A tale of Nine Cities and Many Partons: developments in MPI since the first MPI@LHC.

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Double Parton Scattering (DPS)

Underlying Event (UE)

Minimum Bias (MB)

High multiplicity (HM)

Scale of additional interaction(s)

High scale, rare

Lower $x$

Low scale, common

Number of interactions considered

MPI ‘landscape’
Double Parton Scattering (DPS)

High scale, rare

Low scale, common

Affects any measurement involving jets/hadrons

Cleanest MPI system

Qualitatively new behaviour. Collectivity/flow – link to heavy ion?

Learn about structure and interactions of hadrons

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Part I. Hard MPI: The Double Parton Scattering (DPS)
Double Parton Scattering (DPS) = two separate hard interactions in a single proton-proton collision.

In terms of total cross section for production of AB, DPS is power suppressed with respect to single parton scattering (SPS) mechanism:

\[
\frac{\sigma_{DPS}}{\sigma_{SPS}} \sim \frac{\Lambda^2}{Q^2}
\]

DPS populates the final state phase space in a different way from SPS. In particular, it tends to populate the region of small \(q_A, q_B\) – competitive with SPS in this region.

e.g. \(pp \rightarrow e^+ e^- e^+ e^-\)

DPS becomes more important relative to SPS as the collider energy grows, and we probe smaller \(x\) values where there is a larger density of partons.
Inclusive cross section for DPS

Postulated form for integrated double parton scattering cross section based on analysis of lowest order Feynman diagrams / parton model considerations:

\[ \sigma_{D}^{(A,B)} = \frac{m}{2} \sum_{i,j,k,l} \int F_{h}^{ik}(x_1, x_2, y; Q_A, Q_B) F_{h}^{jl}(x'_1, x'_2, y; Q_A, Q_B) \]

\[ \times \hat{\sigma}_{ij}^{A}(x_1, x'_1) \hat{\sigma}_{kl}^{B}(x_2, x'_2) \, dx_1 \, dx'_1 \, dx_2 \, dx'_2 \, d^2 y \]

Collinear double parton distribution (DPD)

Symmetry factor

Parton level cross sections

\( y = \) separation in transverse space between the two partons

Diehl, Ostermeier and Schäfer (JHEP 1203 (2012))
Simplifying assumptions for DPS cross section

If one ignores correlations between partons in the proton:

\[ F^{ij}(x_1, x_2, y) = \int d^2 b \ D^i(x_1, b) \ D^j(x_1, b + y) \]

Impact parameter dependent PDFs (FT of GPD)

Common ‘lore’: approximately valid at low \( x \), due to the large population of partons at such \( x \) values.

Further approximation that is often made:

\[ D^i(x_1, b) = D^i(x_1) G(b) \]

\[ F^{ij}(x_1, x_2, y) = D^i(x_1) \ D^j(x_2) \int d^2 b G(b)(b + y) \]

\[ \sigma_{D}^{(A,B)} = \frac{\sigma_{S}^{(A)} \sigma_{S}^{(B)}}{\sigma_{eff}} \]

“Pocket formula”

Almost all phenomenological estimates of DPS use this equation
Consider adding the effects of QCD evolution in DPS, going backwards from the hard interaction.

Some effects are similar to those encountered in SPS – i.e. (diagonal) emission from one of the parton legs. These can be treated in same way as for SPS.

However, there is a new effect possible here – when we go backwards from the hard interaction, we can discover that the two partons arose from the perturbative '1 → 2' splitting of a single parton.

This 'perturbative splitting' yields a contribution to the DPD of the following form:

\[ F(x_1, x_2, y) \propto \alpha_s \frac{f(x_1 + x_2)}{x_1 + x_2} P \left( \frac{x_1}{x_1 + x_2} \right) \frac{1}{y^2} \]

Diehl, Ostermeier and Schäfer (JHEP 1203 (2012))
Perturbative splitting can occur in both protons (1v1 graph) – gives power divergent contribution to DPS cross section!

\[ \int \frac{d^2 y}{y^4} = ? \]

This is related to the fact that this graph can also be regarded as an SPS loop correction.

\[ \frac{\Lambda^2}{Q^4} \rightarrow \frac{1}{Q^2} \]

Power divergence!

Issue raised MPI@LHC 2010
Single perturbative splitting graphs

Also have graphs with perturbative $1 \rightarrow 2$ splitting in one proton only (2v1 graph).

