Statistical methods in LHC data analysis *part II.2*

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Contents

- Upper limits
- Treatment of background
- Bayesian limits
- Modified frequentist approach (CL_s method)
- Profile likelihood
- Nuisance parameters and Cousins-Highland approach

Significance: claiming a discovery

- If we measure a signal yield s sufficiently inconsistent with zero, we can claim a discovery
- Statistical significance = probability α to observe s or larger signal in the case of pure background fluctuation
- Often preferred to quote "$n\sigma$" significance, where:

$$\alpha = \int_{n\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \mathrm{d}x = 1 - \frac{1}{2}\mathrm{erf}\left(\frac{n}{\sqrt{2}}\right)$$

n = TMath::NormalQuantile(alpha)

- Usually, in literature:
  - If the significance is > 3 ("3σ") one claims "*evidence of*"
  - If the significance is > 5 ("5σ") one claims "*observation*" (discovery!)
    - probability of background fluctuation = 2.87×10^{-7}

When using upper limits

- Not always experiments lead to discoveries ☹
- What information can be determined if no evidence of signal is observed?
- One possible definition of upper limit:
  - *"largest value of the signal s for which the probability of a signal under-fluctuation smaller or equal to what has been observed is less than a given level α (usually 10% or 5%)"*
  - Upper limit @ x Confidence Level = s such that $\alpha(s) = 1 - x$
  - Similar to confidence interval with a central value, but the interval is fully asymmetric now
- Other approaches are possible:
  - Bayesian limits: extreme of an interval over which the poster probability of $[0, s]$ is $1-\alpha$
    - It's a different definition!
  - Unified Feldman-Cousins frequentist limits

Frequentist vs Bayesian intervals

- The two approaches address different questions
- Frequentist:
  - Probability that a fixed $\mu \in [\mu_1, \mu_2] = 1-\alpha$
  - The interval $[\mu_1, \mu_2]$ is a random variable interval determined from the experiment response
  - Choosing the interval requires an ordering principle (fully asymmetric, central, unified)
- Bayesian:
  - The a posterior probability (degree of belief) of μ in the interval $[\mu_1, \mu_2]$ is equal to α

Event counting

- Assume we have a signal process on top of a background process that lead to a Poissonian number of counts:

$$P_s(n_s) = \frac{s^{n_s} e^{-s}}{n_s!} \qquad P_b(n_b) = \frac{b^{n_b} e^{-b}}{n_b!}$$

- If we can't discriminate signal and background events, we will measure:
  - $n = n_s + n_b$
- The distribution of n is again Poissonian!

Simplest case: zero events

- If we observe zero events we can state that:
  - No background events have been observed ($n_b = 0$)
  - No signal events have been observed ($n_s = 0$)
- The Poissonian probability to observe zero events expecting s (assuming the expected background b is negligible) is:
  - $p(n_s\!=\!0) = P_s(0) = e^{-s} = 1 - \mathrm{C.L.} = \alpha$
- So, *naïvely*: $s < s^{\mathrm{up}} = -\ln(1 - \mathrm{C.L.})$:
  - $s < \mathbf{2.9957}$ @ 95% C.L.
  - $s < \mathbf{2.3026}$ @ 90% C.L.

Bayesian interpretation

- We saw that, assuming a uniform prior for s, the posterior PDF for s is:

$$f(s|n) = \frac{s^n e^{-s}}{n!}$$

- The cumulative is:

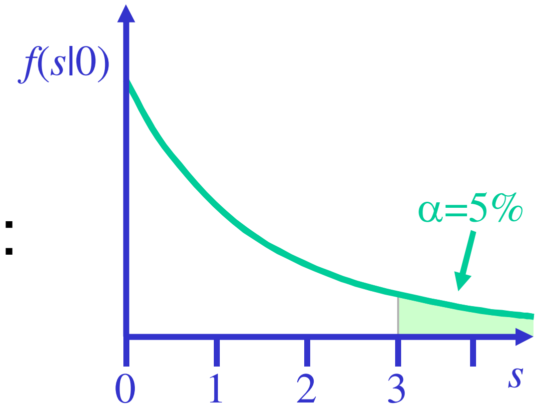
$$F(s|n) = \int_0^s \frac{t^n e^{-t}}{n!} \mathrm{d}t = 1 - e^{-s} \left(\sum_{m=0}^{n} \frac{s^m}{m!} \right)$$

- In particular:

$$F(s|n=0) = 1 - e^{-s}$$

- Which gives *by chance* identical result:

$$P(s < 2.996 | n = 0) = 0.95$$



Upper limits in case of background

- Bayesian approach (flat prior, from 0 to ∞):

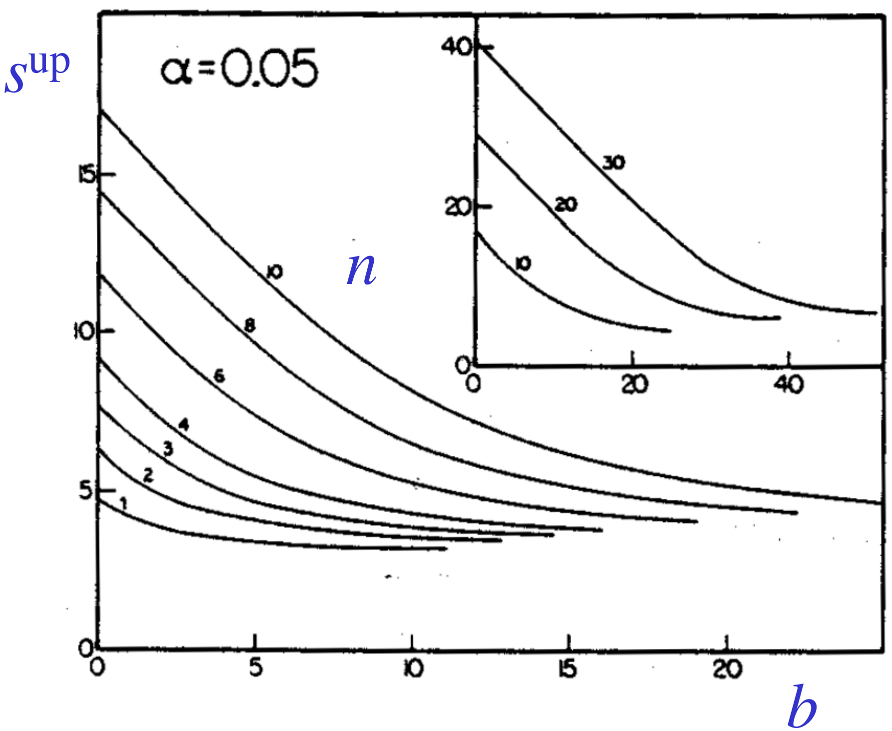
$$1 - \mathrm{C.L.} = \int_0^{s^{\mathrm{up}}} P(s|n)\mathrm{d}n = \frac{\int_0^{s^{\mathrm{up}}} L(n;s)\mathrm{d}s}{\int_0^{\infty} L(n;s)\mathrm{d}s}$$

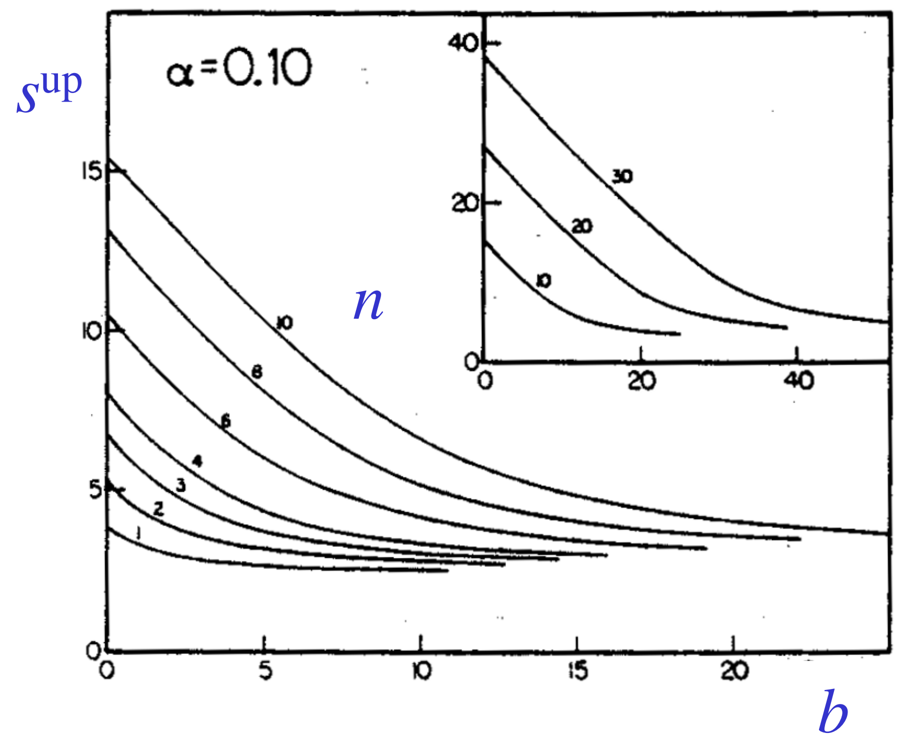
- Where (for fixed b!):

$$L(n;s) = \frac{(s+b)^n}{n!} e^{-(s+b)}$$

- The limit can be obtained inverting the equation: $C.L. = e^{-s^{\mathrm{up}}} \frac{\sum_{m=0}^{n} \frac{(s^{\mathrm{up}}+b)^m}{m!}}{\sum_{m=0}^{n} \frac{b^m}{m!}}$

- Graphical view (O. Helene, 1983)





Bayesian

Upper limit with event counting

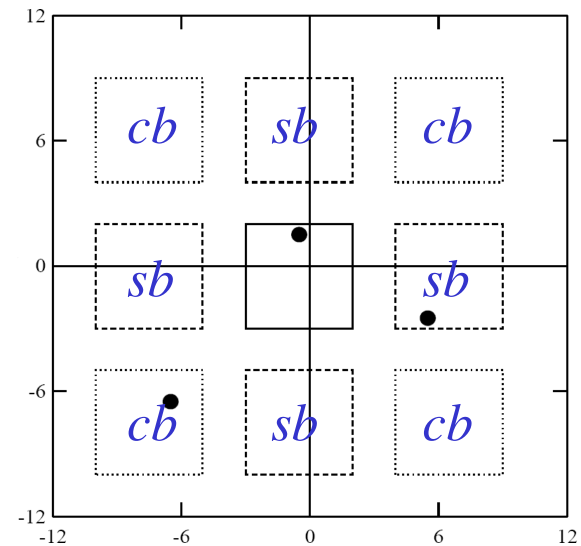
From PDG
in case of no background

"It happens that the upper limit from [central Neyman interval] coincides numerically with the Bayesian upper limit for a Poisson parameter, using a uniform prior p.d.f. for ν."

	$1-\alpha=90\%$		$1-\alpha=95\%$	
n	ν_{lo}	ν_{up}	ν_{lo}	ν_{up}
0	–	2.30	–	3.00
1	0.105	3.89	0.051	4.74
2	0.532	5.32	0.355	6.30
3	1.10	6.68	0.818	7.75
4	1.74	7.99	1.37	9.15
5	2.43	9.27	1.97	10.51
6	3.15	10.53	2.61	11.84
7	3.89	11.77	3.29	13.15
8	4.66	12.99	3.98	14.43
9	5.43	14.21	4.70	15.71
10	6.22	15.41	5.43	16.96

Poissonian background uncertainty

- Assume we estimate the background from sidebands applying scaling factors:
  - $\lambda = s = n - b = n - \alpha\, n_{sb} + \beta\, n_{cb}$
  - sb = “side band”, cb = “corner band”
- Upper limit on s with a C.L. $= \delta$ can be as difficult as:



$$\frac{\int_0^\infty d\lambda_{sb} \int_0^{\gamma\lambda_{sb}} d\lambda_{cb} \frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!} e^{-\lambda_{sb}} \frac{\lambda_{cb}^{N_{cb}}}{N_{cb}!} e^{-\lambda_{cb}} e^{-(\lambda+\alpha\lambda_{sb}-\beta\lambda_{cb})} \sum_{n=0}^{N_{ob}} \frac{(\lambda+\alpha\lambda_{sb}-\beta\lambda_{cb})^n}{n!}}{\int_0^\infty d\lambda_{sb} \int_0^{\gamma\lambda_{sb}} d\lambda_{cb} \frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!} e^{-\lambda_{sb}} \frac{\lambda_{cb}^{N_{cb}}}{N_{cb}!} e^{-\lambda_{cb}} e^{-(\alpha\lambda_{sb}-\beta\lambda_{cb})} \sum_{n=0}^{N_{ob}} \frac{(\alpha\lambda_{sb}-\beta\lambda_{cb})^n}{n!}}$$

$$= \frac{\sum_{n=0}^{N_{ob}} \sum_{k=0}^{n} \alpha^k \frac{\lambda^{n-k} e^{-\lambda}}{(n-k)!} \sum_{j=0}^{k} \frac{(-1)^{k-j}(N_{cb}+k-j)!}{(\gamma-\alpha)^{k-j} j!(k-j)!} \left[\frac{(N_{sb}+j)!}{(1+\alpha)^{N_{sb}+j+1}} - \sum_{i=0}^{N_{cb}+k-j} \frac{(\gamma-\alpha)^i (N_{sb}+j+i)!}{(1+\gamma)^{N_{sb}+j+i+1} i!} \right]}{\sum_{k=0}^{N_{ob}} \alpha^k \sum_{j=0}^{k} \frac{(-1)^{k-j}(N_{cb}+k-j)!}{(\gamma-\alpha)^{k-j} j!(k-j)!} \left[\frac{(N_{sb}+j)!}{(1+\alpha)^{N_{sb}+j+1}} - \sum_{i=0}^{N_{cb}+k-j} \frac{(\gamma-\alpha)^i (N_{sb}+j+i)!}{(1+\gamma)^{N_{sb}+j+i+1} i!} \right]}$$

$$= 1 - \delta,$$

Physical integration bound: $\lambda_{cb} > \frac{\alpha}{\beta}\lambda_{sb} \equiv \gamma\lambda_{sb}$ $\qquad \alpha = 2\frac{A_{sb}}{A_{sig}} \qquad \beta = \frac{A_{cb}}{A_{sig}}$

