# Quasi-PDFs: What they are and how to compute them on the lattice

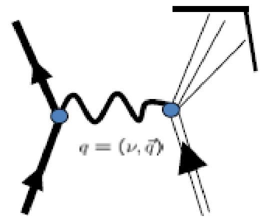

Fernanda Steffens

University of Bonn

In collaboration with: Constantia Alexandrou (Univ. of Cyprus; Cyprus Institute),
Krzysztof Cichy (Adam Mickiewicz, Poland)
Martha Constantinou (Temple University)
Karl Jansen (DESY – Zeuthen)
Aurora Scapellato (HPC-LEAP; Uni. Of Cyprus; Uni. of Wuppertal)

# Quark distributions and quasi-distributions

Cross sections are measured



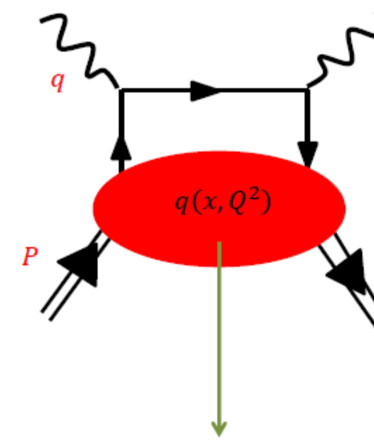

Cross sections written in terms of structure functions: $F_1(x,Q^2), F_2(x,Q^2), g_1(x,Q^2), g_2(x,Q^2), \cdots$

QCD + OPE: $$\int_0^1 dx x^{n-2} F_2(x,Q^2) = \sum_i a_n^{(i)} C_n^{(i)}(Q^2)$$

$$\langle P | \mathcal{O}_{\mu_1 \cdots \mu_n} | P \rangle = a_n P_{\mu_1} \cdots P_{\mu_2}$$

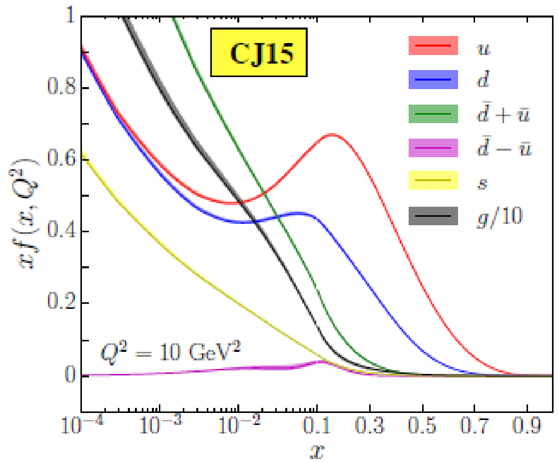
Moments of the parton distributions: $$a_n = \int dx\, x^{n-1} q(x)$$