This has a log divergence:

$$\int d^2 y / y^2 F_{\text{non-split}}(x_1, x_2; y)$$

Related to the fact that this graph can also be thought of as a twist 4 x twist 2 contribution to AB cross section

Blok, Dokshitzer, Frankfurt, Strikman, MPI@LHC 2011
Solving the double counting problem

First approach to solve problem:
- Completely remove 1v1 graphs from DPS cross section, and consider these as pure SPS (no natural part of these graphs to separate off as DPS).
- Put (part of) 2v1 graphs in DPS – sum logs of 1→2 splitting + DGLAP emissions in this contribution.

This scheme comes out if one chooses to regulate \( y \) integral using dim reg:

\[
\int d^2y / y^4 \rightarrow \int d^{2-2\epsilon}y / y^4 = 0
\]

In this case, cross section can no longer be written as parton level cross sections convolved with overall DPD factors for each hadron: appropriate hadronic operators involve both hadrons at once!
DPS framework of Diehl, Gaunt, Schönwald (DGS)

[JHEP 1706 (2017) 083]

Use double parton distributions (DPDs) in $y$ space, insert cut-off into $y$ integration:

$$\sigma_{DPS} = \int d^2y \Phi^2(\nu y) F(x_1, x_2; y) F(\bar{x}_1, \bar{x}_2; y)$$

Cuts off integral for $y \lesssim 1/\nu$, regulates power divergence

Use subtraction term in sum of SPS and DPS to avoid double counting:

$$\sigma_{tot} = \sigma_{DPS} + \sigma_{SPS} - \sigma_{sub}$$

$\nu$ dependence cancelled order by order

DPS cross section with both DPDs replaced by fixed order splitting expression

Advantages:

- Retain concept of the DPD for an individual hadron.
- Resum logs in all diagrams where appropriate (2v2, 2v1 and 1v1).
- Permits all-order formulation. Corrections can be practicably computed.
Transverse momentum in DPS

Small $q_i$ region particularly important for DPS – DPS & SPS same power

Parton model analysis: $\frac{d\sigma^{(A,B)}}{d^2q_1d^2q_2} \sim \int d^2yd^2ze^{-i\cdot q_1\cdot z_1-i\cdot q_2\cdot z_2} F(z_1, z_2, y) \ F(z_1, z_2, y)$

DTMDs

QCD treatment of transverse momentum in DPS (including DGS-style double counting subtraction) pursued in Buffing, Diehl, Kasemets JHEP 1801 (2018) 044

For perturbative $|q_i| \gg \Lambda$ can expand DTMDs in terms of collinear quantities:

Large $y \sim 1/\Lambda$:

Small $y \sim 1/q_T \sim |z_i|$:

Model using DPD

QCD treatment of transverse momentum in DPS (including DGS-style double counting subtraction) pursued in Buffing, Diehl, Kasemets JHEP 1801 (2018) 044

For perturbative $|q_i| \gg \Lambda$ can expand DTMDs in terms of collinear quantities:

Large $y \sim 1/\Lambda$:

Small $y \sim 1/q_T \sim |z_i|$:
Formal factorisation status of DPS producing two colour singlets is very good! [For coloured particle production at measured $p_T$, problems identified even in SPS case.]

Soft and Glauber exchanges

Extra (unphysically polarised) gluon connections to hard

Formal factorization status of DPS

$\sigma \sim F \otimes F \otimes \hat{\delta} \otimes \hat{\delta}$


(Vladimirov, JHEP 1804 (2018) 045)
Interference contributions to proton-proton DPS

SPS: One parton per proton ‘leaves’, interacts and ‘returns’.

To reform proton, parton must return with same quantum numbers.

No interference contributions to SPS cross section.

DPS: Here we have two partons per proton interacting.

Interference contributions to total cross section in which quantum numbers are swapped between parton legs. Complementary swap is required in other proton.

Can get interference contributions in colour, spin, flavour, and quark number. Also distributions associated with parton correlations!
Numerics of polarised distributions

Model calculations indicate spin correlations are large at large $x$


Evolution decreases relative importance of polarised distributions, especially at small $x$

Diehl, Kasemets, Keane, JHEP 1405 (2014) 118

Recently shown that polarisation effects can have an important effect on $\eta$ distributions of leptons in same-sign WW.