Zech's "frequentist" interpretation

- Restrict the probability to the observed condition that the number of background events does not exceed the number of observed events
- *"In an experiment where b background events are expected and n events are found, $P(n_b;b)$ no longer corresponds to our improved knowledge of the background distributions. Since n_b can only take the numbers $n_b \le n$, it has to be renormalized to the new range of n_b:*

$$P'(n; s+b) = \sum_{n_b=0}^{n} \sum_{n_s=0}^{n-n_b} P'(n_b; b) P(n_s; s)$$

$$P'(n_b; b) = P(n_b; b) / \sum_{n_b=0}^{n} P(n_b; b)$$

- Leads to a result identical to the Bayesian approach!

$$C.L. = e^{-s^{\mathrm{up}}} \frac{\sum_{m=0}^{n} \frac{(s^{\mathrm{up}}+b)^m}{m!}}{\sum_{m=0}^{n} \frac{b^m}{m!}}$$

- Zech's frequentist derivation attempted was criticized by Highlands: does not respect the coverage requirement
- Often used in a "pragmatic" way, and recommended for some time by the PDG

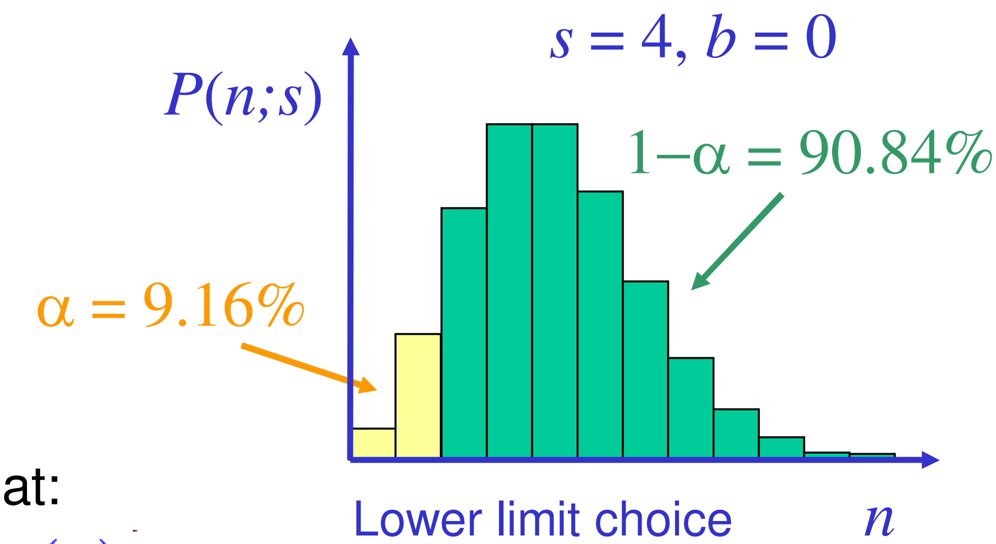
Zech's derivation references

- Bayesian solution found first proposed by O.Helene
  - O. Helene. Nucl. Instr. and Meth. A 212 (1983), p. 319 *Upper limit of peak area* (Bayesian)
- Attempt to derive the same conclusion with a frequentist approach
  - G. Zech, Nucl. Instr. and Meth. A 277 (1989) 608-610 *Upper limits in experiments with background or measurement errors*
- Frequentist validity criticized by Highland
  - V.L. Highland Nucl. Instr. and Meth. A 398 (1989) 429-430 *Comment on "Upper limits in experiments with background or measurement errors" [Nucl. Instr. and Meth. A 277 (1989) 608–610]*
- Zech agreement that his derivation is not rigorously frequentist
  - *G. Zech,* Nucl. Instr. and Meth. A 398 (1989) 431-433 *Reply to 'Comment on "Upper limits in experiments with background or measurement errors" [Nucl. Instr. and Meth. A 277 (1989) 608–610]'*
- Cousins overview and summary of the controversy
  - Workshop on Confidence Limits, 27-28 March, 2000, Fermilab

Frequentist (classic) upper limits

- Construct of Neyman belt with discrete values, can't exactly satisfy coverage:
- $P(s \in [s_1, s_2]) \geq 1-\alpha$
- Build for all s; asymmetric in this example
- Determine the upper limit on given the observed n_{obs}
- The limit s^{up} on s is such that:

$$s^{\mathrm{up}} = \min_{\sum_{n=0}^{n_{\mathrm{obs}}} P(n;s) < \alpha} (s)$$



- In case of $n_{\mathrm{obs}} = 0$ the simple formula holds:

$$s^{\mathrm{up}} = \min_{P(0;s)<\alpha} (s) = \min_{e^{-s}<\alpha} (s) = -\ln\alpha$$

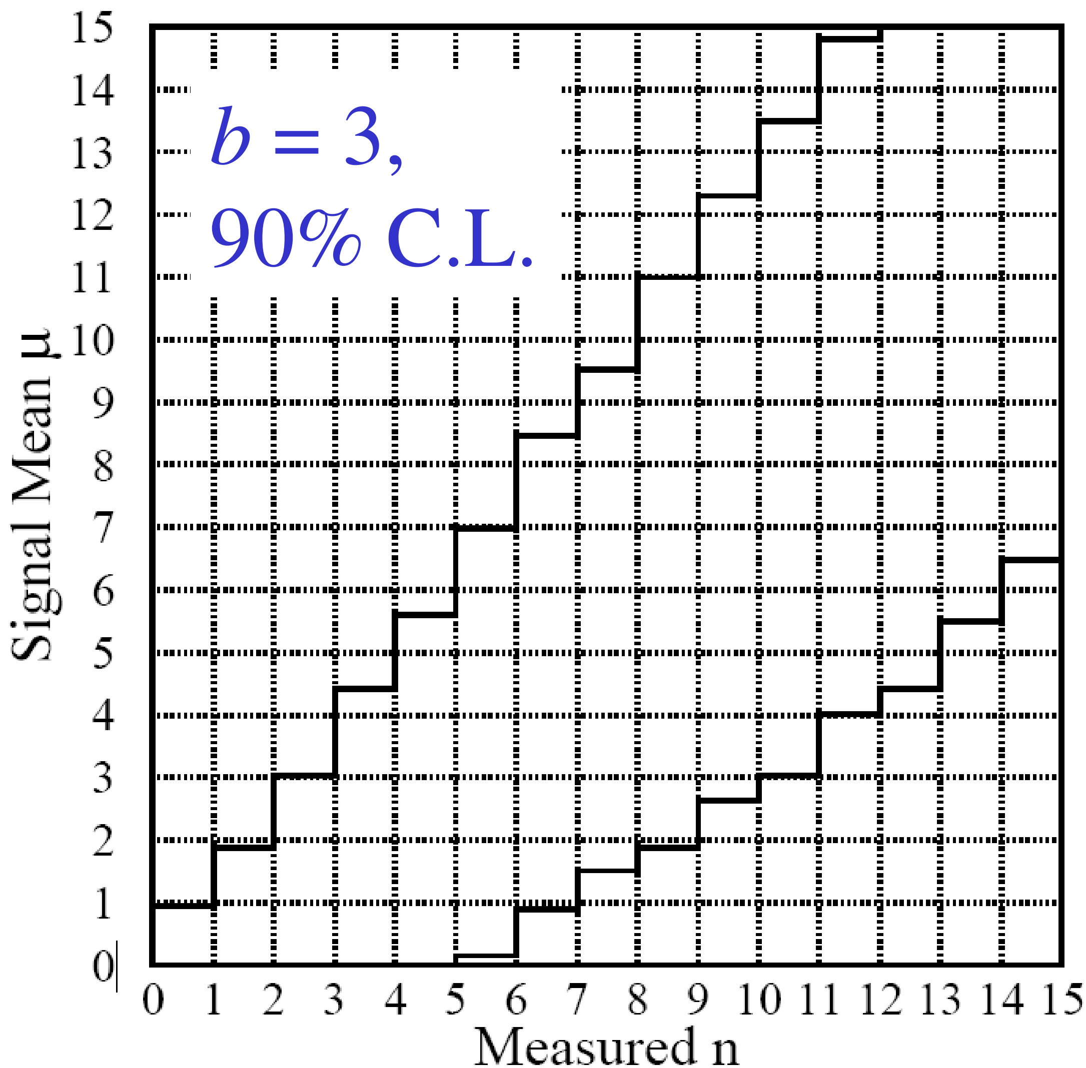
Identical to Bayesian limit for this simple case

Poissonian using Feldman-Cousins

Frequentist

ordering based on likelihood ratio (see slides from part I)

Belt depends on b, of course

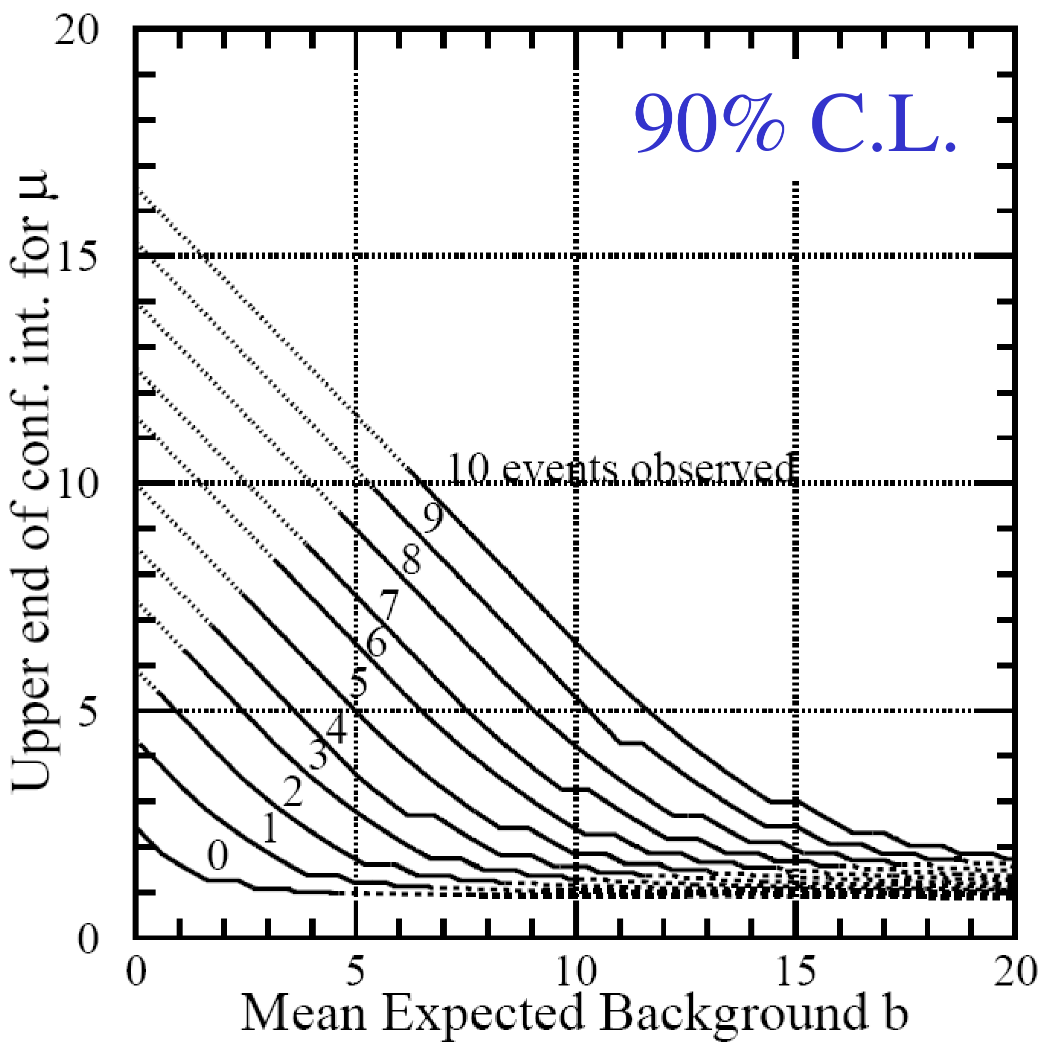


G.Feldman, R.Cousins, Phys.Rev.D,57(1998), 3873

Limits using Feldman-Cousins

Note that the curve for n = 0 decreases with b, while the result of the Bayesian calculation is constant at 2.3