At leading order (LO) in pQCD:, $$F_2(x,Q^2) = x \sum_q e_q^2 q(x,Q^2)$$

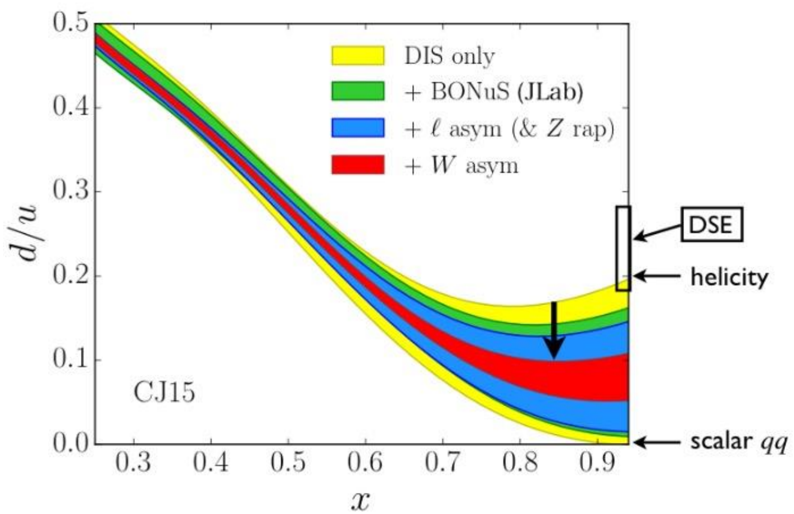



Parton distributions

# The individual distributions




Giving a closer look




From W. Melnitchouk, presentation at QCD Down Under 2017

SU(6) symmetry: $d/u \to 1/2$

$S = 0$ $qq$ dominance (colour-hyperfine interaction): $d/u \to 0$

$S_z = 0$ $qq$ dominance (perturbative gluon exchange): $d/u \to 1/5$

DSE with $qq$ correlations: $d/u \to$ 0.18-0.28

Extrapolated ratio at $x = 1$: $0.09 \pm 0.03$

No model can account for it

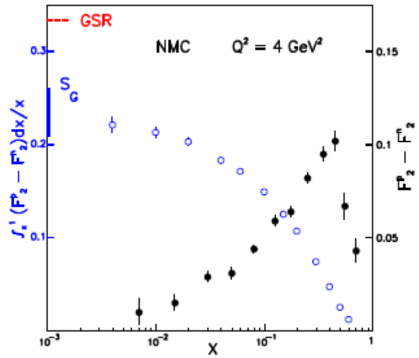
Can lattice say something about the large $x$ region? Or the $x$ dependence in general?

# Antiquarks are not symmetric

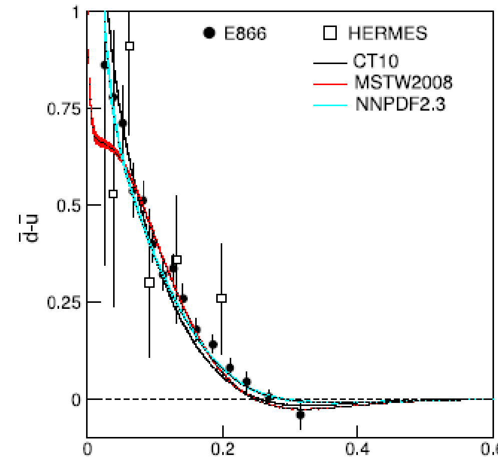

Expect $\bar{d} = \bar{u}$ if sea quarks are produced in $g \to q\bar{q}$

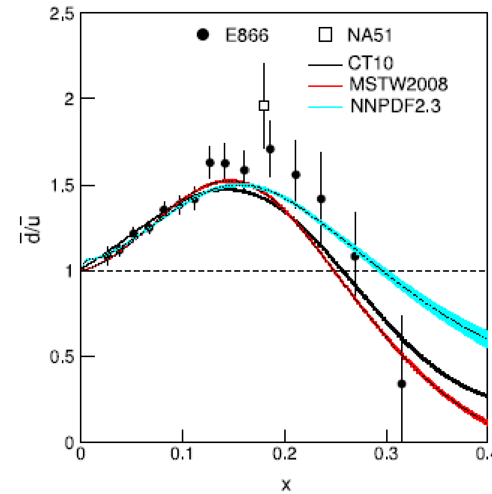
The Gottfried Sum Rule

$$S_G = \int_0^1 [(F_2^p(x) - F_2^n(x))/x]\,dx$$
$$= \frac{1}{3} + \frac{2}{3}\int_0^1 (\bar{u}_p(x) - \bar{d}_p(x))\,dx$$
$$= \frac{1}{3} \quad (if\ \bar{u}_p = \bar{d}_p)$$




From JC Peng, EINN2015




New Muon Collaboration (NMC) obtains

$S_G = 0.235 \pm 0.026$

**( Significantly lower than 1/3 ! )** $\Rightarrow \bar{d} \neq \bar{u}$ ?

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Polarized sector: STAR data also consistent with an asymmetry in favor of u antiquarks

Can we explain these curves from first principles?

# Light-cone quark distributions

The most general form of the matrix element is:

$$\langle P|O^{\mu_1\mu_2\cdots\mu_n}|P\rangle = 2a_n^{(0)}\Pi^{\mu_1\mu_2\cdots\mu_n}$$

$$\Pi^{\mu_1\mu_2\cdots\mu_n} = \sum_{j=0}^{k}(-1)^j\frac{(2k-j)!}{2^j(2k)!}\{g\cdots gP\cdots P\}_{k,j}(P^2)^j$$

