Statistical Issues on the Neutrino Mass Hierarchy Determination

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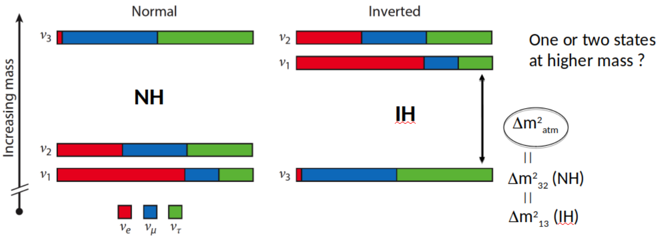
The standard 3 neutrino Mass Hierarchy (MH) issue



About a 15 year story and we are still missing:

1) A common agreed statistical strategy,
2) a robust certified statistical technique.

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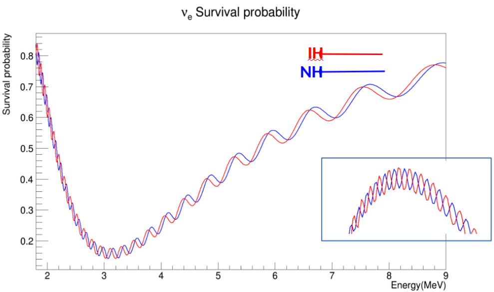
Neutrino MH Determination Using Reactor Spectrum

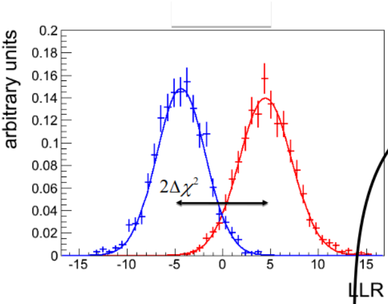


$$p^{\lambda}(\bar{\nu}_e \rightarrow \bar{\nu}_e) = 1 - \frac{1}{2}\sin^2 2\theta_{13}(1 - cos\frac{\Delta m^2_{atm} L}{2E}) - \frac{1}{2}\cos^4\theta_{13}\sin^2 2\theta_{12}(1 - cos\frac{\delta m^2_{sol} L}{2E})$$
$$+ \frac{1}{2}\sin^2 2\theta_{13}\left[\cos^2\left(\theta_{12} + \frac{\pi}{2}\lambda\right)\right](cos\frac{L}{2E}(\Delta m^2_{atm} - \delta m^2_{sol}) - cos\frac{L\Delta m^2_{atm}}{2E})$$

$$\begin{cases} \lambda = 0 \longrightarrow p^{\lambda}(\bar{\nu}_e \rightarrow \bar{\nu}_e) = p_{IH}(\bar{\nu}_e \rightarrow \bar{\nu}_e) \\ \lambda = 1 \longrightarrow p^{\lambda}(\bar{\nu}_e \rightarrow \bar{\nu}_e) = p_{NH}(\bar{\nu}_e \rightarrow \bar{\nu}_e) \end{cases}$$

The "standard" $\Delta\chi^2$ method

$$\Delta\chi^2 = \chi^2_{min(NH)} - \chi^2_{min(IH)}$$



$$n"\sigma" = \frac{2\Delta\chi^2}{2\sqrt{\Delta\chi^2}} = \sqrt{\Delta\chi^2}$$

$$n\sigma = \sqrt{\chi^2_{\min}(NH) - \chi^2_{\min}(NH)}$$

50% prob to reject the wrong MO with nσ

median sensitivity

General result: sigma of each Gaussian = 2 √(Δχ²) (arXiv:1210.8141)

Investigation of the sources of the approximations that brings to

$$\sigma_{\Delta\chi^2} = 2\sqrt{\Delta\chi^2}$$

- ✔ 1) Gaussian statistical errors
- ✔ 2) Gaussian systematic errors
- ✔ 3) Dropping higher orders based on $\mu_i^{IH} >> \left|\mu_i^{NH} - \mu_i^{IH}\right|$
- ! 4) No correlation between number of events in each bin

This is based on well know relations:

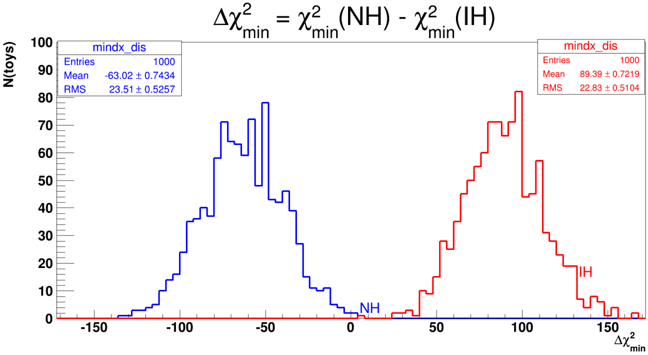
Be x_1, x_2, x_3 ... x_n *n* independent random variables, each with Gaussian distribution. Then it holds

$$y = \sum_{i=1}^{n} x_i\,, \quad \sigma_y^2 = \sum_{i=1}^{n} \sigma_i^2$$

$$\sigma_E = 3\%\sqrt{E}$$

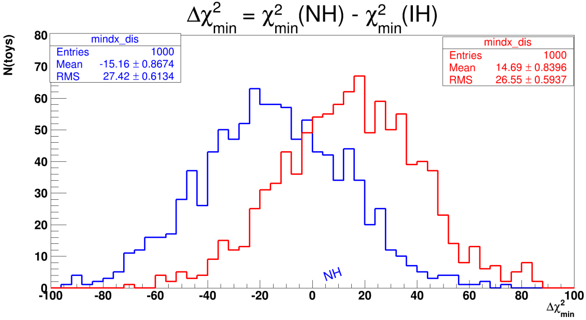
The energy error introduces strong correlations between bins. That is the major limit to the approximation, destroying e.g. the JUNO standard sensitivity

The Validation of Statistical Approximation at Infinity Energy Resolution



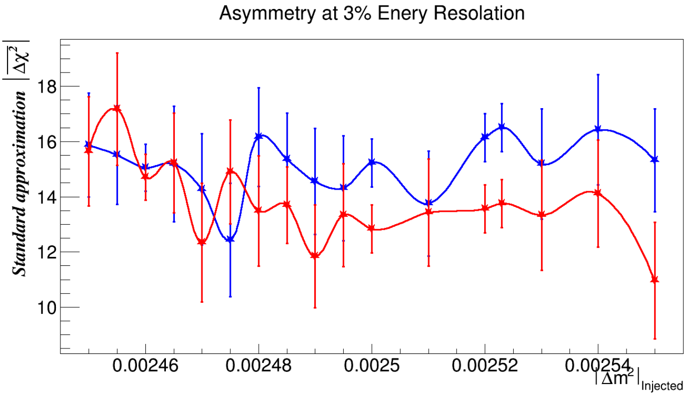
$\Delta\chi^2$ for 1000(NH) + 1000 (IH)JUNO-toy simulations that generated at $\Delta m^2 = 2.500 \times 10^{-3}$ for NH hypothesis (blue distribution) and $\Delta m^2 = -2.460 \times 10^{-3}$ for IH hypothesis (red distribution) with 1 banchmark and **infinity energy resolution**.

The Validation of Statistical Approximation at 3% Relative Energy Resolution



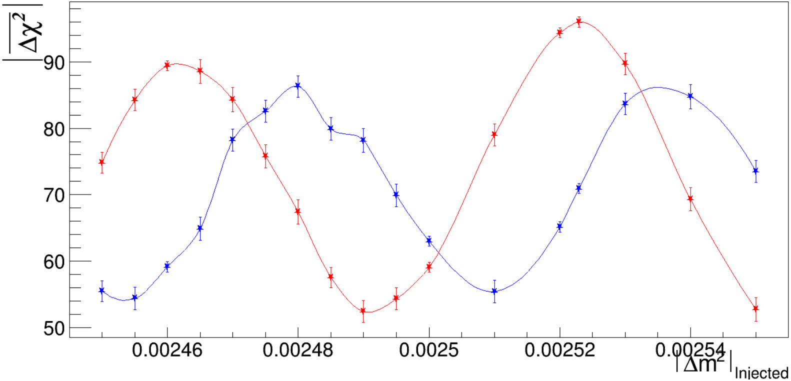
relative energy resolution3%		
μ_{NH}	-15.21	
σ_{NH}	27.52	
μ_{IH}	14.69	
σ_{IH}	26.55	
n"σ"(NH)	1.086(without)	3.9(with)
n"σ"(IH)	1.120(without)	3.8(with)

