\text{MHV}} = 0, \quad G = \{ P, M, K, Q, \bar{Q}, S, \bar{S}, \ldots \} \in \tilde{SU}(2,2|4) \]

Dual superconformal symmetry is a property of all tree-level amplitudes $(\text{MHV}, \text{NMHV}, \mathcal{N}^2 \text{MHV}, \ldots)$ in $\mathcal{N} = 4$ SYM theory

[Drummond, Henn, GK, Sokatchev], [Brandhuber, Heslop, Travaglini]
Symmetries of tree amplitudes

✔ The relationship between conventional and dual superconformal $su(2, 2|4)$ symmetries:

\[ \begin{align*}
    p & \quad q \quad \bar{q} \\
    q & \quad \bar{q} \\
    s & \quad \bar{s} \\
    \bar{s} & \quad P
\end{align*} \]

[Drummond,Henn,GK,Sokatchev]

✔ The same symmetries appear at strong coupling from invariance of AdS$_5 \times$ S$^5$ sigma model under bosonic [Kallosh,Tseytlin] + fermionic T-duality [Berkovits,Maldacena],[Beisert,Ricci,Tseytlin,Wolf]

✔ (Infinite-dimensional) closure of two symmetries has Yangian structure [Drummond,Henn,Plefka]

Are tree level amplitudes completely determined by the symmetries?
Integrability of tree amplitudes

- General expression for the tree amplitude dictated by the symmetries
  \[ A_{n}^{NP\text{MHV}} = A_{n}^{\text{MHV}} \times \sum_{\alpha} c_{\alpha} R_{n}^{(\alpha)} \quad (c_{\alpha} \text{ arbitrary constants}) \]

- \( R_{n}^{(\alpha)} \) are (super) invariants of both conventional \((g)\) and dual \((G)\) symmetries:
  \[
  \begin{cases}
  g \cdot R_{n} = G \cdot R_{n} = 0 \\
  R_{n} = \text{Polynomial in Grassmann } \theta \text{'s of degree } 4p
  \end{cases}
  \]

- General form of \( R \)– invariants is known \[\text{[Arkani-Hamed et al],[Mason,Skinner],[Drummond,Ferro],[GK,Sokatchev]}\]

- \( c_{\alpha} \) are fixed from analytic properties: \[ A_{n}^{NP\text{MHV}} = \text{meromorphic functions of } S_{i..j} \]

- Example: \( n \)– particle NMHV tree amplitude \[\text{[Drummond,Henn,GK,Sokatchev]}\]

- \[ A_{n}^{\text{NMHV}} = A_{n}^{\text{MHV}} \sum_{4 \leq s+1 < t \leq n} R_{1st} \]

- \[ R_{rst} = \frac{\langle s - 1s \rangle \langle t - 1t \rangle \delta^{(4)}(\langle r | x_{rs} x_{st} | \theta_{tr} \rangle + \langle r | x_{rt} x_{ts} | \theta_{sr} \rangle)}{x_{st}^{2} \langle r | x_{rs} x_{st} | t - 1 \rangle \langle r | x_{rs} x_{st} | t \rangle \langle r | x_{rt} x_{ts} | s - 1 \rangle \langle r | x_{rt} x_{ts} | s \rangle} \]

**All tree \( N = 4 \) amplitudes are uniquely fixed by symmetries + analyticity condition**
Do symmetries survive loop corrections?

- Loop corrections to the amplitudes necessarily induce infrared divergences
- IR divergences preserve supersymmetry but break conformal + dual conformal symmetry
- Symmetries $(p, q, \bar{q}, P, Q, \bar{S}, \ldots)$ survive loop corrections, other $(s, \bar{s}, k, K, S, \bar{Q}, \ldots)$ are broken

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Dual conformal $K$-anomaly is *universal* for all amplitudes (MHV, NMHV, ...)

\[
K^\alpha\dot{\alpha} A_n \equiv \sum_{i=1}^{n} \left[ 2x_i^{\alpha\dot{\alpha}} (x_i \cdot \partial x_i) - x_i^2 \partial x_i^{\alpha\dot{\alpha}} \right] A_n = \frac{1}{2} \Gamma_{\text{cusp}}(g^2) \sum_{i=1}^{n} x_{i,i+1}^{\alpha\dot{\alpha}} \ln \left( \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} \right) A_n
\]

The $s$– and $\bar{Q}$–anomalies are hard to control

[Drummond, Henn, GK, Sokatchev]
Dual conformal anomaly at work

Consequences of the dual conformal $K$-anomaly for the finite part of MHV amplitude:

✓ $n = 4, 5$ are special: the Ward identity has a unique all-loop solution

\[
\ln A_{\text{MHV}}^4 = \frac{1}{4} \Gamma_{\text{cusp}}(g^2) \ln^2 \left( \frac{x_{13}^2}{x_{24}^2} \right) + \text{const},
\]

\[
\ln A_{\text{MHV}}^5 = -\frac{1}{8} \Gamma_{\text{cusp}}(g^2) \sum_{i=1}^{5} \ln \left( \frac{x_{i,i+2}^2}{x_{i,i+3}^2} \right) \ln \left( \frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2} \right) + \text{const}
\]

Exactly the ABDK/BDS ansatz for the 4- and 5-point MHV amplitudes!

✓ Starting from $n = 6$ there are conformal invariants in the form of cross-ratios $u_{ijkl} = \frac{x_{i,l}^2 x_{j,k}^2}{x_{i,k}^2 x_{j,l}^2}$

General solution to the Ward identity contains an arbitrary function of the conformal cross-ratios

✓ The function is identified at two loops [Drummond,Henn,GK,Sokatchev] [Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]

✗ Analytical expression at weak coupling [Del Duca,Durer,Smirnov],[Zhang],[Goncharov,Spradlin,Vergu,Volovich]

✗ Strong coupling prediction [Alday, Gaiotto,Maldacena]

✗ Rich structure at strong coupling (integrability, Y-system, TBA) [Alday, Gaiotto,Maldacena,Sever,Viera]
Conclusions and open questions

✔ Scattering amplitudes in planar $\mathcal{N} = 4$ SYM possess dual superconformal symmetry:

[Drummond, Henn, G, Sokatchev]

✗ Relates various particle amplitudes with different helicity configurations (MHV, NMHV,...)

✗ Uniquely fixes all tree amplitudes (under appropriate analyticity conditions)

✗ Imposes non-trivial constraints on the loop corrections

✔ The symmetry becomes manifest through MHV amplitude/correlation function duality

[Alday, Eden, Korchemsky, Maldacena, Sokatchev]

Questions:

✗ What are integrable structures underlying scattering amplitudes in $\mathcal{N} = 4$ SYM?

✗ ‘Bethe ansatz’ for all-loop amplitudes?