We use the following four-vectors

$$P = (P_0,0,0,P_3) \quad \lambda = (1,0,0,-1)/\sqrt{2} \quad \longrightarrow \quad \boxed{\lambda\cdot P = (P_0+P_3)/\sqrt{2} = P_+}$$

$$\lambda_{\mu_1}\lambda_{\mu_2}\left\langle P\middle|O^{\mu_1\,\mu_2}\middle|P\right\rangle = 2a_n^{(0)}\left(P^+P^+ - \lambda^2\frac{M^2}{4}\right) = 2a_n^{(0)}P^+P^+$$

In general, we have

$$\lambda_{\mu_1}\cdots\lambda_{\mu_n}\Pi^{\mu_1\cdots\mu_n} = (P^+)^n \quad \longrightarrow \quad \boxed{\langle P|O^{+\cdots+}|P\rangle = 2a_n^{(0)}(P^+)^n}$$

Matrix elements projected on the light-cone are protected from target mass corrections

Taking the inverse Mellin transform

$$a_n^{(0)} = \int dx\, x^{n-1} q(x) \qquad\qquad q(x) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn\, x^{-n} a_n^{(0)}$$

Using $a_n^{(0)} = \langle P|O^{+\cdots+}|P\rangle / 2(P^+)^n$

$$\Rightarrow \quad q(x) = \int_{-\infty}^{+\infty} \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-} \langle P|\bar{\psi}(\xi^-)\gamma^+ W(\xi^-,0)\psi(0)|P\rangle$$

$$W(\xi^-,0) = e^{-ig\int_0^{\xi^-} A^+(\eta^-)d\eta^-}$$ (Wilson line)

- Light cone correlations
- Equivalent to the distributions in the Infinite Momentum Frame
- Light cone dominated $\xi^2 = t^2 - z^2 \sim 0$
- Not calculable on Euclidian lattice $t^2 + z^2 \sim 0$
- Moments, however, can be calculated

# Quasi Distributions

X. Ji, “Parton Physics on a Euclidean Lattice,” PRL 110 (2013) 262002.

Suppose we project outside the light-cone:

$$\lambda = (0,0,0,-1) \qquad P = (P_0, 0, 0, P_3) \qquad \lambda \cdot P = P_3$$

For example, for n=2

$$\lambda_{\mu_1}\lambda_{\mu_2}\left\langle P\middle|O^{\mu_1\,\mu_2}\middle|P\right\rangle = 2a_n^{(0)}\left(P_3^2 - \lambda^2\frac{M^2}{4}\right) = 2a_n^{(0)}\left(P_3^2 + \frac{M^2}{4}\right)$$

$\lambda^2$ = -1

Mass terms contribute

After the inverse Mellin transform,

$$\tilde{q}(x, P_3) = \int_{-\infty}^{+\infty} \frac{dz}{4\pi} e^{-izxP_3} \langle P|\bar{\psi}(z)\gamma^3 W(z,0)\psi(0)|P\rangle + \mathcal{O}\left(\frac{M^2}{P_3^2}, \frac{\Lambda_{QCD}^2}{P_3^2}\right)$$

Higher twist

- Nucleon moving with finite momentum in the z direction
- Pure spatial correlation
- Can be simulated on a lattice

The light cone distributions:

$$x = \frac{k^+}{P^+}$$

$$0 \leq x \leq 1$$

Distributions can be defined in the infinite momentum frame: $P_3, P^+ \to \infty$

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Quasi distributions:

$P_3$ large but finite

Usual partonic interpretation is lost

$x < 0$ or $x > 1$ is possible

But they can be related to each other!

# Extracting quark distributions from quark quasi-distributions

Infrared region untouched when going from finite to infinite momentum

Infinite momentum: $p_3 \to \infty$ (before integrating over the quark transverse momentum $k_T$)

$$q(x,\mu) = q_{bare}(x)\left\{1 + \frac{\alpha_s}{2\pi} Z_F(\mu)\right\} + \frac{\alpha_s}{2\pi}\int_x^1 \Gamma\left(\frac{x}{y},\mu\right) q_{bare}(y)\frac{dy}{y} + \mathcal{O}(\alpha_s^2)$$