The Δm^2_{inj} Issue



$|\overline{\Delta\chi^2}|$ varies with $|\Delta m^2|_{inj}$ for 200(NH) + 200 (IH) JUNO-toy like simulations for 1 banchmark assuming 3% relativity energy resolution where blue line for NH sample and red line for IH sample.

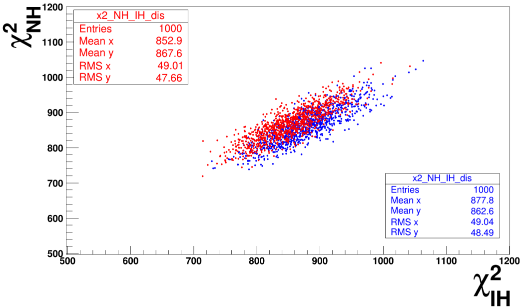
$|\overline{\Delta\chi^2}|$ vs Δm^2_{inj} at Infinity Energy Resolution



$|\overline{\Delta\chi^2}|$ varies with $|\Delta m^2|_{inj}$ for 200(NH) + 200 (IH) JUNO-toy like simulations for 1 banchmark assuming an **infinite energy resolution** where blue line for NH sample and red line for IH sample.

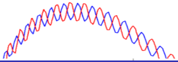
The Sensitivity Results using χ^2 as a Bi-Dimensional

Workshop for Advanced Statistics for Physics Discovery, Statistics Department, Padova University, September 24-25, 2018



	NH	IH
μ_{NH}	853.5	828
σ_{NH}	44	44
μ_{IH}	862	867
σ_{IH}	43	42
r	0.85	0.85
p-Value(NH)	0.331	
nσ(NH)	0.437 σ	
p-Value(IH)	0.310	
nσ(IH)	0.496 σ	

$2D$- F Estimator



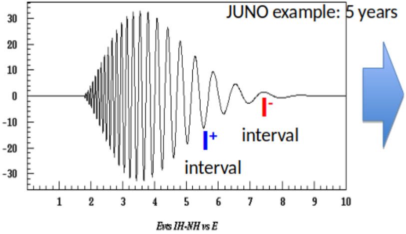
L. Stanco, G. Salamanna, A. Lokhov, C. Sirignano and F. Sawy; A new way to determine the neutrino mass hierarchy at reactors. arXiv:1707.07651v3 [hep-ph]

Instead of keeping separated the two universes NH and IH, construct an estimator based on both of them, optimized to get the maximum separation sensitivity between NH and IH

Price to pay: get degenerate solutions for different Δm^2_{atm} values

$$F = \sum_{\lambda=0}^{1} F^{\lambda} \quad (1)$$

Amount of difference in events for 180 GW year



$$F^{\lambda} = \sum_{i \in I^+} \Delta_i^+ + \sum_{i \in I^-} \Delta_i^- \quad (2)$$

$$= \sum_{i \in I^+} \left(n_{obs} - \mu_i^{\lambda}\right) + \sum_{i \in I^-} \left(\mu_i^{\lambda} - n_{obs}\right) \quad (3)$$