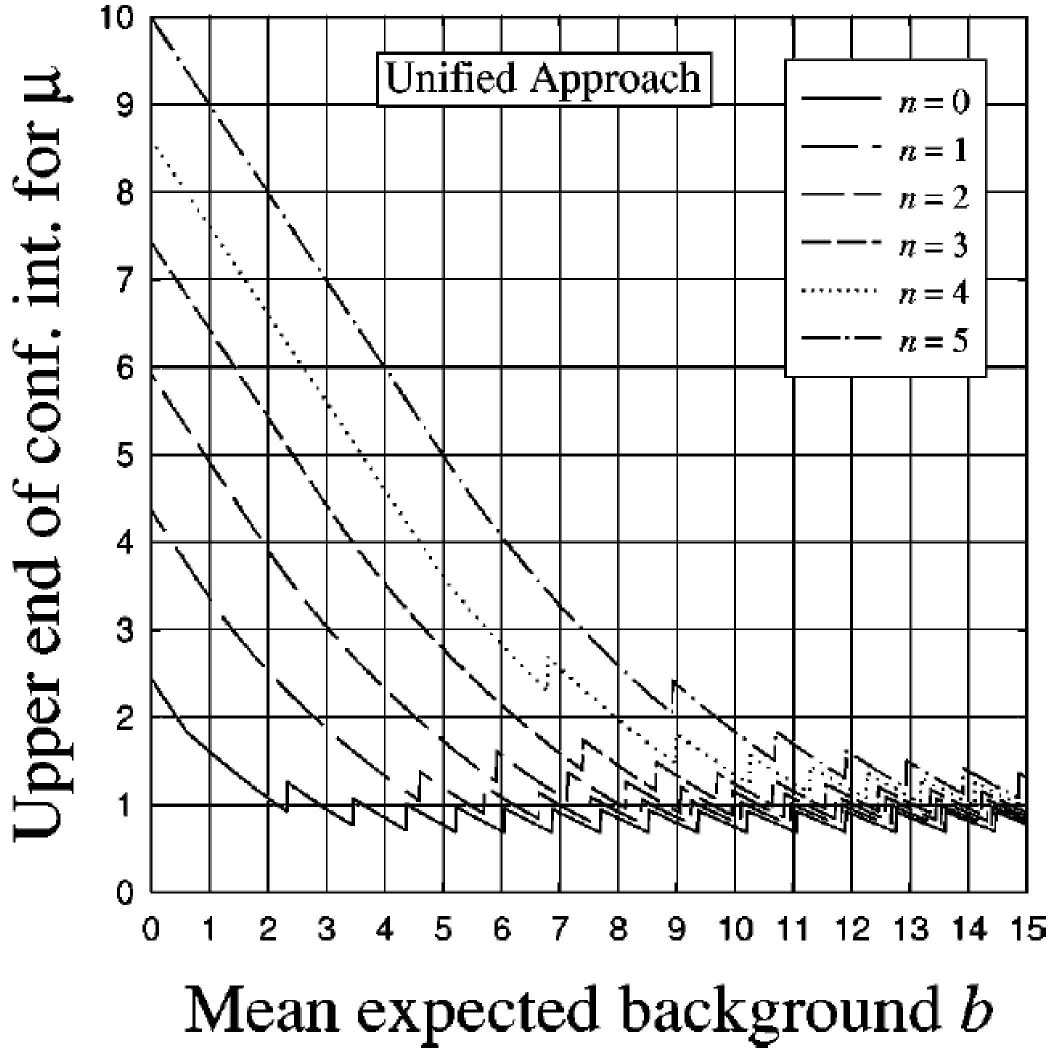
Frequentist interval do not express $P(\mu|x)$!



G.Feldman, R.Cousins, Phys.Rev.D,57(1998), 3873

A Close-up

Note the 'ripple' structure



C. Giunti, Phys.Rev.D,59(1999), 053001

Limits in case of no background

From PDG

“Unified” (i.e.: Feldman-Cousins) limits for Poissonian counting in case of no background

Larger than Bayesian limits

	$1-\alpha=90\%$		$1-\alpha=95\%$	
n	ν_1	ν_2	ν_1	ν_2
0	0.00	2.44	0.00	3.09
1	0.11	4.36	0.05	5.14
2	0.53	5.91	0.36	6.72
3	1.10	7.42	0.82	8.25
4	1.47	8.60	1.37	9.76
5	1.84	9.99	1.84	11.26
6	2.21	11.47	2.21	12.75
7	3.56	12.53	2.58	13.81
8	3.96	13.99	2.94	15.29
9	4.36	15.30	4.36	16.77
10	5.50	16.50	4.75	17.82

From PDG Review…

"The intervals constructed according to the unified procedure for a Poisson variable n consisting of signal and background have the property that for n = 0 observed events, the upper limit decreases for increasing expected background. This is counter-intuitive, since it is known that if n = 0 for the experiment in question, then no background was observed, and therefore one may argue that the expected background should not be relevant. The extent to which one should regard this feature as a drawback is a subject of some controversy"

Problems of "classic"(frequentist) methods

- The presence of background may introduce problems in interpreting the meaning of upper limits
- A statistical under-fluctuation of the background may lead to the exclusion of a signal of zero at 95% C.L.
  - Unphysical estimated "negative" signal?
- *"tends to say more about the probability of observing a similar or stronger exclusion in future experiments with the same expected signal and background than about the non-existence of the signal itself"* [*]
- What we should derive, is just that there is not sufficient information to discriminate the b and $s+b$ hypotheses

- When adding channels that have low signal sensitivity may produce upper limits that are severely worse than without adding those channels

[*] A. L. Read, Modified frequentist analysis of search results (the CLs method), 1st Workshop on Confidence Limits, CERN, 2000

CL$_s$: Higgs search at LEP-II

- Analysis channel separated by experiment (Aleph, Delphi, L3, Opal) and separate decay modes
- Using the Likelihood ratio discriminator:

$$Q(m_H) = \frac{L(s+b)}{L(b)}$$

- Confidence levels estimator (→different from Feldman-Cousins):

$$\boxed{\mathrm{CL}_s = \frac{\mathrm{CL}_{s+b}}{\mathrm{CL}_b} = \frac{N_{Q_{s+b} \le Q}}{N_{Q_b \le Q}}}$$

  - Gives over-coverage w.r.t. classical limit (CL$_s$ > CL$_{s+b}$: conservative)
  - Similarities with Bayesian C.L.

- Identical to Bayesian limit for Poissonian counting!

$$C.L. = e^{-s^{\mathrm{up}}} \frac{\sum_{m=0}^{n} \frac{(s^{\mathrm{up}}+b)^m}{m!}}{\sum_{m=0}^{n} \frac{b^m}{m!}}$$

- *"approximation to the confidence in the signal hypothesis, one might have obtained if the experiment had been performed in the complete absence of background"*

- No problem when adding channels with low discrimination

Observations on CL_s method

- *"A specific modification of a purely classical statistical analysis is used to avoid excluding or discovering signals which the search is in fact not sensitive to"*
- *"The use of CLs is a conscious decision not to insist on the frequentist concept of full coverage (to guarantee that the confidence interval doesn't include the true value of the parameter in a fixed fraction of experiments)."*
- *"confidence intervals obtained in this manner do not have the same interpretation as traditional frequentist confidence intervals nor as Bayesian credible intervals"*

A. L. Read, Modified frequentist analysis of search results (the CLls method), 1st Workshop on Confidence Limits, CERN, 2000

Profile Likelihood

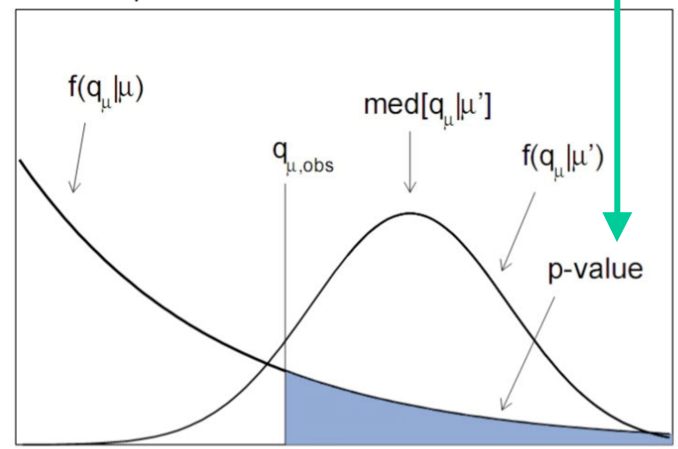
- Use a different test statistics:

$$\lambda(\mu) = \frac{L(\mu, \hat{\hat{\theta}})}{L(\hat{\mu}, \hat{\theta})}$$

Fix μ, fit θ

Fit both μ and θ

$$p_\mu = \int_{q_\mu^{\mathrm{obs}}}^{\infty} f(q_\mu|\mu)\mathrm{d}q_\mu$$



$$q_\mu = -2\ln\lambda(\mu)$$

RooStats::ProfileLikelihoodCalculator

- Wilk's theorem ensures asymptotical χ^2 distribution
- Popular technique in ATLAS
- CMS and LEP use more frequently:

$$Q = \frac{L_{s+b}}{L_b} = \frac{L(\mu=1)}{L(\mu=0)}$$

New signal

SM only

What if s is exactly zero

- If a signal does not exist, s is exactly zero
  - E.g.: searches for non existing exotic processes
- In that case, experiments will observe no signal events (within background fluctuation), but the finite size of the experimental sample will always determine an upper limit greater than zero (at, say, 95% C.L.)
  - Feldman-Cousins gives a C.L. lower than e^{-s} when zero events are seen
- So, the number of experiments where s is greater than the upper limit is zero, not 5%!
- This is a reason of criticism of frequentist upper limits from Bayesians

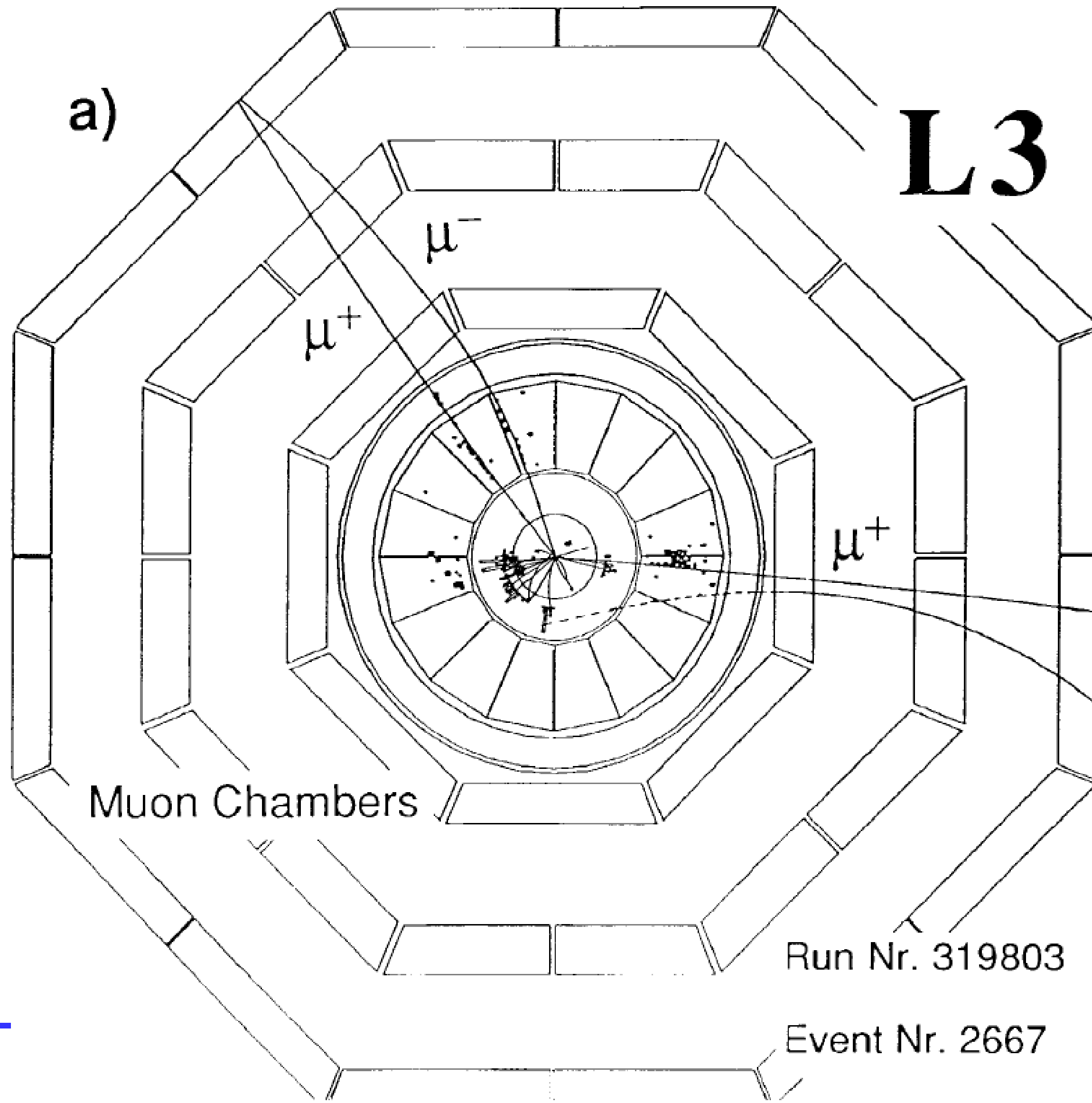
Concrete example I

Higgs search at LEP I (L3)