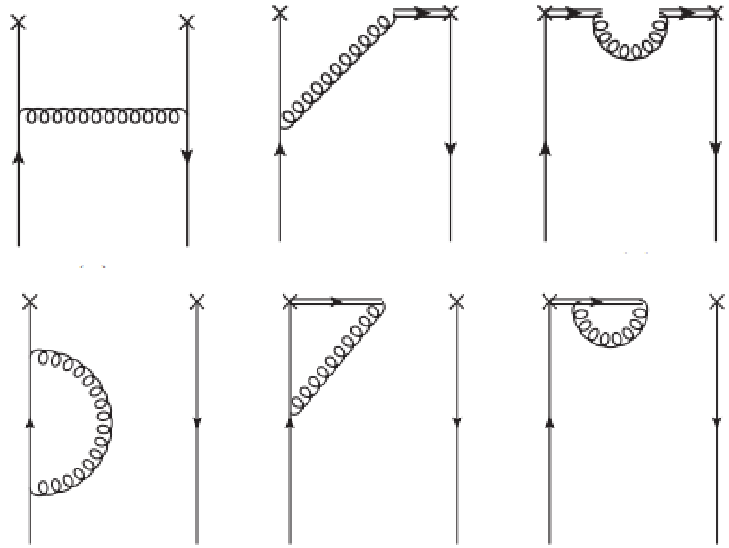
Finite momentum: $p_3$ fixed

$$\tilde{q}(x,P_3) = q_{bare}(x)\left\{1 + \frac{\alpha_s}{2\pi} \tilde{Z}_F\,(P_3)\right\} + \frac{\alpha_s}{2\pi}\int_{x/y_c}^1 \tilde{\Gamma}\left(\frac{x}{y},P_3\right) q_{bare}(y)\frac{dy}{y} + \mathcal{O}(\alpha_s^2)$$

$$\tilde{q}(\pm y_c) = 0$$

In principle, $y_c \to \infty$

# Perturbative QCD in the continuum



**Vertex: $\Gamma$ or $\tilde{\Gamma}$**

**Self-energy**: $Z_F$ or $\tilde{Z}_F$

$$q(x,\mu) = \tilde{q}(x,p_3) - \frac{\alpha_s}{2\pi}\tilde{q}(x,p_3)\delta Z_F\left(\frac{\mu}{p_3},x_c\right) - \frac{\alpha_s}{2\pi}\int_{-x_c}^{-|x|/y_c}\delta\Gamma\left(y,\frac{\mu}{p_3}\right)\tilde{q}\left(\frac{x}{y},p_3\right)\frac{dy}{|y|}$$
$$-\frac{\alpha_s}{2\pi}\int_{+|x|/y_c}^{+x_c}\delta\Gamma\left(y,\frac{\mu}{p_3}\right)\tilde{q}\left(\frac{x}{y},p_3\right)\frac{dy}{|y|}$$

$$\delta\Gamma = \tilde{\Gamma} - \Gamma$$

**Matching equation**

$$\delta Z_F = \tilde{Z}_F - Z_F$$


X. Xiong, X. Ji, J. H. Zhang and Y. Zhao, PRD 90 014051 (2014)
C.Alexandrou, K.Cichy, V.Drach, E.Garcia-Ramos, K.Hadjiyiannakou, K.Jansen, F.Steffens and C.Wiese, PRD 92 014502 (2015)
W. Wang, S. Zhao and R. Zhu, Eur. Phys. J. C78 (2018) 147;
W. Stewart, Y. Zhao, PRD 97 054512 (2018
T.Izubuchi, X.Ji, L.Jin, I.W.Stewart and Y.Zhao, arXiv:1801.03917
C.Alexandrou, K.Cichy, M.Constantinou, K.Jansen, A.Scapellato and F.Steffens, arXiv:1803.02685, to appear in PRL

# Matching kernel for unpolarized case

$$\delta\Gamma^R\left(y,\frac{\mu}{p_3}\right) = \begin{cases} -\frac{1+y^2}{1-y}ln\frac{y-1}{y}+1+\frac{3}{2y} & y>1 \\ -\frac{1+y^2}{1-y}ln\frac{\mu^2}{4p_3^2y(1-y)}-\frac{y+y^2}{1-y} & 0<y<1 \\ -\frac{1+y^2}{1-y}ln\frac{y}{y-1}-1+\frac{3}{2(1-y)} & y<0 \end{cases}$$