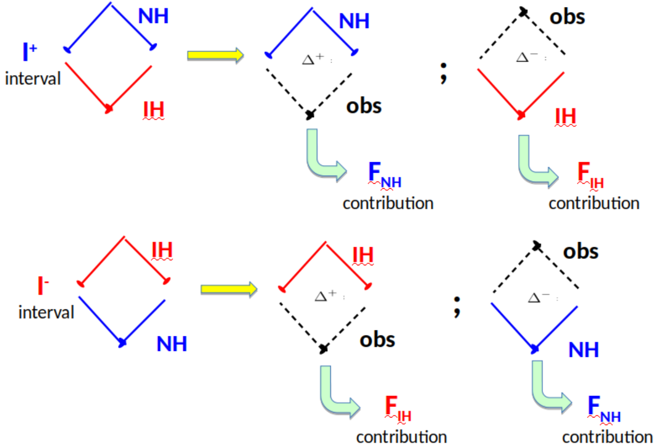
$$I^+ \quad intervals \quad when \quad \mu_i^{NH} > \mu_i^{IH} \quad (4)$$

$$I^- \quad intervals \quad when \quad \mu_i^{NH} < \mu_i^{IH} \quad (5)$$

$$\Delta^+ = n_{obs} - \mu_i^{\lambda} \quad (6)$$

$$\Delta^- = \mu_i^{\lambda} - n_{obs} \quad (7)$$

F computation emphasize the energy intervals where one of the two mass hierarchies is expected to produce more/less events than the opposite one.

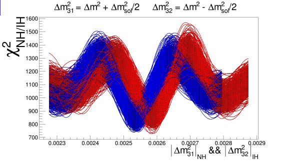


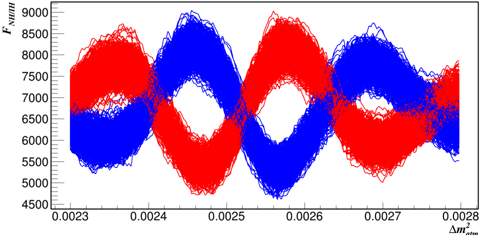
The Δm^2_{atm} Issue

F owns a degeneracy on Δm^2_{atm} !
Two different solutions:

1. one for NH at Δm^2_{32}
2. one for IH at $\Delta m^2_{31} \neq |\Delta m^2_{32}|$

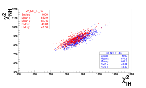
BUT the degeneracy can be overcome by the external information

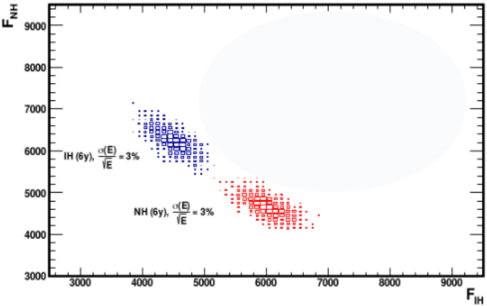




F_{MO} varies with $|\Delta m^2|_{atm}$ for 200(NH) JUNO-toy like simulations for 1 banchmark assuming 3% relativity energy resolution where blue curves for F_{NH} and red curves for F_{IH}.

The Sensitivity Results Using F





	NH	IH
μ_{NH}	4667.2 ± 7.0	6242.9 ± 7.9
σ_{NH}	227.9 ± 4.3	248.4 ± 4.9
μ_{IH}	6006.1 ± 7.4	4491.3 ± 7.0
σ_{IH}	252.0 ± 4.6	219.2 ± 4.3
r	-0.682 ± 0.015	-686 ± 0.14
p-Value(NH)	3.98×10^{-8}	
$n\sigma$(NH)	5.49σ	
p-Value(IH)	3.12×10^{-8}	
$n\sigma$(IH)	5.53σ	

Even with degeneracy $\sim 12 \times 10^{-5} ev^2$, the significance results are still high $\approx 5\sigma$

MHD Using Quasi-Optimal Weights Method:

arXiv:physics/0604127v3,1111.4835v1,and physics/0108030v1; **Fyodor Tkachov**

The method is derived from Pearson's generalized weights. Assuming that there is set of events N_i binned in bin number i and their assumed probability $\pi_i(N_i)$ follows Poisson distribution with the mean μ_i.

$$\pi_i(N_i) = \frac{\mu_i^{N_i}}{N_i!} e^{-\mu_i} \quad (8)$$

The optimal weight $\phi^i_{opt}(N_i)$ for parameter λ:

$$\phi^i_{opt}(N_i) = \frac{\partial ln\pi_i(N_i)}{\partial\lambda} = \frac{\partial}{\partial\lambda}(N_i ln\mu_i - \mu_i) = \left(\frac{\partial ln\mu_i}{\partial\lambda}\right)(N_i - \mu_i) \quad (9)$$