Cotogno, Kasemets, Myska, 1809.09024
Knowledge of nonperturbative DPDs

Model calculations:

MPI@LHC 2013,…

Bag model

Light-front CQM
[Rinaldi, Scopetta, Traini, Vento, JHEP 12 (2014) 028]

AdS/QCD

Momentum and number sum rules:
[JG, Stirling, JHEP 1003 (2010) 005
Diehl, Plößl, Schafer, arXiv:1811.00289]

MPI@LHC 2010,…

Lattice calculations (so far only for pion) → see talk by Christian Zimmermann
Experimental DPS extractions

Experimental extractions so far typically limited to one number, with large errors (and this is already tough!):

\[ \sigma_{\text{eff}} \equiv \frac{\sigma_{D}^{(A,B)}}{\sigma_{S}^{(A)} \sigma_{S}^{(B)}} \]

DPS characteristics simulated either using MC event generators, or independent SPS events overlaid (latter assumes no correlations).

Typical observables considered:

\[ \Delta S = \arccos \left[ \frac{p_T(jet1, jet2) \cdot p_T(jet1, jet2)}{|p_T(jet1, jet2)| \cdot |p_T(jet3, jet4)|} \right] \]

\[ \Delta p_T^{ij} = \frac{|p_{T,i} + p_{T,j}|}{|p_{T,i}| - |p_{T,j}|} \]

\[ \Delta y = |y_A - y_B| \]

DPS has relatively flat distribution, SPS peaked near \( \pi \)

DPS peaked towards \( \Delta p_T^{ij} = 0 \)

Generally wider distributions for DPS


JHEP 06, 047, (2017)
Experimental DPS extractions

Two principal methods:

‘Template method’
- Generate SPS and DPS ‘templates’ in observables considered
- Extract DPS fraction
- Convert to $\sigma_{\text{eff}}$

‘Inclusive-fit method’
- Based on MC generator tuning. Use DPS-sensitive data as input, and fit only MPI-sensitive parameters
- Convert fitted MPI-parameters in MC to $\sigma_{\text{eff}}$
Results for $\sigma_{\text{eff}}$

CMS ($\sqrt{s} = 7$ TeV, 4 jets, 2016)
- CMS ($\sqrt{s} = 8$ TeV, $T(1S) + T(1S)$, 2016)
- LHCb ($\sqrt{s} = 13$ TeV, $J/\psi + J/\psi$, 2017)
- CMS + Lansberg, Shao ($\sqrt{s} = 7$ TeV, $J/\psi + J/\psi$, 2014)

4 jets

V + jets

MPI@LHC 2011

Same-sign WW

MPI@LHC 2015

Heavy flavour

MPI@LHC 2012

4 jets

MPI@LHC 2015

Plot from MPI book chapter by I. Belyaev and D. Savrina

Plot taken from talk by A. Mehta at MPI@LHC 2017
Future prospects at HL-LHC

In same-sign WW, important observable
\[ a_{\eta l} \neq 0 \] only if there are correlations in DPS

Various effects can cause a few per cent asymmetry:
- \( x \)-space correlations, valence number effects
- 1 → 2 splitting effects

Asymmetry of a few per cent is observable at the HL-LHC – would be definitive sign of correlations!

\[ a_{\eta l} \approx 0.03 \]


Cotogno, Kasemets, Myska, 1809.09024

JG et al., CERN HL/HE-LHC YR

CMS-TDR-017-003

10/12/2018

MPI@LHC 2018, Perugia
Part II. Soft MPI: Phenomenology and Description in MC Generators
MB/UE: Many scatters at low scale (nonperturbative physics)!

Handle with Monte Carlo event generators. Physics-inspired models with several adjustable parameters.

Treating MB/UE is some mixture of physics input and fitting

### MCEG Parameters

- **MPI**
  - $\alpha_{\text{strong}}$ value at scale $Q^2 = M^2$
  - $p_T$ cutoff scale
  - Energy extrapolation exponent
  - Impact parameter/inverse hadron radius dependence

- **BBR**
  - Primordial/intrinsic $k_T$
  - Strength of color reconnection
  - Length of color strings

... ...
Handling MB/UE

Monte Carlo Event Generators

Multiple Interaction Model

Colour Reconnection

Hadronisation model

MCEG Parameters

- MPI: $\alpha_{strong}$ value at scale $Q^2 = M^2$
- MPI: $p_T$ cutoff scale
- MPI: Energy extrapolation exponent
- MPI: Impact parameter/inverse hadron radius dependence
- BBR: Primordial/intrinsic $k_T$
- BBR: Strength of color reconnection
- BBR: Length of color strings

MB/UE Data

Parameterisation of MC response

By hand/intuition

(R. Field, … )

Fitted MC parameters

Professor


10/12/2018 MPI@LHC 2018, Perugia 25
Handling MB/UE

Many distributions quite well described by the main MC event generators + tuning.