$$\delta Z_F^R\left(\frac{\mu}{p_3}\right) = \begin{cases} \int_{-\infty}^{+\infty}d\eta\left(\frac{1+\eta^2}{1-\eta}ln\frac{\eta-1}{\eta}-1-\frac{3}{2\eta}\right) & \eta>1 \\ \int_{-\infty}^{+\infty}d\eta\left(\frac{1+\eta^2}{1-\eta}ln\frac{\mu^2}{4p_3^2\eta(1-\eta)}+\frac{\eta+\eta^2}{1-\eta}\right) & 0<\eta<1 \\ \int_{-\infty}^{+\infty}d\eta\left(\frac{1+\eta^2}{1-\eta}ln\frac{\eta}{\eta-1}+1-\frac{3}{2(1-\eta)}\right) & \eta<0 \end{cases}$$

Regions outside the physical region, $0 < x < 1$, are not equal to zero!

Infrared divergences are the same in the quasi and light-cone PDFs: same splitting functions;

Automatically preserves quark number in all stages of the computation;

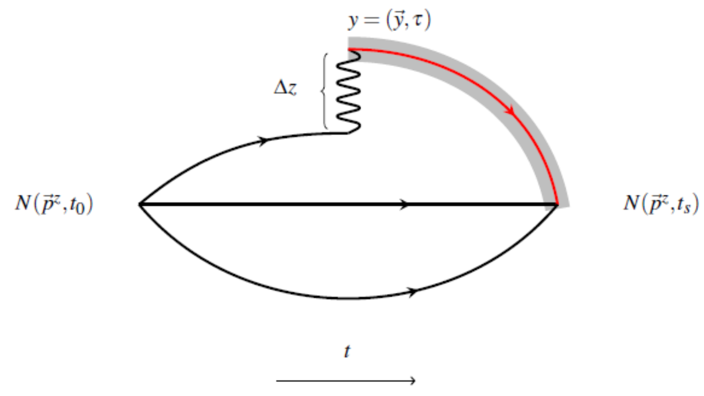
For the helicity case, similar as above, obtained by adding a factor $2\,(1+y)$ in the physical region.

## Main steps of the procedure:

1. Compute the matrix elements between proton states with finite $P_3$;
2. Non-perturbative renormalization of the matrix elements;
3. Fourier transform to obtain the quasi-PDF $\tilde{q}(x, P_3, \mu)$;
4. Matching procedure to obtain the light-cone PDF $q(x, \mu)$;
5. Apply Target Mass Corrections (TMCs) to correct for the powers of $M^2/P_3^2$ .

# Computation of matrix elements

$$\frac{C^{3pt}(T_s,\tau,0;P_3)}{C^{2pt}(T_s,0;P_3)} \propto \Delta h(P_3,z), \qquad 0 \ll \tau \ll T_s$$




With the 3 point function given by:

$$C^{3pt}(t,\tau,0) = \left\langle N_\alpha(\vec{P},t)\mathcal{O}(\tau)\overline{N_\alpha}(\vec{P},0)\right\rangle$$

And

$$\mathcal{O}(z,\tau,Q^2=0) = \sum_{\vec{y}} \bar{\psi}(y+z)\gamma^3\gamma^5 W\,(y+z,y)\psi(y)$$

Where the matrix elements (ME) are: $\Delta h(P_3,z) = \left\langle P\middle|\bar{\psi}(z)\gamma^3\gamma^5 W\,(z,0)\psi(0)\middle|P\right\rangle$

Setup: $N_f = 2, \qquad \beta = \frac{6}{g_0^2} = 2.10, \qquad a = 0.0938(3)(2)fm$

$$48^3 \times 96, \qquad L = 4.5fm, \qquad m_\pi = 0.1304(4)\ GeV, \qquad m_\pi L = 2.98(1)$$

$$P_3 = \frac{6\pi}{L}, \frac{8\pi}{L}, \frac{10\pi}{L} = 0.84, 1.11, 1.38 \text{ GeV}$$