The average number of events in each bin under λ hypothesis is μ^λ:

$$\mu^\lambda = \sigma_E \times \phi_E \times p^\lambda(\bar{\nu}_e \to \bar{\nu}_e) \quad (10)$$

$$\begin{aligned}
F^\lambda &= \sum_i \phi^i_{opt}(N_i) = \sum_i \left(\frac{\partial ln\mu_i^\lambda}{\partial\lambda}\right)\left(N_i - \mu_i^\lambda\right) && (11)\\
&= \sum_i \left[\begin{cases} +1, & in \quad I^+, \quad where \quad \mu^{NH} > \mu^{IH} \\ -1, & in \quad I^-, \quad where \quad \mu^{NH} < \mu^{IH} \end{cases}\right]\left(N_i - \mu_i^\lambda\right) && (12)\\
&= \sum_{i\epsilon I^+}\left(N_i - \mu_i^\lambda\right) + \sum_{i\epsilon I^-}\left(\mu_i^\lambda - N_i\right) && (13)\\
&= \sum_{i\epsilon I^+}\Delta_i^+ + \sum_{i\epsilon I^-}\Delta_i^- && (14)
\end{aligned}$$

To conclude

After the comparison between two different statistical estimators in view of: construction, implementation and sensitivity results, we come up with

1. The statistical significance using $1D - \Delta\chi^2$ depends on the $|\Delta m^2_{inj}|$ value.
2. The $\Delta\chi^2$ is controlled by the statistical assumptions $\rightarrow$ low significance.
3. A new estimator $2D - F$ is introduced.
4. $2D - F$ gives significance results ($\approx 5\sigma$), although a degeneracy on Δm^2_{atm} has to be taken in .

THANK YOU

The BackUP

1. What is meant by a common agreed statistical strategy?
2. What is MHD in physics and statistics?
3. Methodology for p-value Calculation for Bi-Dimensional estimator
4. How are the toy simulations done?
5. How are the fitting procedures?
6. The theoretical derivation of F

Strategy. Results could be:

- compatible* with both NH and IH → wrong experiment
- reject* both NH and IH → wrong data analysis, or new Physics
- compatible* with a MO and reject the other MO' with some significance → good experiment and data analysis, observe MO with that significance

- "borderline" results → wait for more data
- significance at 5 σ → end of the story

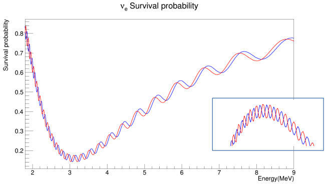
SENSITIVITY= probability (significance) to reject the wrong MH, once exp is compatible* with true MH

* compatible at 95% C.L.
* reject at 95% C.L.

LS, Rev. in Phys. 1 (2016) 90

Neutrino Mass Hierarchy Determination (MHD) Problem

$$\begin{aligned} P^{MH}_{\bar{\nu}_e \longrightarrow \bar{\nu}_e} &= 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2(\Delta_{21}) \\ &- \cos^2\theta_{12}\sin^2 2\theta_{13}\sin^2(\Delta_{31}) \\ &- \sin^2\theta_{12}\sin^2 2\theta_{13}\sin^2(\Delta_{32}) \end{aligned}$$



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✓ **From the physics point of view,** MH is

$$\text{order}\begin{cases} NH \rightarrow m_3^2 > m_2^2 > m_1^2 \;\rightleftharpoons\; \Delta m^2_{31}(NH)\ \textit{has positive value} \\ IH \rightarrow m_2^2 > m_1^2 > m_3^2 \;\rightleftharpoons\; \Delta m^2_{32}(IH)\ \textit{has negative value} \end{cases}$$

$$\Delta m^2_{31}(NH) = \mid \Delta m^2_{32} \mid (IH)$$

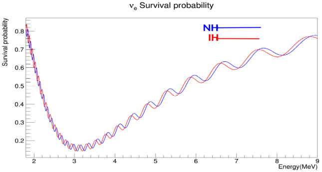
✓ **From the statistics point of view**, MH is a test to distinguish between two hypothesis; normal (NH) and inverted hierarchies (IH).