Higgs search at LEP

- Production via $e^+e^- \rightarrow HZ^* \rightarrow bb l^+ l^-$
- Higgs candidate mass measured via missing mass to lepton pair
- Most of the background rejected via kinematic cuts and isolation requirements for the lepton pair
- Search mainly dominated by statistics
- A few background events survived selection (first observed in L3 at LEP-I)

First Higgs candidate ($m_H \cong 70$ GeV)



Extended likelihood approach

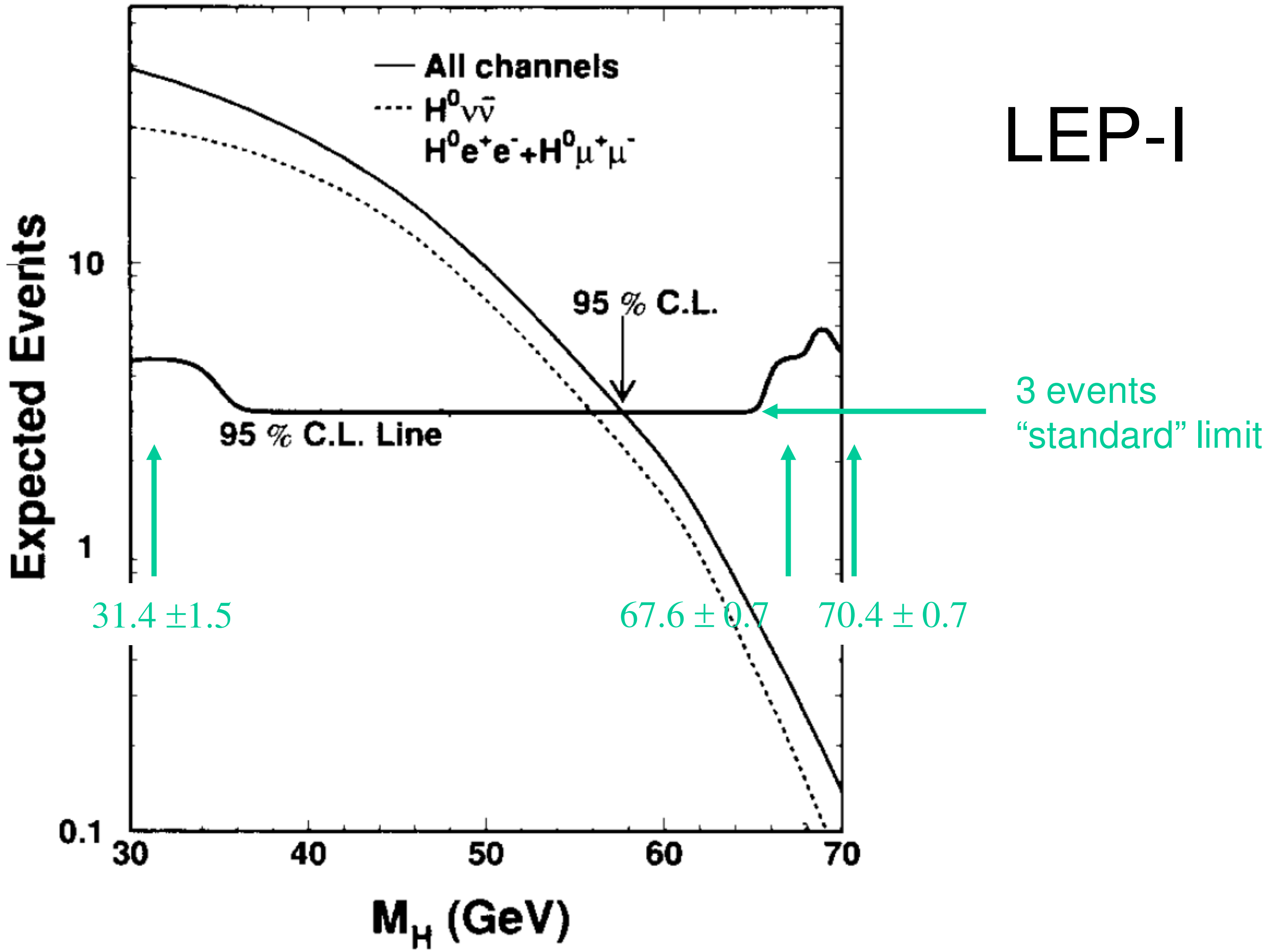
- Assume both signal and background are present, with different PDF for mass distribution: Gaussian peak for signal, flat for background (from Monte Carlo samples):

$$L = e^{-(s+b)} \prod_{i=1}^{n} (sP_s(m_i) + bP_b(m_i))$$

- Bayesian approach can be used to extract the upper limit (with flat prior):

$$1 - \mathrm{C.L.} = \frac{\displaystyle\int_{s^{\mathrm{up}}}^{\infty} e^{-s} \prod_{i=1}^{n} (sP_s(m_i) + bP_b(m_i))\mathrm{d}s}{\displaystyle\int_{0}^{\infty} e^{-s} \prod_{i=1}^{n} (sP_s(m_i) + bP_b(m_i))\mathrm{d}s}$$

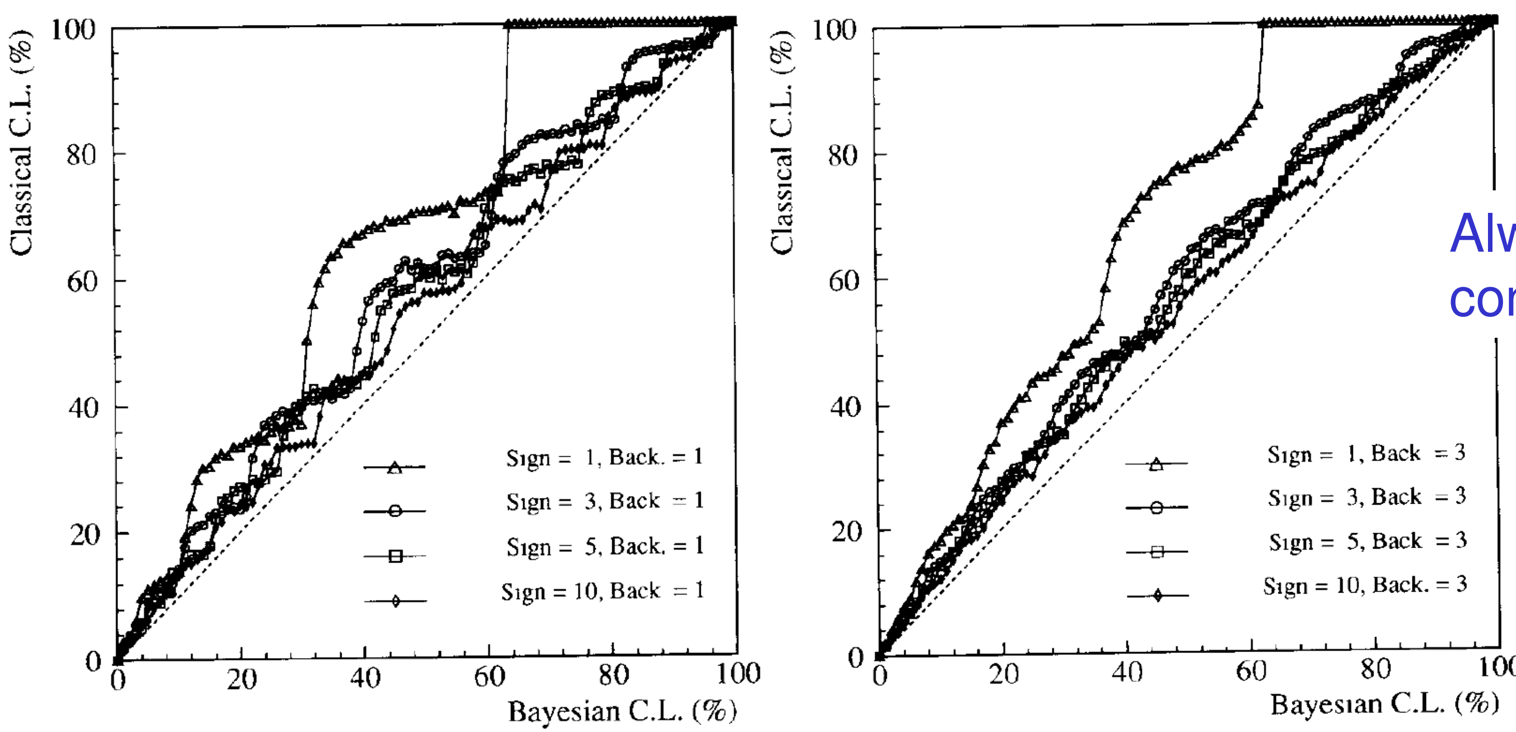
Application to Higgs search at L3



LEP-I

Comparison with frequentist C.L.

- Toy MC can be generated for different signal and background scenarios
- “classic” C.L. can be computed counting the fraction of toy experiments above/below the Bayesian limit



Always conservative!

Concrete example II

Combined Higgs search
at LEP II

Combined Higgs search at LEP-II

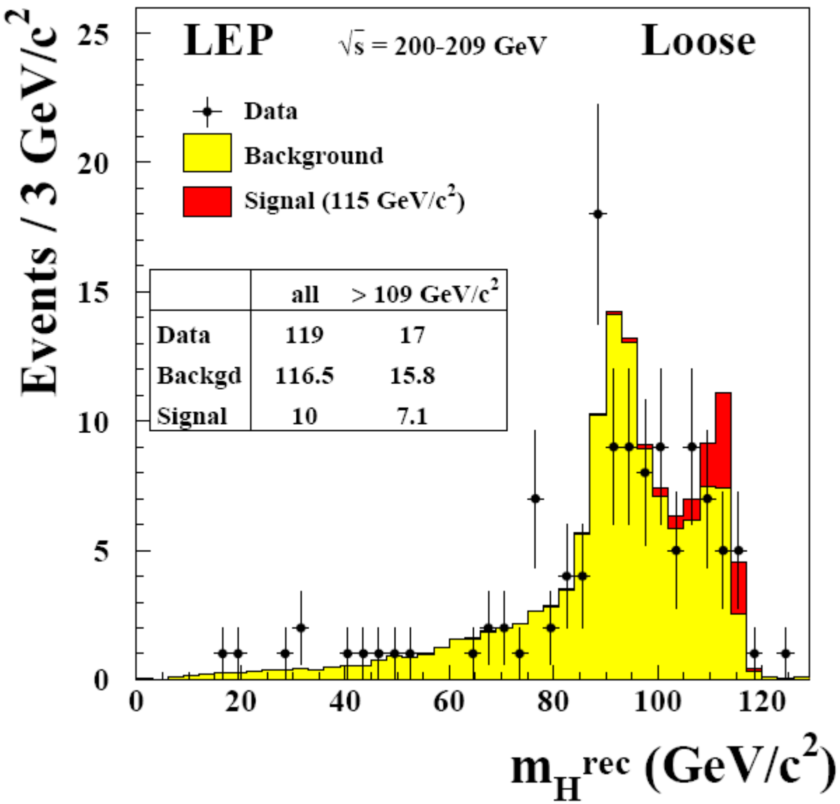
- Extended likelihood definition:

$$L(\eta) = \prod_{k=1}^{n_{\mathrm{ch}}} \frac{e^{-(\eta s_k(m_H)+b_k)}\cancel{(\eta s_k(m_H)+b_k)^{n_k}}}{n_k!} \times \prod_{j=1}^{n_k} \frac{\eta s_k(m_H) S_k(\vec{x_{jk}}; m_H) + b_k B_k(\vec{x_{jk}})}{\cancel{\eta s_k(m_H)+b_k}}$$