6 directions of Wilson line: $\pm x, \pm y, \pm z$

16 source positions

Separation $T_s \approx 1.1$ fm as the lowest safe choice

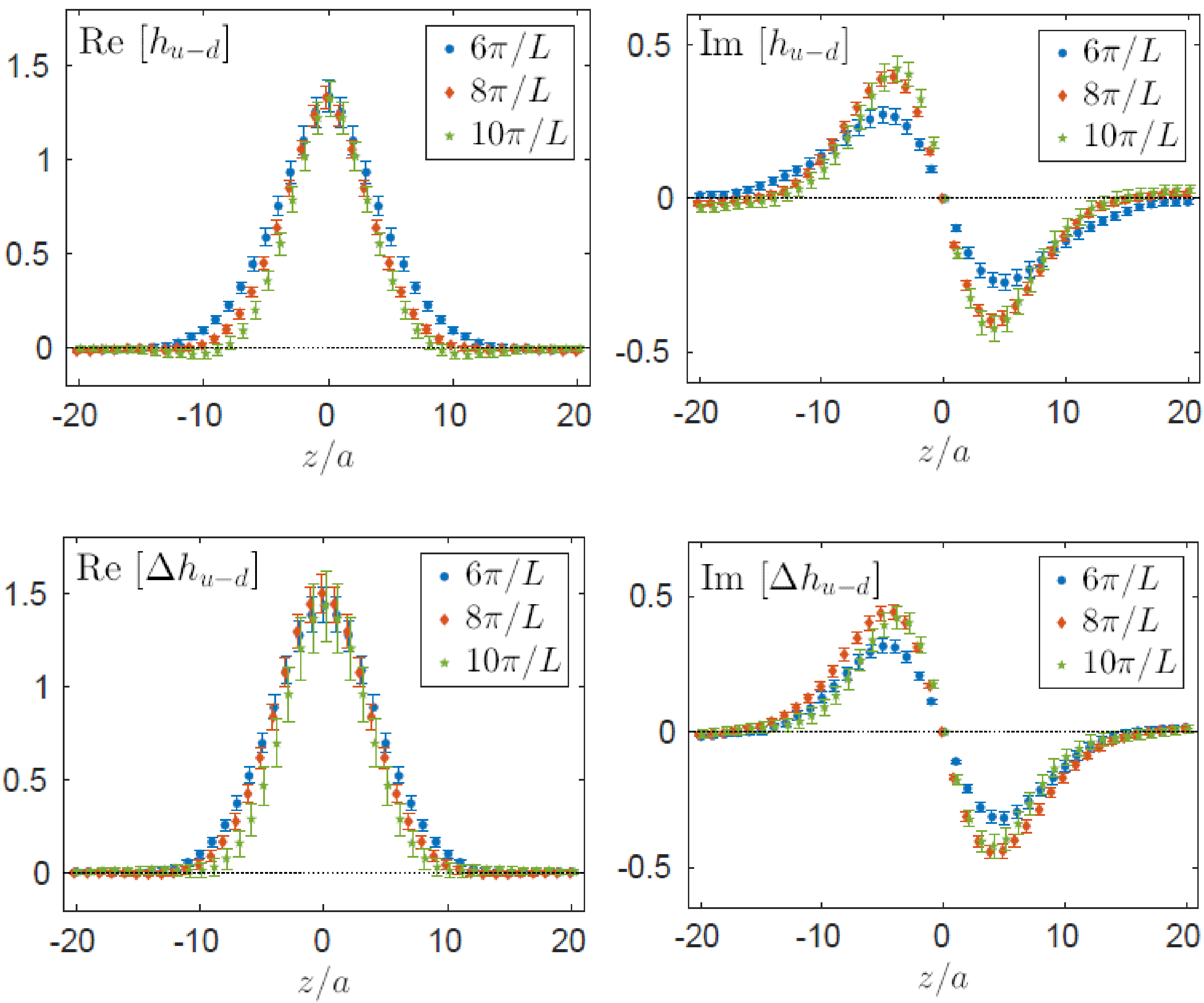
| $P_3 = \frac{6\pi}{L}$ | | | $P_3 = \frac{8\pi}{L}$ | | | $P_3 = \frac{10\pi}{L}$ | | |
|---|---|---|---|---|---|---|---|---|
| Ins. | $N_{\text{conf}}$ | $N_{\text{meas}}$ | Ins. | $N_{\text{conf}}$ | $N_{\text{meas}}$ | Ins. | $N_{\text{conf}}$ | $N_{\text{meas}}$ |
| $\gamma_0$ | 50 | 4800 | $\gamma_0$ | 425 | 38250 | $\gamma_0$ | 655 | 58950 |
| $\gamma_5\gamma_3$ | 65 | 6240 | $\gamma_5\gamma_3$ | 425 | 38250 | $\gamma_5\gamma_3$ | 655 | 58950 |