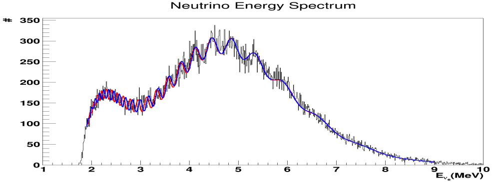
Neutrino MH Determination Using Reactor Spectrum

$$p^{\lambda}(\bar{\nu}_e \to \bar{\nu}_e) = 1 - \frac{1}{2}\sin^2 2\theta_{13}\left(1 - \cos\frac{\Delta m^2_{atm} L}{2E}\right) - \frac{1}{2}\cos^4\theta_{13}\sin^2 2\theta_{12}\left(1 - \cos\frac{\delta m^2_{sol} L}{2E}\right) + \frac{1}{2}\sin^2 2\theta_{13}\left[\sin^2\left(\theta_{12} + \frac{\pi}{2}\lambda\right)\right]\left(\cos\frac{L}{2E}(\Delta m^2_{atm} - \delta m^2_{sol}) - \cos\frac{L\Delta m^2_{atm}}{2E}\right)$$

$$\begin{cases} \lambda = 0 \longrightarrow p^{\lambda}(\bar{\nu}_e \to \bar{\nu}_e) = p_{NH}(\bar{\nu}_e \to \bar{\nu}_e) \\ \lambda = 1 \longrightarrow p^{\lambda}(\bar{\nu}_e \to \bar{\nu}_e) = p_{IH}(\bar{\nu}_e \to \bar{\nu}_e) \end{cases}$$

One benchmark is referring to 6 years running at a distance $\sim$ 52.5 km with a total power 36 GW and relative energy resolution $\frac{3\%}{\sqrt{E}}$. A total of 108357 signal events has been used in our simulations with a 10 keV bin energy width. All the oscillation parameters are in their best fitting values.





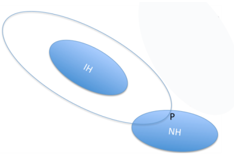
Methodology for p-value Calculation for Bi-Dimensional estimator

$$p_{val}(IH = null\ hypothesis)_{weighted} = p_{val}(IH) \otimes Weight = \int_{\Omega_{NH}} d\bar{x} \left[\int_{\Omega_{NH}(\bar{x})} d\bar{x} f_{IH}(\bar{x}) \bullet f_{NH}(\bar{x}) \right]$$

$$\Omega_{NH} \perp \int_{\Omega_{NH}} d\bar{x} f_{NH}(\bar{x}) = C.L. \quad (15)$$

C.L. = 99.7% is the expected, alternative, hypothesis confidence level.

$$\Omega_{NH}(\bar{x}) \ni f_{IH}(\bar{x'}) \leqslant f_{IH}(\bar{x});\ \bar{x'} \in \Omega_{NH} \quad (16)$$



The Toy Simulations

A toy simulations were based on a single event basis and the expected systematic errors via a Gaussian distribution centered at the expected mean and with the standard deviation of the estimated uncertainty can be added. For JUNO, a global $3\%/E$(MeV) resolution on the energy reconstruction is expected. The oscillation parameters have been taken from the most recent global fits.

	best-fit	3σ region
Sin^2_{12}	0.2970	0.2500 - 0.3540
$Sin^2_{13}(NH)$	0.02140	0.0185 - 0.0246
$Sin^2_{13}(IH)$	0.02180	0.0186 - 0.0248
δm^2_{sol}	7.37×10^{-5}	$6.93 \times 10^{-5} - 7.97 \times 10^{-5}$
$\Delta m^2(NH)$	2.500×10^{-3}	2.37×10^{-3} - 2.63×10^{-3}
$\Delta m^2(IH)$	2.460×10^{-3}	-2.60×10^{-3} to -2.33×10^{-3}

The Poisson statistical fluctuation is automatically included. Version "J17v1r1" of official JUNO Software is used for date simulations.