A few problematic distributions, for example related to strange particle production:

Motivates further development of MC models!

Anyway, always better to have more physics in the models – should enable more accurate extrapolations + ability to fit various different types of distributions.

Much recent work focused on improving colour reconnection in particular.

\[
\Lambda = uds \\
K_s^0 = \frac{d\bar{s} + s\bar{d}}{\sqrt{2}}
\]

CMS, JHEP 05 (2011) 064

MPI@LHC 2011, 12
MPI model in Pythia

Use perturbative $2 \rightarrow 2$ QCD scatters for additional scatters, regularise at low $p_{\perp}^2$:

$$\frac{d\hat{\sigma}}{dp_{\perp}^2} = \frac{8\pi\alpha_s^2(p_{\perp}^2)}{9p_4^4} \rightarrow \frac{8\pi\alpha_s^2(p_{\perp}^2 + p_{\perp 10}^2)}{9(p_{\perp}^2 + p_{\perp 10}^2)^2}$$

Evolution of MPI interleaved with initial-state radiation:

$$\frac{dP}{dp_{\perp}} = \left(\frac{dP_{\text{MI}}}{dp_{\perp}} + \sum \frac{dP_{\text{ISR}}}{dp_{\perp}}\right) \exp \left(-\int_{p_{\perp}}^{p_{\perp i-1}} \left(\frac{dP_{\text{MI}}}{dp'_{\perp}} + \sum \frac{dP_{\text{ISR}}}{dp'_{\perp}}\right) dp'_{\perp}\right)$$

Adjust parton densities ‘after’ each additional scattering to take into account removal of flavour/momentum. E.g. valence quarks:

$$u_{i,\text{val}}(x, Q^2) = \frac{N_{u,\text{val,remain}}}{N_{u,\text{val,original}}} \frac{1}{X} u_{\text{val}} \left(\frac{x}{X}, Q^2\right)$$

with $X = 1 - \sum_{j=1}^{i-1} x_j$, $Mtm$ fraction left after higher-scale scatters

Also possibility for an $x$-dependent proton size, with Gaussian width $a(x) = a_0 (1 - a_1 \ln x)$

[Corke, Sjöstrand, JHEP 1105 (2011) 009]
Assume partons uncorrelated & x and y dependence factorizable → scatters Poissonian at given impact parameter.

Two components – ‘hard’ and ‘soft’, applicable above and below $p_t^{\text{min}}$:

$$\tilde{n}(b, s) = A(b, \mu)\sigma_{\text{hard}}^{\text{inc}}(s, p_t^{\text{min}}) + A(b, \mu_{\text{soft}})\sigma_{\text{soft}}^{\text{inc}}$$

Width parameter

Use $\sigma_{\text{tot}}$ and $\frac{d}{dt}\left(\ln \frac{d\sigma_{\text{el}}}{dt}\right)_{t=0}$ to fix 3, 4

Recent development: soft component altered to distribute particles evenly in rapidity according to multiperipheral model + model of diffraction added.

Solved ‘bump problem’
MPI models in SHERPA

AMISIC++ model: simple version of Pythia MPI model

SHRiMPS model: designed to describe MB data

Uses Khoze-Martin-Ryskin (KMR) model, multi-pomeron scattering

Reasonable description of MB data with only few parameters.
Original motivation for colour reconnection: rising $\langle p_T \rangle(n_{ch})$ at $Sp\bar{p}S$.

If all MPIs completely independent, each MPI contributes same $\langle p_T \rangle$ and $\langle n_{ch} \rangle \rightarrow$ predicts flat $\langle p_T \rangle(n_{ch})$

If MPIs can ‘reconnect’ in colour in ways such that string length is reduced, each additional MPI contributes same amount to $p_T$ but less and less to $n_{ch} \rightarrow d\langle p_T \rangle/dn_{ch} > 0$

Figure credit: Sjöstrand, arXiv:1706.02166
Colour reconnection in Pythia 8

‘MPI-based model’

MPI with hardness scale $p_{\perp}$ reconnected with harder system with probability:

$$P = \frac{p_{\perp,\text{Rec}}^2}{(p_{\perp,\text{Rec}}^2 + p_{\perp}^2)}$$

$O(1)$ parameter

$$p_{\perp,\text{Rec}} = RR \times p_{\perp}^{\text{MI}}$$

Soft systems reconnect more

Gluons from softer MPI inserted into harder MPI dipole in such a way to minimise increase in string length $\lambda$

‘QCD-based model’

Try to incorporate more of the structure of QCD into the description, go beyond leading colour.