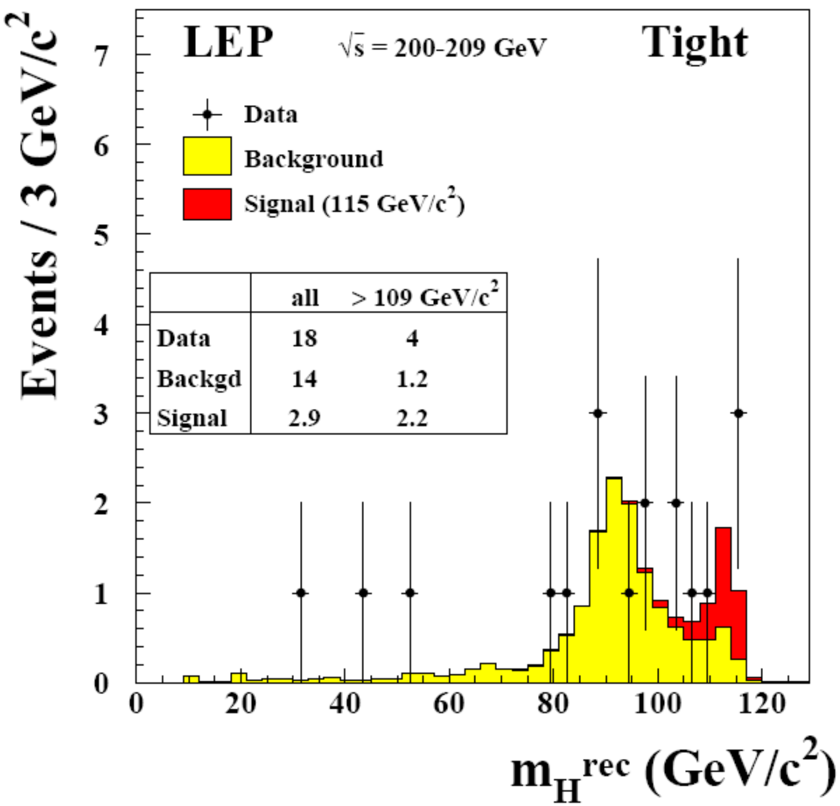
- η = 0 for *b* only, 1 for *s* + *b* hypotheses
- Likelihood ratio:

$$-2\ln Q(m_H) = 2\sum_{k=1}^{n_{\mathrm{ch}}}\left[s_k(m_H) - \sum_{j=1}^{n_k}\ln\left(1+\frac{s_k(m_H)S_k(\vec{x_{jk}}; m_k)}{b_k B_k(\vec{x_{jk}})}\right)\right]$$