With these configurations, we compute the corresponding matrix elements

$$h(P_3, z) = \langle P|\bar{\psi}(z)\gamma^0 W(z,0)\psi(0)|P\rangle$$ Unpolarized

$$\Delta h(P_3, z) = \langle P|\bar{\psi}(z)\gamma^3\gamma^5 W(z,0)\psi(0)|P\rangle$$ Helicity

C. Alexandrou et al., 1803.02685, PRL121, 112001 (2018)




**Unpolarized**

**Helicity**

The bare matrix elements $\langle P|\bar{\psi}(z)\Gamma W(z,0)\psi(0)|P\rangle$, however, contain divergences:

Renormalization is necessary!

# Renormalization

$$\Delta h^{R,u-d} = Z_{\Delta h} M \Delta h^{u-d} = (Re[Z_{\Delta h}] + i\, Im[Z_{\Delta h}])\left(Re\left[\Delta h^{u-d}\right] + i\, Im\left[\Delta h^{u-d}\right]\right)$$

$Z_{\Delta h}$ renormalizes both the usual log divergence
and the estra linear divergence associated with the Wilson line

Nonperturbative renormalization using the RI'-MOM to remove both divergences

C. Alexandrou et al., NPB 923 (2017) 394 (Frontier Article)
J-W. Chen et al., PRD 97 014505 (2018)
C. Alexandrou et al., 1807.00232

Convert the ME from RI'-MOM to $\overline{MS}$ using 1-loop perturbation theory

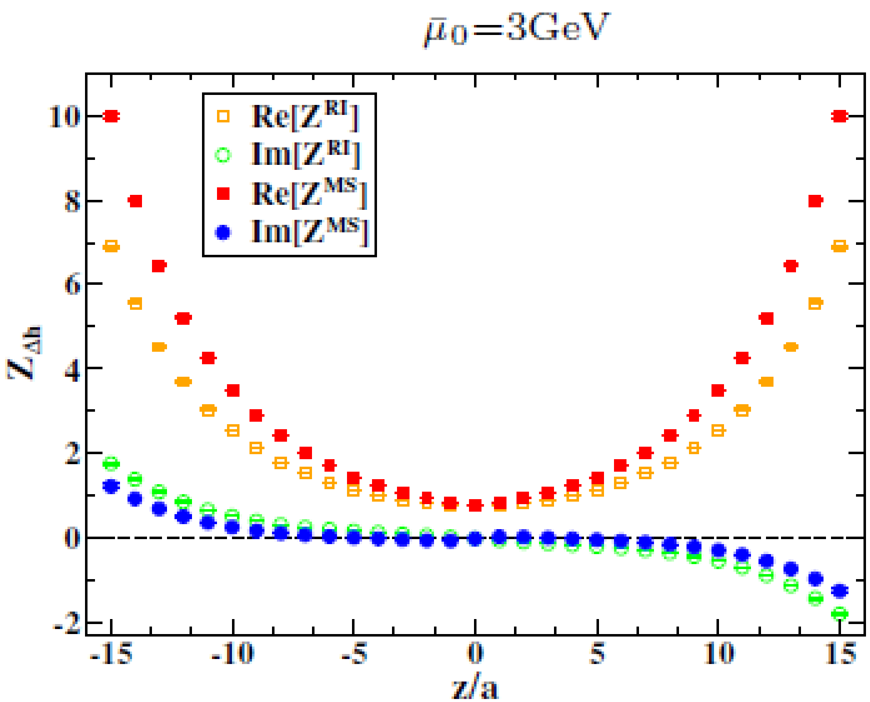
M. Constantinou, H. Panapaulos, PRD (2017)054506

We present results for the $\overline{MS}$ scheme

# Renormalization factor for helicity

RI'-MOM scheme at the scale $\bar{\mu}_0 = 3$ GeV

Perturbative conversion to $\overline{MS}$ scheme at the scale 2 GeV




$$Z_q^{-1}\, Z_{\mathcal{O}} \frac{1}{12} Tr[\nu(p,z)(\nu^{Born}(p,z))^{-1}]|_{p^2=\bar{\mu}_0^2} = 1$$

$$Z_q = \frac{1}{12} Tr[\left(S(p)\right)^{-1} S^{Born}(p)]|_{p^2=\bar{\mu}_0^2}$$

The vertex function $\nu$ contains the same divergences as the nucleon matrix elements

The factor $Z_{\mathcal{O}}$ subtracts both the linear and log divergences.