- Establish what kinds of reconnections are allowed, using a simplified model approximating SU(3)

Christiansen, Skands, JHEP 1508 (2015) 003

MPI@LHC 2014
Colour reconnection in Pythia 8

- Use simplistic space-time picture to determine if two strings coexist
- Reconnect colours to minimise string length $\lambda$ (local deterministic minimisation)

New type of string connection: junction

Enables new types of colour reconnection, e.g.

Better description of $\Lambda/K$ for $p_T < 5$ GeV

Number of baryons increases faster with multiplicity

Christiansen, Skands, JHEP 1508 (2015) 003
Colour reconnection in Herwig

Plain colour reconnection: consider reconnecting $q\bar{q}$ cluster to all other clusters in the event, and accept reconnection that gives minimal cluster masses with probability $p_{\text{reco}}$

Statistical colour reconnection: similar to PCR, but allows reconnections which increase colour length $\lambda \equiv \sum_i m_i^2$ with probability $\exp(-\Delta\lambda/T)$ (simulated annealing).

Recent developments:

Rapidity-based reconnection algorithm, with baryonic clusters:

Consider for mesonic reconnection

Clusters with maximal $|y_q| + |y_{\bar{q}}|$ considered for reconnection

New, more elementary pathway to produce baryons


MPI@LHC 2012

MPI@LHC 2017
Colour reconnection in Herwig

Improved description of $p/\pi$ with baryonic clusters

Need nonperturbative $g \rightarrow s\overline{s}$ splittings to improve agreement for measurements involving strange particles

Some data still not described well

Investigation using perturbative picture extended into NP region – reconnections achieved by soft gluon exchanges:

\[ U(\{p\}, \mu^2, \{M_{ij}^2\}) = \exp \left( \sum_{i \neq j} T_i \cdot T_j \frac{\alpha_s}{2\pi} \left( \frac{1}{2} \ln^2 \frac{M_{ij}^2}{\mu^2} - i\pi \ln \frac{M_{ij}^2}{\mu^2} \right) \right) \]

Evolution of a few clusters investigated – perturbative evolution generically reduces \( \lambda \) and \( \Delta Y \)

Not feasible to use on realistic events with many clusters, but recombination probabilities computed using few clusters can be used as input in other models.
High-multiplicity collisions

Several interesting phenomena in high-multiplicity pp collisions – seem to resemble those found in heavy ion collisions!

Near-side ridge in two particle correlations

Increasing strangeness fraction with multiplicity

- Near-side ridge in two particle correlations
- Low multiplicity pp
- High multiplicity pp
- CMS pp 7 TeV, Minimum Bias
  $1 < p_T < 3$ GeV/c
- CMS pp 7 TeV, $N_{pp} > 110$
  $1 < p_T < 3$ GeV/c
- MPI@LHC 2010

- Herwig plot: new CR model seems to help
  somewhat
- Bad description by Pythia

- Pb-Pb
- p-Pb
- $\Lambda+\Lambda$ ($>2$)
- $\Xi^-+\Xi^-$ ($>8$)
- $\Omega+\Omega$ ($>16$)

- Near side ridge!
First step: interactions described using ‘Parton Based Gribov Regge Theory’ (PBGRT) – language in terms of cut and uncut pomerons

Each ‘pomeron’ is a DGLAP ladder with hard scattering in the centre. Saturation effects incorporated in effective way by modifying \( G \)

Parton ladders = colour strings/flux tubes

Second step: string segments organised into ‘core’ and ‘corona’ pieces, depending on transverse momentum \( p_t \) and local string density \( \rho \)

High \( \rho \), small \( p_t \) \( \rightarrow \) core. Hydrodynamic evolution + hadronic cascade.
Low \( \rho \), large \( p_t \) \( \rightarrow \) corona. Simply escape as hadrons.

EPOS has core contributions even in high multiplicity \( pp \)
Achieve good description of minimum bias data:

Underlying event also fair. Strangeness production enhanced by collective effects.