Higgs candidates mass

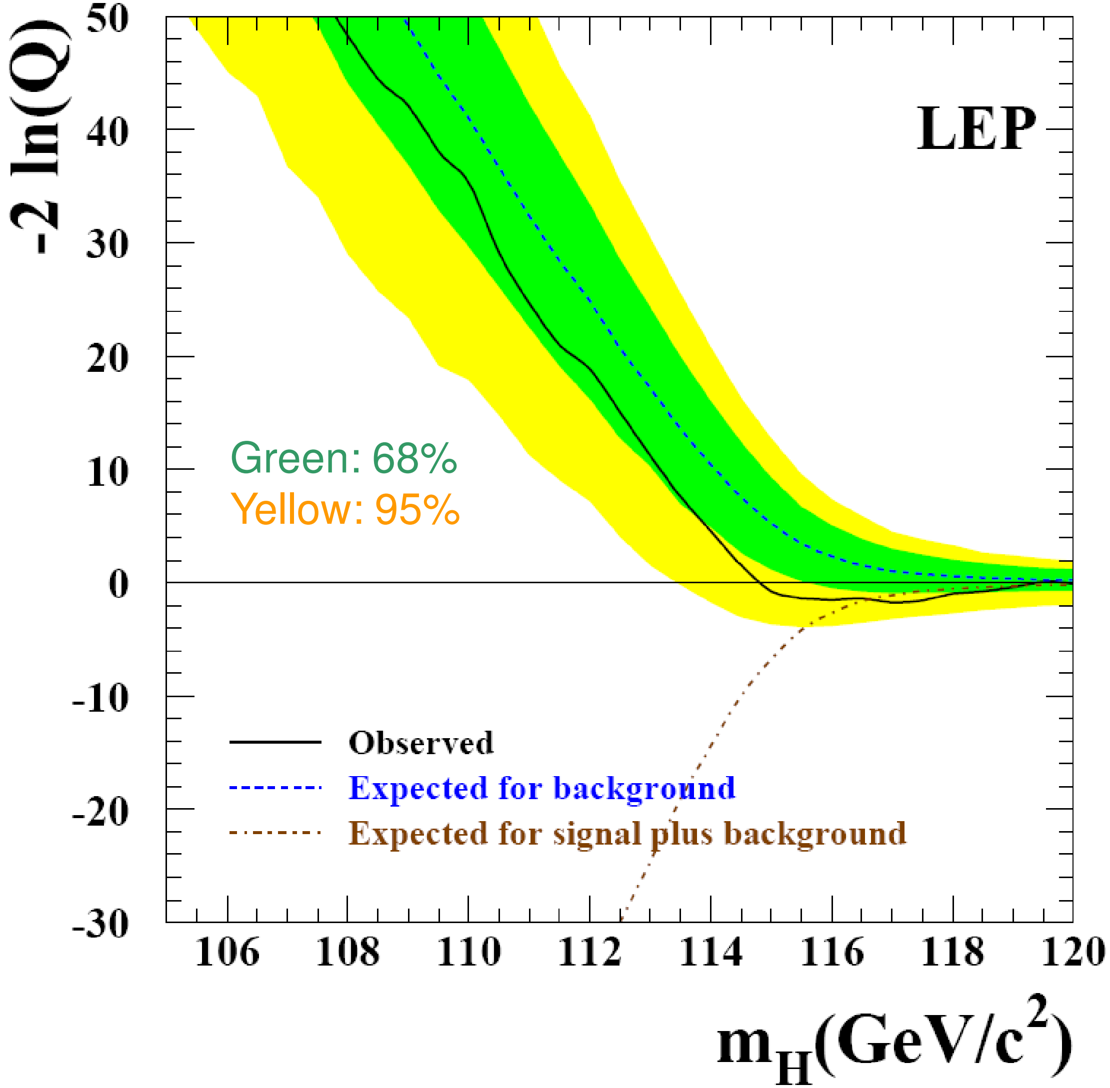


	all	> 109 GeV/c²
Data	119	17
Backgd	116.5	15.8
Signal	10	7.1

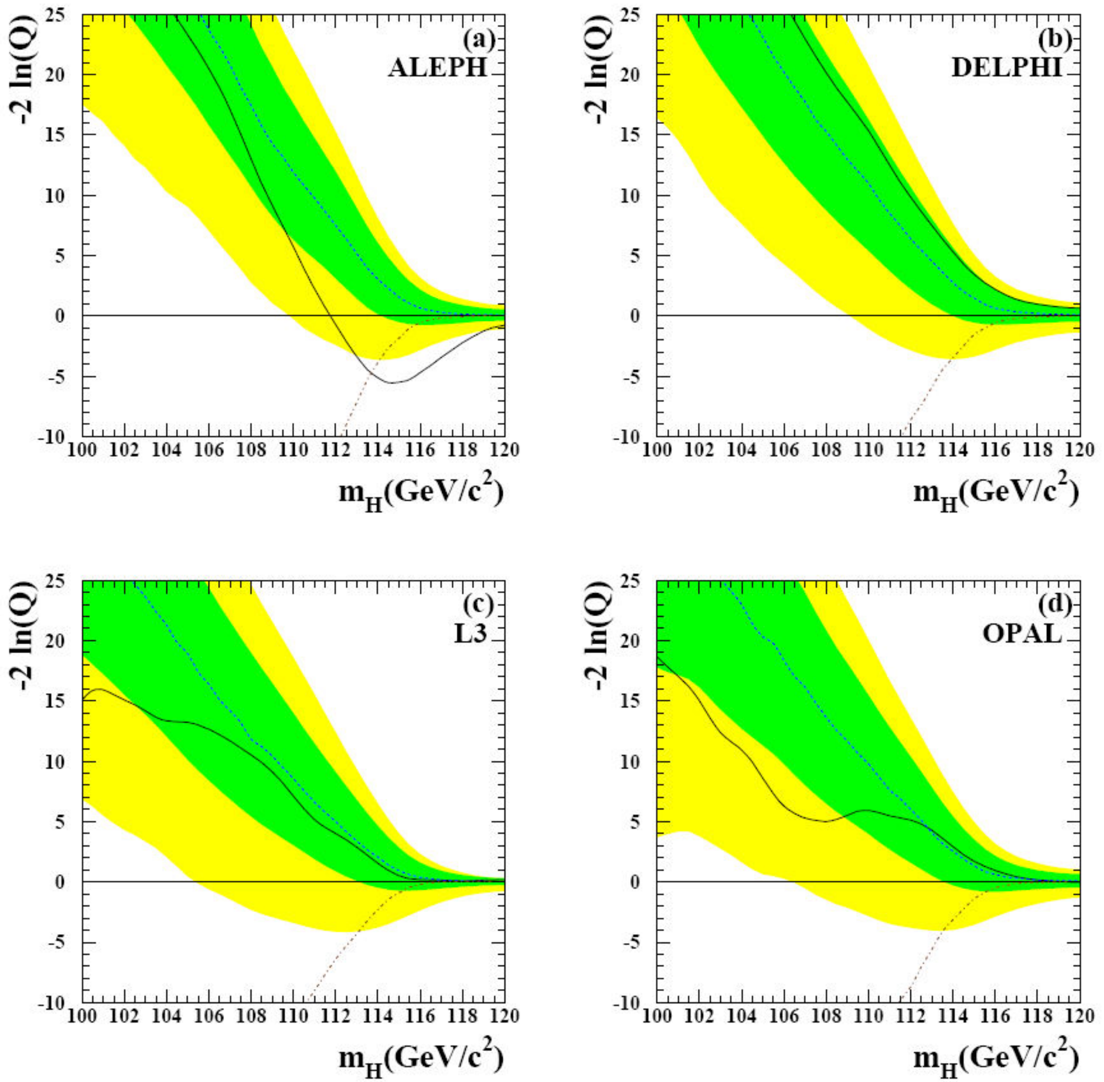


	all	> 109 GeV/c²
Data	18	4
Backgd	14	1.2
Signal	2.9	2.2

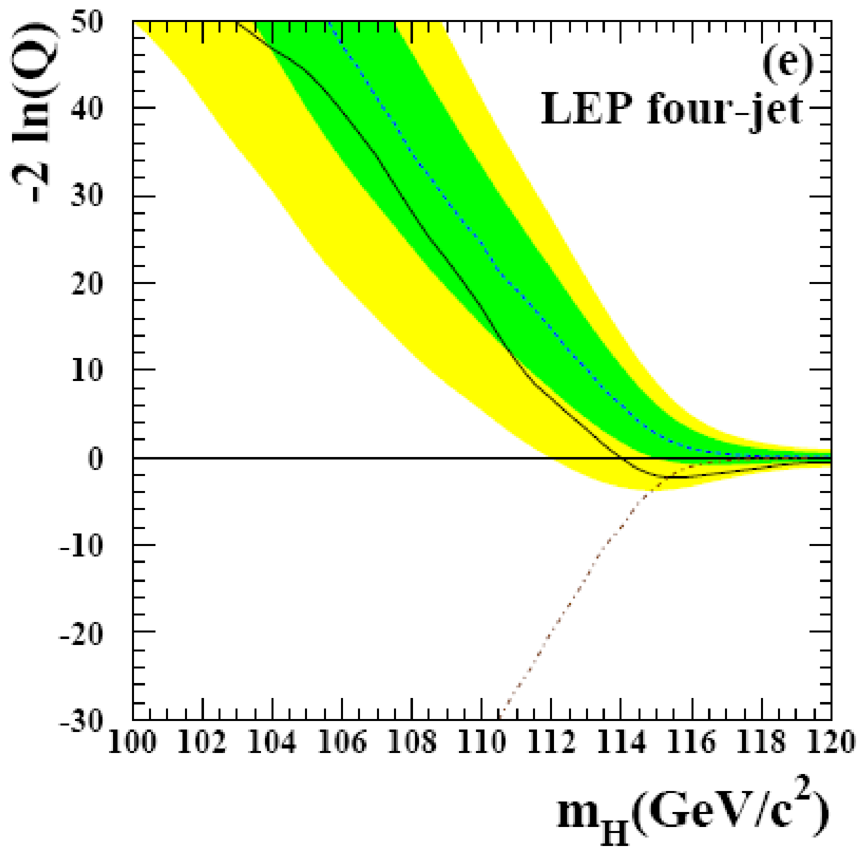
Mass scan plot

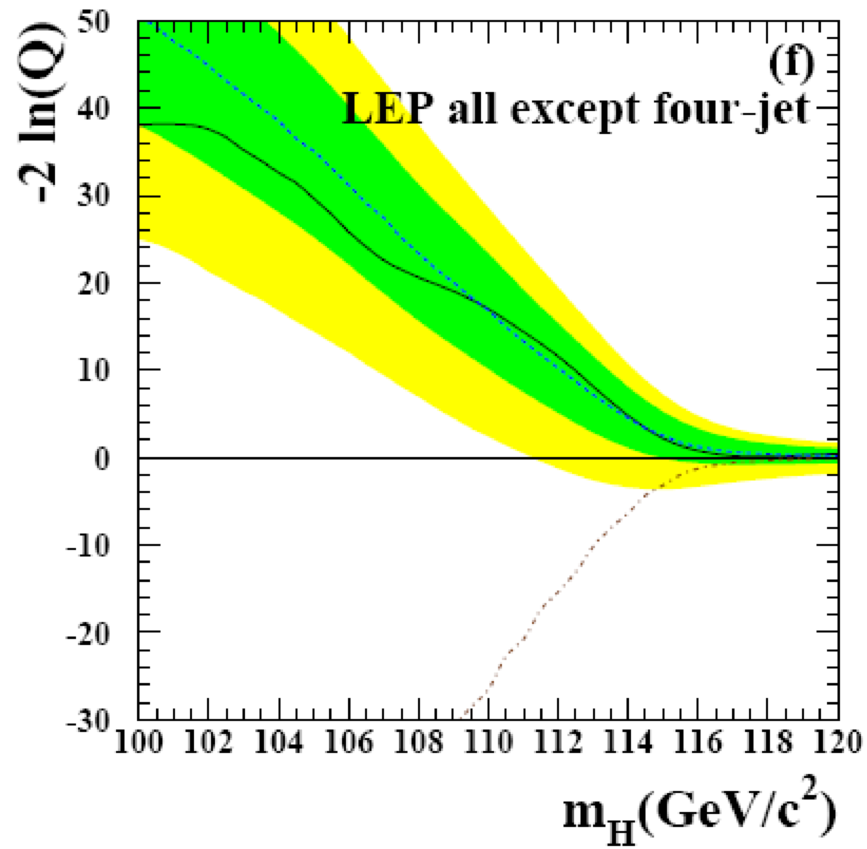


Mass scan by experiment

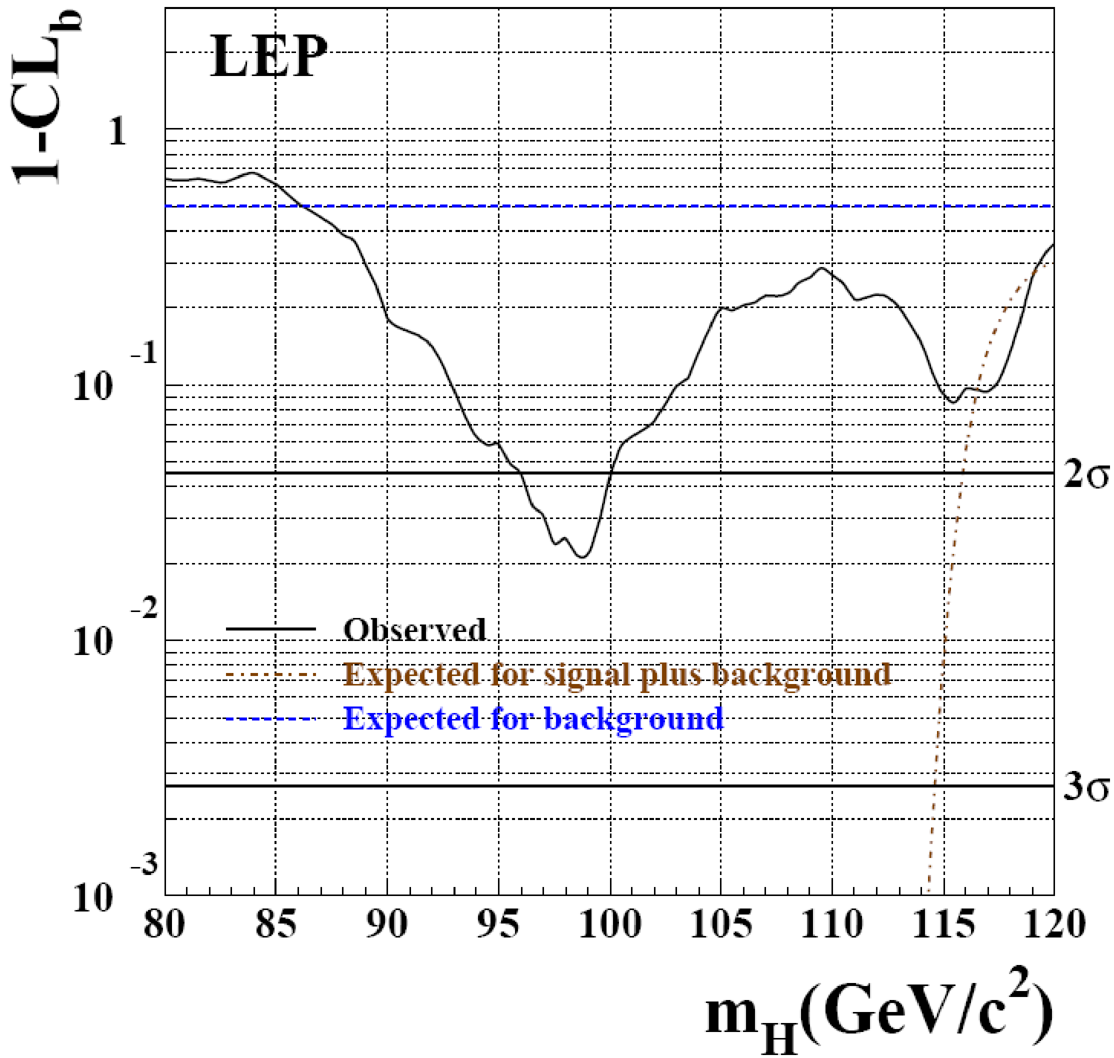


Mass scan by channel

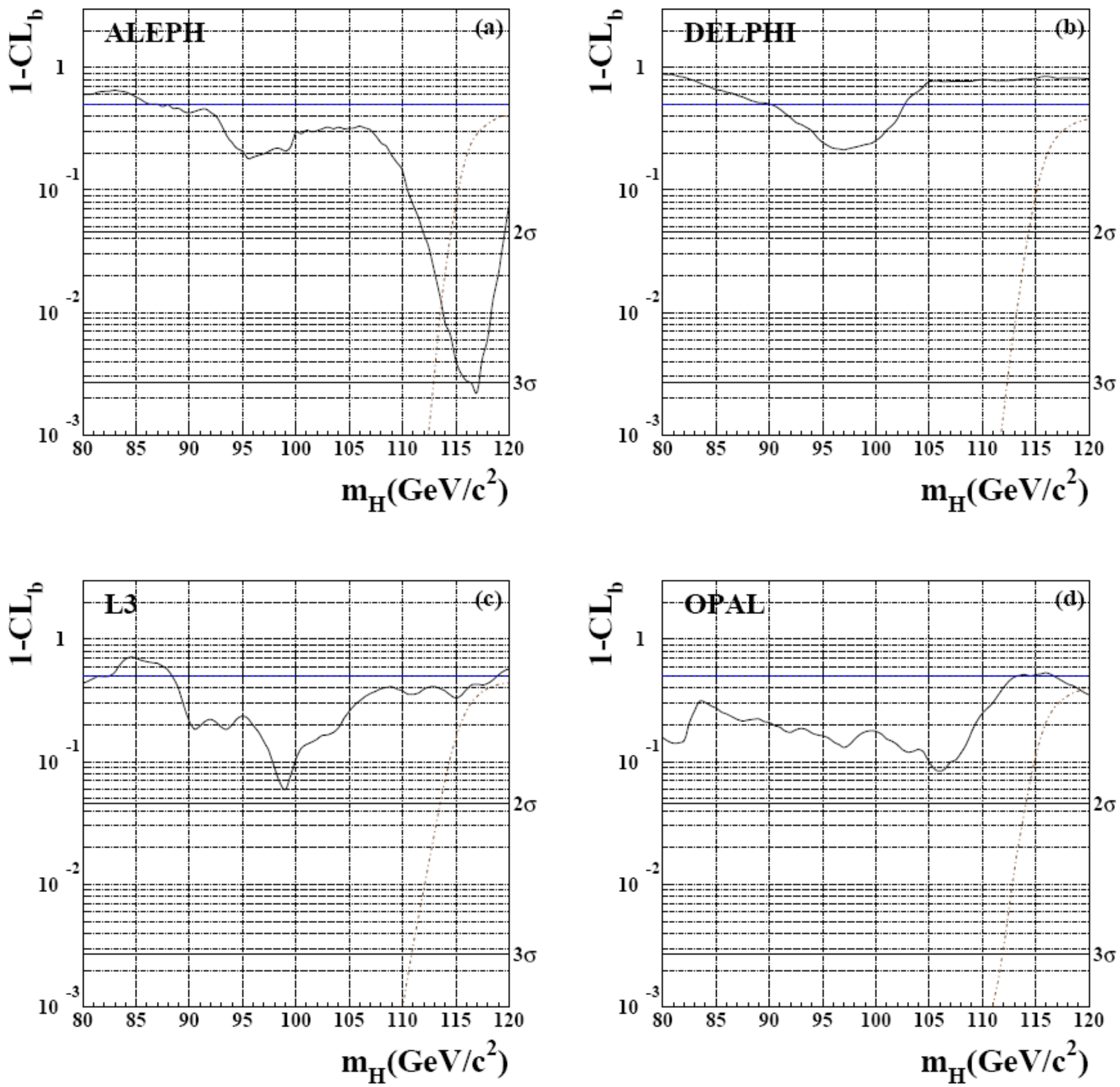




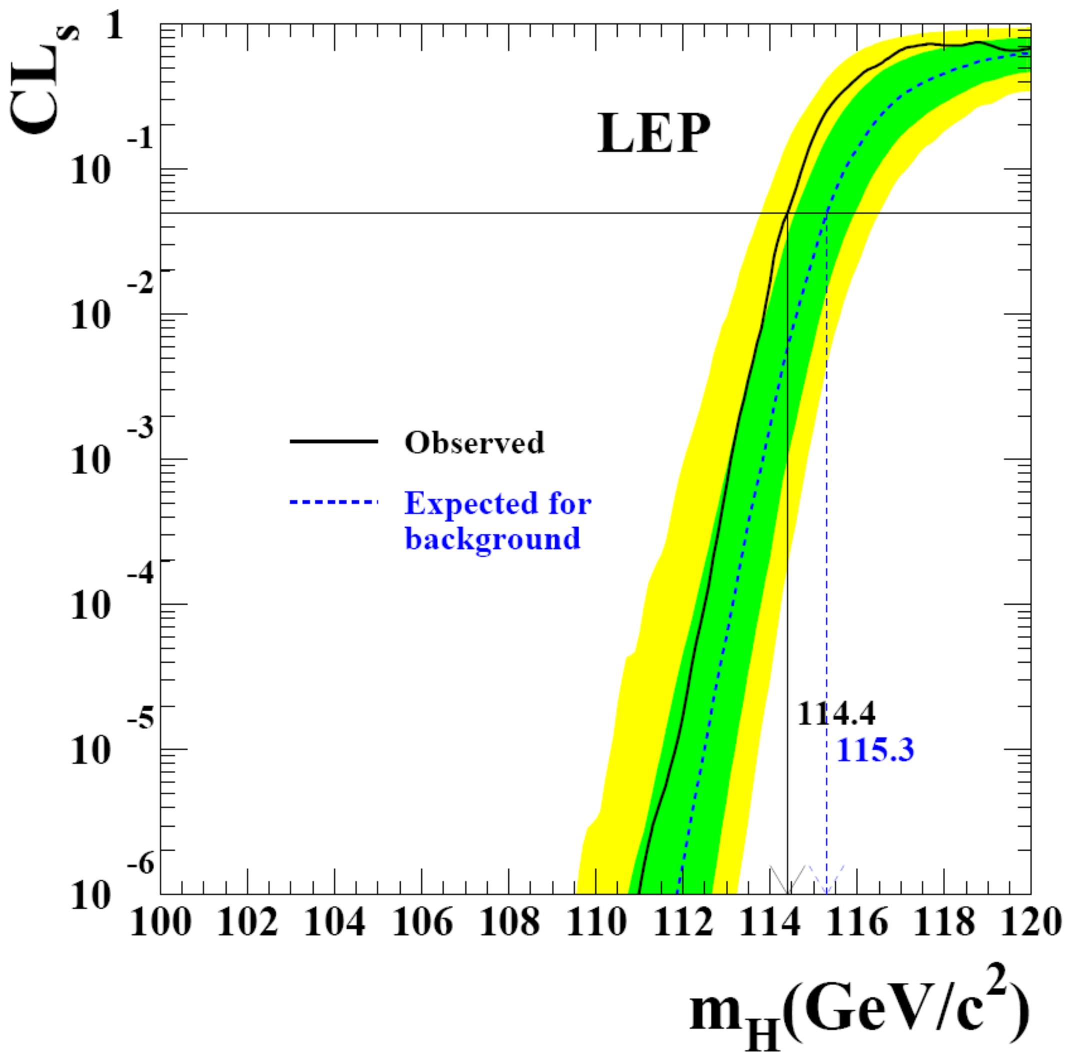
Background hypothesis C.L.