The Fitting Procedures

The fitting and minimization of χ^2 has required to use directly the ROOT minimization libraries, in particular the TMinuit algorithm. In the minimization procedure all the parameters were fixed to the best values that are indicated in assuming a very small error on it.

MHD Using Quasi-Optimal Weights Method

arXiv:physics/0604127v3, arXiv:1111.4835v1, arXiv:physics/0108030v1

The method is derived from Pearson's generalized weights. Assuming that there is set of events N_i binned in bin number i and their assumed probability $\pi_i(N_i)$ follows Poisson distribution with the mean μ_i.

$$\pi_i(N_i) = \frac{\mu_i^{N_i}}{N_i!} e^{-\mu_i} \tag{17}$$

The optimal weight $\phi^i_{opt}(N_i)$ for parameter λ:

$$\phi^i_{opt}(N_i) = \frac{\partial ln\pi_i(N_i)}{\partial\lambda} = \frac{\partial}{\partial\lambda}(N_i ln\mu_i - \mu_i) = \left(\frac{\partial ln\mu_i}{\partial\lambda}\right)(N_i - \mu_i) \tag{18}$$

The average number of events in each bin under λ hypothesis is μ^λ:

$$\mu^\lambda = \sigma_E \times \phi_E \times p^\lambda(\bar{\nu}_e \to \bar{\nu}_e) \tag{19}$$

$$p^\lambda(\bar{\nu}_e \to \bar{\nu}_e) = 1 - \frac{1}{2}\sin^2 2\theta_{13}(1 - cos\frac{\Delta m^2_{atm} L}{2E}) \tag{20}$$

$$- \frac{1}{2}\cos^4\theta_{13}\sin^2 2\theta_{12}(1 - cos\frac{\delta m^2_{sol} L}{2E}) \tag{21}$$

$$+ \frac{1}{2}\sin^2 2\theta_{13}\left[\cos^2\left(\theta_{12} + \frac{\pi}{2}\lambda\right)\right]\left(cos\frac{L}{2E}(\Delta m^2_{atm} - \delta m^2_{sol}) - cos\frac{L\Delta m^2_{atm}}{2E}\right) \tag{22}$$

$$\text{if} \begin{cases} \lambda = 0 \longrightarrow p^{\lambda}(\bar{\nu}_e \to \bar{\nu}_e) = p_{IH}(\bar{\nu}_e \to \bar{\nu}_e) \\ \lambda = 1 \longrightarrow p^{\lambda}(\bar{\nu}_e \to \bar{\nu}_e) = p_{NH}(\bar{\nu}_e \to \bar{\nu}_e) \end{cases}$$

$$\frac{\partial \mu}{\partial \lambda} = \sigma_E \times \phi_E \times \frac{1}{2}\sin^2 2\theta_{13} \times \left[\frac{\pi}{2}\sin^2(2\theta_{12} + \pi\lambda)\right] \times \left(cos\frac{L}{2E}(\Delta m^2_{atm} - \delta m^2_{sol}) - cos\frac{L\Delta m^2_{atm}}{2E}\right) \quad (23)$$

$$F^{\lambda} = \sum_i \phi^i_{opt}(N_i) \quad (24)$$

$$= \sum_i \left(\frac{\partial ln\mu^{\lambda}_i}{\partial\lambda}\right)\left(N_i - \mu^{\lambda}_i\right) \quad (25)$$

$$= \sum_i \left[\begin{cases} +1, & in \quad I^+, \quad where \quad \mu^{NH} > \mu^{IH} \\ -1, & in \quad I^-, \quad where \quad \mu^{NH} < \mu^{IH} \end{cases}\right]\left(N_i - \mu^{\lambda}_i\right) \quad (26)$$

$$= \sum_{i\epsilon I^+}\left(N_i - \mu^{\lambda}_i\right) + \sum_{i\epsilon I^-}\left(\mu^{\lambda}_i - N_i\right) \quad (27)$$

$$= \sum_{i\epsilon I^+}\Delta^+_i + \sum_{i\epsilon I^-}\Delta^-_i \quad (28)$$