EPOS produces near-side ridge in high multiplicity $pp$, but only with hydro

Random distribution of flux tubes in transverse space $\rightarrow$ azimuthal asymmetry (ellipse), but longitudinal invariance $\rightarrow$ ridge after hydro
Based on Mueller’s dipole model (LL BFKL in transverse coordinate space), but with various improvements beyond LL:

- Energy-momentum conservation, projectile-target symmetry, running coupling…
- Subleading $N_c$ corrections – the dipole ‘swing’
- Confinement effects

Naturally includes saturation effects and MPI

Only few tuneable parameters

Fair description of minimum bias data (nb DIPSY has only gluons)

Easily generalised to nuclear collisions
Density of strings very high in LHC environment – possibility for strings to interact and form ‘colour ropes’ of higher colour representation.

Larger Casimir for higher colour rep → larger string tension → more strange quarks and baryons, in agreement with data.

String radius = 0.1fm
Rope shoving

Higher energy density in string overlap region than outside $\rightarrow$ dynamically generates pressure $\rightarrow$ flow effects (but without thermal equilibrium)

Can induce a near-side ridge!

Bierlich, Gustafson, Lönnblad
CGC + Lund model

Separation of dynamics between fast and slow partons

\[ J^\nu = \delta^{\nu \pm} \rho_{A(B)}(x^+, x_-) \]

Classical colour field associated with fast partons used as source for classical gluon field of slow partons

\[ [D_\mu, F_{\mu\nu}] = J_\nu \]

Gluon multiplicity distribution from CGC calculation

Generate gluons from distribution

Connect gluons with strings, fragment using Pythia

Increase in \( \langle p_T \rangle \) with \( n_{ch} \) mainly from CGC

Ridge effect from initial state correlations

Schenke, Schlichting, Tribedy, Venugopalan, Phys.Rev.Lett. 117 (2016) no.16, 162301
Hydrodynamics/QGP?

String shoving?

Ridge effect in $pp$

MPI@LHC 2017
Partonic transport?

MPI@LHC 2017
Quantum-mechanical interference effects?

Greif, Greiner, Schenke, Schlichting, Xu, Phys. Rev. D 96, 091504 (2017)

Initial-state correlations in CGC?

Blok, Jäkel, Strikman, Wiedemann JHEP 1712 (2017) 074
Many interesting developments in MPI physics in the last 10 years.

We look forward to hearing more at this workshop!
Correlated parton contributions to DPS

There are also contributions to the unpolarised p-p DPS cross section associated with correlations between partons:

e.g.  \[ \Delta q_1 \Delta q_2 = q_1 \uparrow q_2 \uparrow + q_1 \downarrow q_2 \downarrow - q_1 \uparrow q_2 \downarrow - q_1 \downarrow q_2 \uparrow \]

Same spin  Opposing spin

Can construct limits on size of colour/spin correlated distributions based on probabilistic picture of parton densities (‘positivity bounds’):

Simple case:  \[ |f_{\Delta q \Delta q}| \leq f_{qq} \]

More general:

\[
\rho = \frac{1}{4} \left( \begin{array}{cccc}
 f_{qq} + f_{\Delta q \Delta q} & -i e^{i \varphi_y} y M f_{\delta q \delta q} & -i e^{i \varphi_y} y M f_{q \delta q} & 2 e^{2i \varphi_y} y^2 M^2 f^t_{\delta q \delta q} \\
 i e^{-i \varphi_y} y M f_{\delta q \delta q} & f_{qq} - f_{\Delta q \Delta q} & 2 f_{\delta q \delta q} & -i e^{i \varphi_y} y M f_{q \delta q} \\
 i e^{-i \varphi_y} y M f_{q \delta q} & 2 f_{\delta q \delta q} & f_{qq} - f_{\Delta q \Delta q} & -i e^{i \varphi_y} y M f_{q \delta q} \\
 2 e^{-2i \varphi_y} y^2 M^2 f^t_{\delta q \delta q} & i e^{-i \varphi_y} y M f_{q \delta q} & i e^{-i \varphi_y} y M f_{q \delta q} & f_{qq} + f_{\Delta q \Delta q} \\
\end{array} \right) \sum_{\lambda_1' \lambda_2' \lambda_1 \lambda_2} v^*_{\lambda_1' \lambda_2'} \rho_{(\lambda_1' \lambda_2') (\lambda_1 \lambda_2)} v_{\lambda_1 \lambda_2} \geq 0
\]