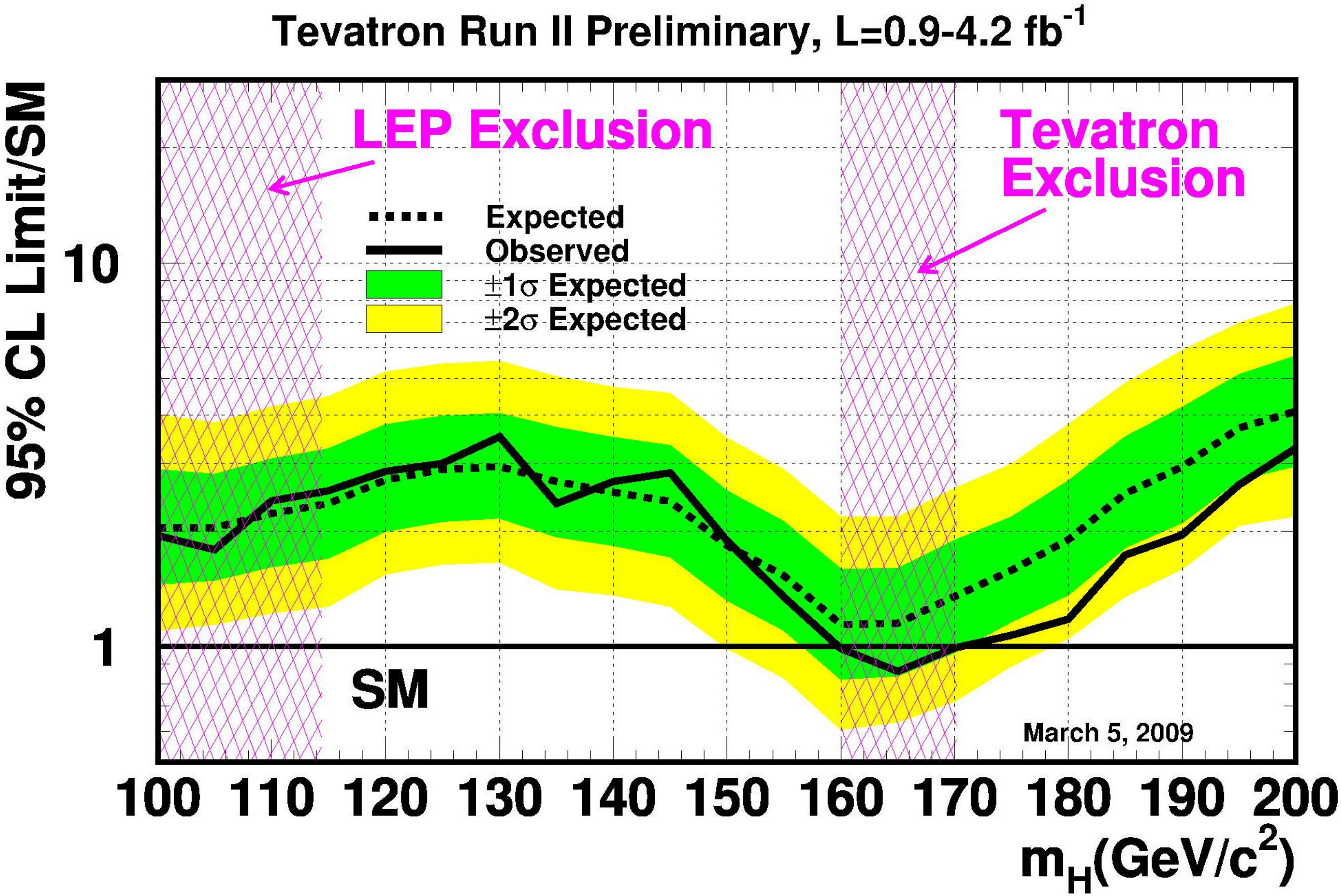
Background C.L. by experiment



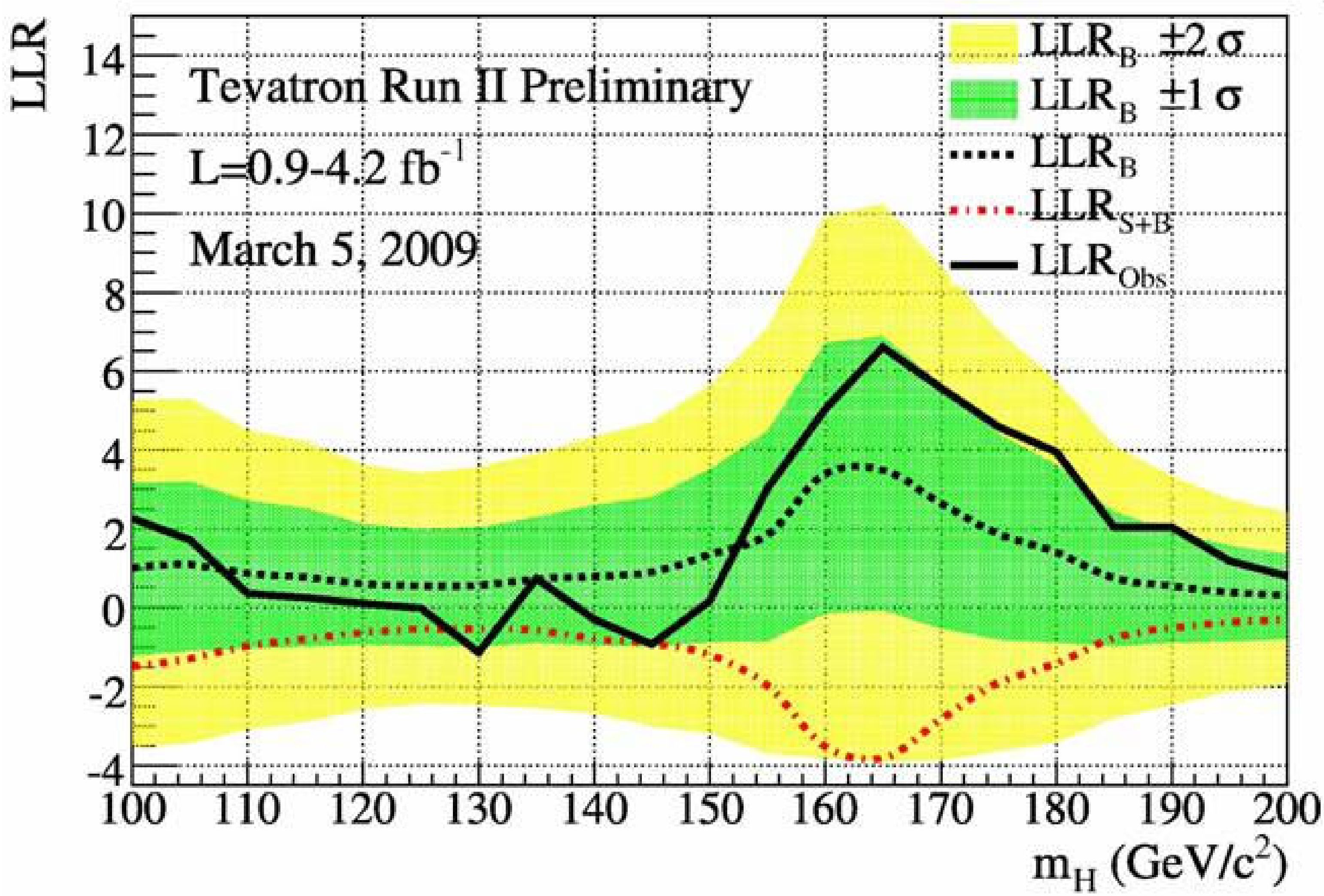
Signal hypothesis C.L.



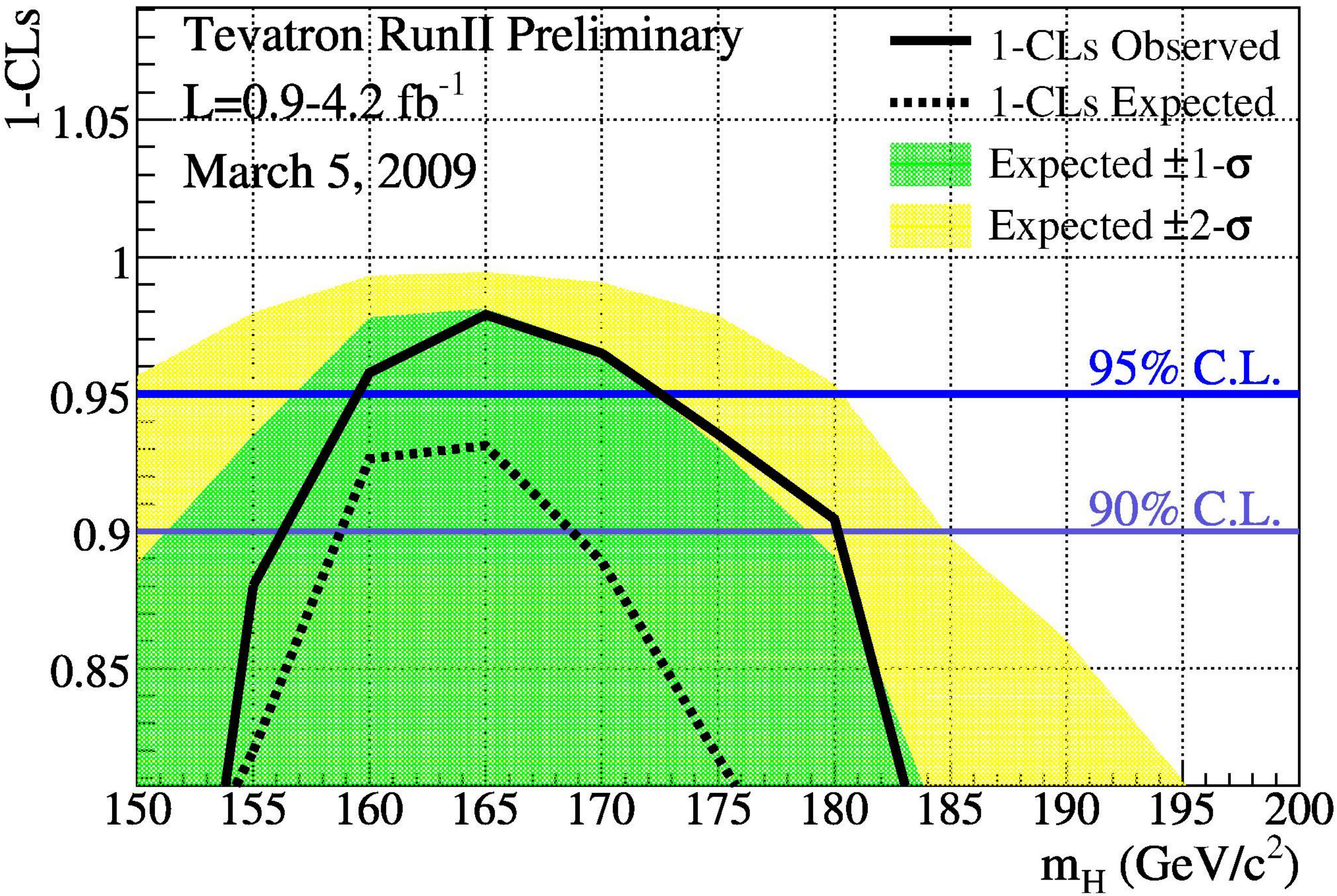
The new limit from Tevatron (I)



The new limit from Tevatron (II)



The new limit from Tevatron (III)



"Look elsewhere" effect

- When searching for a signal over a wide range of unknown parameters (e.g.: the Higgs mass), the chance that an over-fluctuation may occur on *at least* one point increases with the searched range
  - Average fraction of false discoveries at a fixed point (False Discovery Rate): $\mathrm{FDR} = (1-\mathrm{CL}_b) / (1-\mathrm{CL}_{s+b})$
- Not sufficient to test the significance at the most likely point!
  - Significance would be overestimated
- Magnitude of the effect: roughly proportional to the ratio of resolution over the search range
  - Better resolution = less chance to have more events compatible with the same mass value
- The effect can be evaluated with brute-force Toy Monte Carlo
  - Run N experiments with background-only, find the largest '*local*' significance over the whole search range, and get its distribution to determine '*overall*' significance

Nuisance parameters

- So called "nuisance parameters" are unknown parameters that are not interesting for the measurement
  - E.g.: detector resolution, uncertainty in backgrounds, other systematic errors, etc.
- Two main possible approaches:
- Add the nuisance parameters together with the interesting unknown to your likelihood model
  - But the model becomes more complex!
  - Easier to incorporate in a fit than in upper limits
- "Integrate it away" (→ Bayesian)

Nuisance pars in Bayesian approach

- No particular treatment:

$$P(\theta, \nu | x) = \frac{L(x; \theta, \nu) P(\theta, \nu)}{\int \mathrm{d}\theta L(x; \theta, \nu) P(\theta, \nu)}$$

- $P(\theta|x)$ obtained as marginal PDF, "integrating out" ν:

$$P(\theta|x) = \int P(\theta, \nu | x) \mathrm{d}\nu = \int \mathrm{d}\nu \frac{L(x; \theta, \nu) P(\theta, \nu)}{\int \mathrm{d}\theta L(x; \theta, \nu) P(\theta, \nu)}$$

Cousins - Highland hybrid approach

- No fully solid background exists on how to incorporate nuisance parameters within a frequentist approach
- Hybrid approach proposed by Cousins and Highland
  - Integrate the posterior PDF over the nuisance parameters (Nucl.Instr.Meth.A320 331-335, 1992)
  - Some Bayesian approach in the integration…: "*seems to be acceptable to many pragmatic frequentists*" (G. Zech, Eur. Phys. J. C 4 (2002) 12)
    - Bayesian integration of PDF, then likelihood used in a frequentist way
  - Some numerical studies with Toy Monte Carlo showed that the frequentist calculation gives very similar results in many cases

RooStats::HybridCalculator

Concrete example III

Search for $B \to \tau\nu$ at BaBar

Upper limits to $B \to \tau\nu$ at BaBar

- Reconstruct one B with complete hadronic decays
- Look for a tau decay on other side with missing energy (neutrinos)
  - Five decay channels used: $\mu^-\nu\nu$, $e^-\nu\nu$, $\pi^-\nu$, $\pi^-\pi^0\nu$, $\pi^-\pi^+\pi^-\nu$
- Likelihood function: product of Poissonian likelihoods for the five channels
- Background is known with finite uncertainties from side-band applying scaling factors (taken from MC)

Combined likelihood

- Combine the five channels with likelihood:

$$L(s+b) = \prod_{i=1}^{n_{\mathrm{ch}}} \frac{e^{-(s_i+b_i)}(s_i+b_i)^{n_i}}{n_i!} \qquad L(b) = \prod_{i=1}^{n_{\mathrm{ch}}} \frac{e^{-b_i} b_i^{n_i}}{n_i!}$$

- Define likelihood ratio estimator, as for combined Higgs search:

$$Q = \frac{L(s+b)}{L(b)} = e^{-\sum_{i=0}^{n_{\mathrm{ch}}} s_i} \prod_{i=0}^{n_{\mathrm{ch}}} \left(1+\frac{s_i}{b_i}\right)^{n_i}$$

- In case Q shows a significant minimum a non-null measurement of s can be determined
- More discriminating variables may be incorporated in the likelihood definition

Upper limit evaluation

- Use toy Monte Carlo to generate a large number of counting experiments
- Evaluate the C.L. for a signal hypothesis defined as the fraction of C.L. for the $s+b$ and b hypotheses:

$$\mathrm{CL}_s = \frac{\mathrm{CL}_{s+b}}{\mathrm{CL}_b} = \frac{N_{Q_{s+b} \le Q}}{N_{Q_b \le Q}}$$

- Frequentist, so far!

Including (Gaussian) uncertainties

- Nuisance parameters are the backgrounds b_i known with some uncertainty from side-band extrapolation
- Convolve likelihood with a Gaussian PDF

$$L(s+b) = \prod_{i=1}^{n_{\mathrm{ch}}} \int_{-\infty}^{+\infty} \mathrm{d}b' \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-(b'_i - b_i)^2/2\sigma_i^2} \frac{e^{-(s_i+b'_i)}(s_i+b'_i)^{n_i}}{n_i!}$$

$$L(b) = \prod_{i=1}^{n_{\mathrm{ch}}} \int_{-\infty}^{+\infty} \mathrm{d}b' \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-(b'_i - b_i)^2/2\sigma_i^2} \frac{e^{-b'_i}{b'_i}^{n_i}}{n_i!}$$

  - Note: b_i is the estimated background, not the “true” one!
- … but C.L. evaluated anyway with a frequentist approach (Toy Monte Carlo)!
- Analytical integrability leads to huge CPU saving!
  (L.L., A 517 (2004) 360–363)

Analytical expression

- Simplified analytic Q derivation:

$$Q = \frac{L(s+b)}{L(b)} = e^{-\sum_{i=0}^{n_{\mathrm{ch}}} s_i} \prod_{i=0}^{n_{\mathrm{ch}}} \frac{p_{n_i}(s_i + b_i - \sigma_i^2, \sigma_i)}{p_{n_i}(b_i - \sigma_i^2, \sigma_i)}$$

- Where $p_n(\alpha, \beta)$ are polynomials defined with a recursive relation:

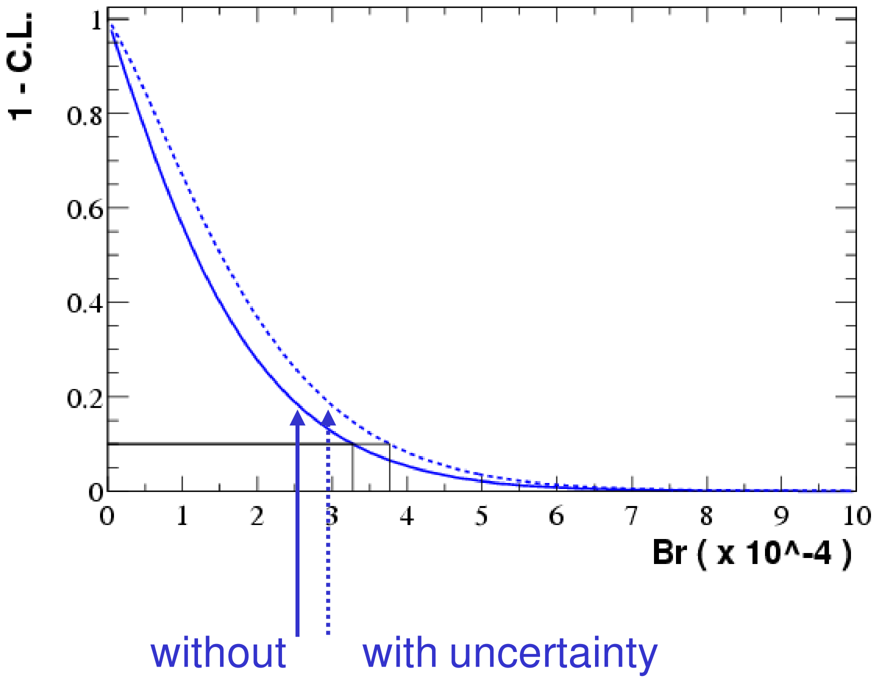
$$\begin{cases} p_0(\alpha, \beta) = 1, \\ p_1(\alpha, \beta) = \alpha, \\ p_2(\alpha, \beta) = \alpha^2 + \beta^2, \\ p_3(\alpha, \beta) = \alpha^3 + 3\alpha\beta^2, \\ p_4(\alpha, \beta) = \alpha^4 + 6\alpha^2\beta^2 + 3\beta^4, \\ p_5(\alpha, \beta) = \alpha^5 + 10\alpha^3\beta^2 + 15\alpha\beta^4, \\ \vdots \end{cases}$$

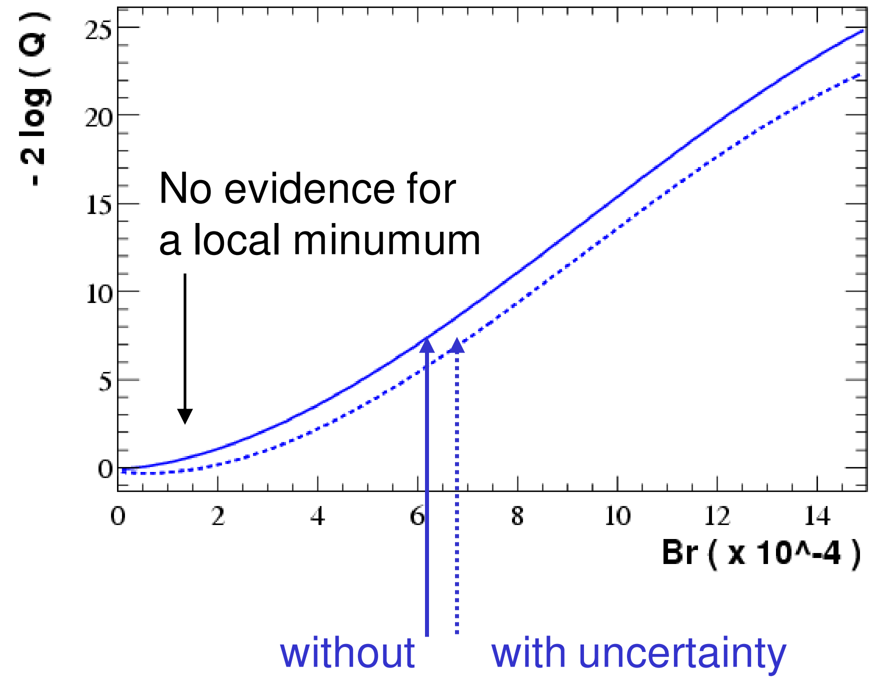
$$p_n(\alpha, \beta) = \alpha p_{n-1}(\alpha, \beta) + (n+1)\beta^2 p_{n-2}(\alpha, \beta)$$

… but in many cases it' hard to be so lucky!

Branching ratio: $\int L dt = 82\ \mathrm{fb}^{-1}$

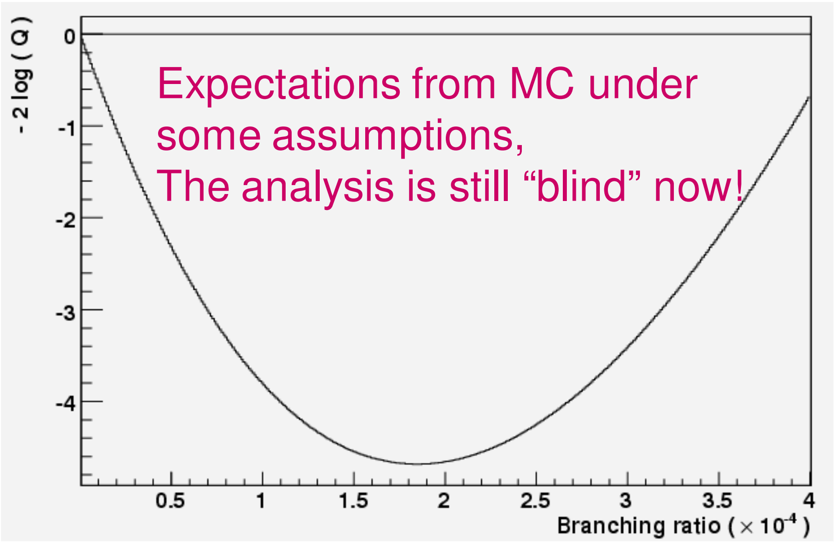
- Low statistics scenario





RooStats::HypoTestInverter

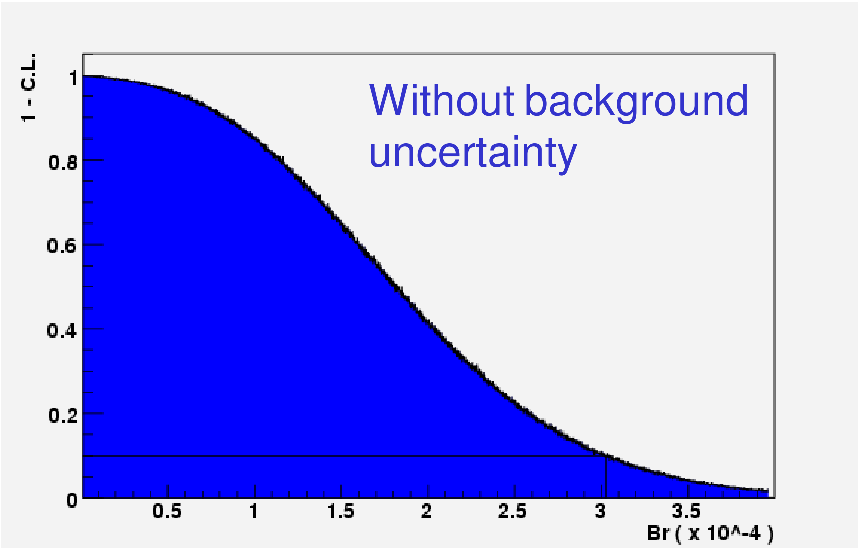
Branching ratio: $\int L dt = 350\ \text{fb}^{-1}$

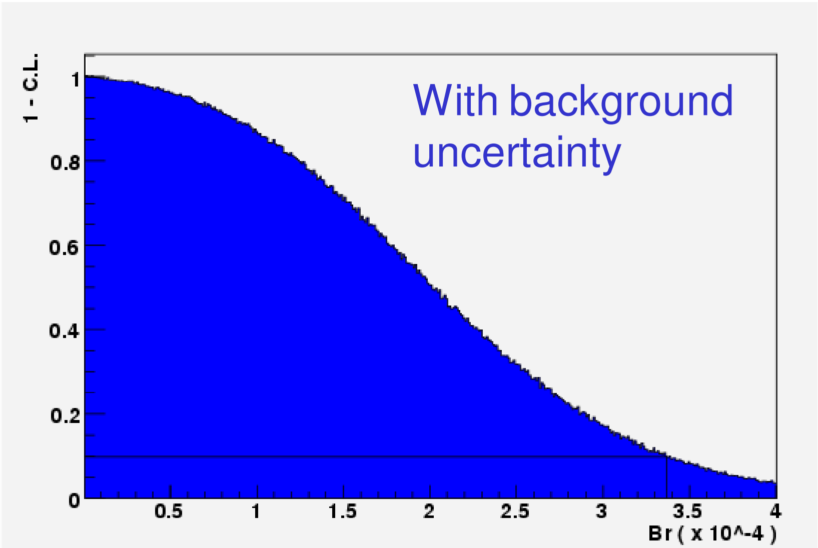


$$B(B \to \tau\nu) = 1.8 \pm^{1.0}_{0.9} \times 10^{-4}$$

Significance: $\Sigma = \sqrt{-2 \ln Q_{min}}$

Σ = 2.2σ (with uncertainties)
2.7σ (without uncert.)





In conclusion

- Many recipes and approaches available
- Bayesian and Frequentist approaches lead to similar results in the easiest cases, but may diverge in frontier cases
- Be ready to master both approaches!
- Bayesian and Frequentist limits have very different meanings
- Unified Feldman-Cousins approach or its modified frequentist versions (CLs, profile likelihood) address in a unified way upper limit and central values + confidence interval cases
  - Bayesian approach still needed for “hybrid” treatment of nuisance parameters (Highlands-Cousins)

- If you want your paper to be approved soon:
  - Be consistent with your assumptions
  - Understand the meaning of what you are computing
  - Try to adopt a popular and consolidated approach (an better, software tools!), wherever possible
  - Debate your preferred statistical technique in a statistics paper, not a physics result publication!

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