The linear divergence associated with the Wilson line makes $Z_{\mathcal{O}}$ to grow very fast for large $z$;

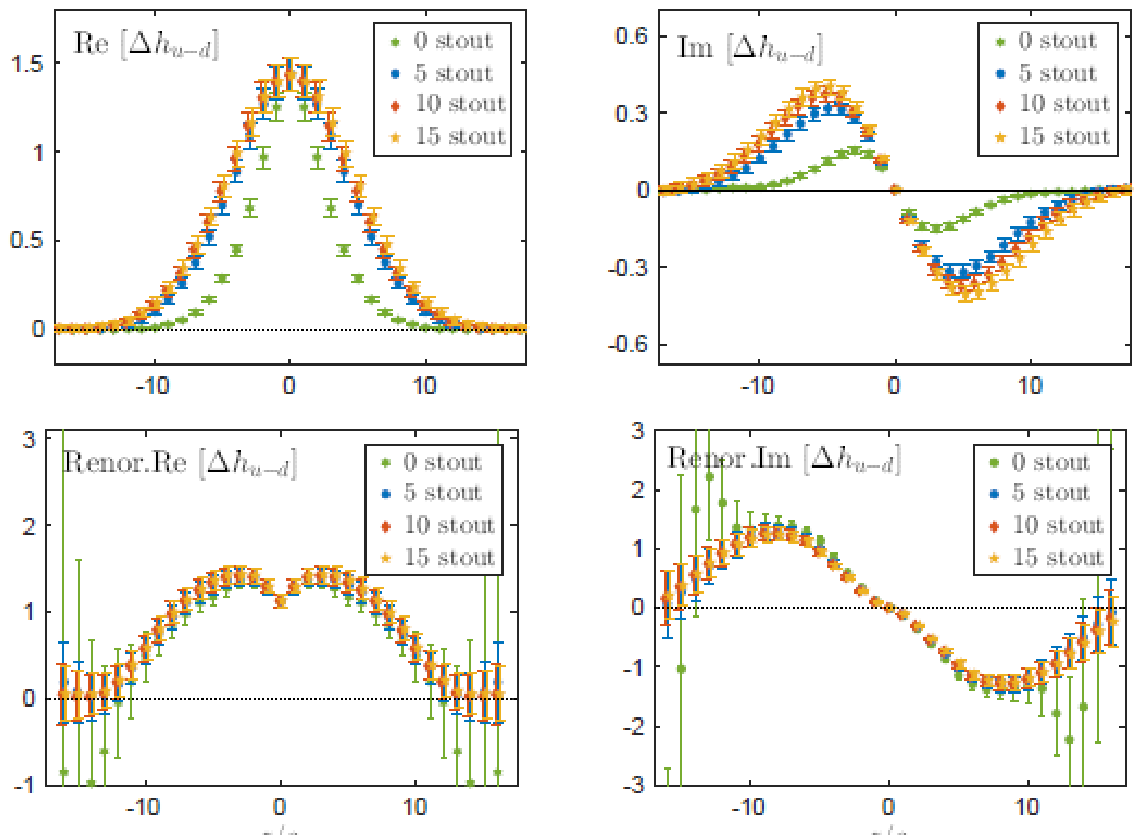
That makes the renormalized ME to have amplified errors at large $z$;

We thus apply smearing to the Wilson lines only in order to smooth the divergence;

In the end, if the procedure is consistent, the resulting renormalized ME should be the same, independent of the smearing applied

# Renormalized ME for the helicity case

$P_3 \approx 0.83\ GeV$




**Bare ME**

**Renormalized ME**

$Re[\Delta h^{u-d}]\ Re[Z_{\Delta h}] - Im[\Delta h^{u-d}]\ Im[Z_{\Delta g}]$

$Re[\Delta h^{u-d}]\ Im[Z_{\Delta h}] + Im[\Delta h^{u-d}]\ Re[Z_{\Delta h}]$

ME sit on top of each other after renormalization

Renormalization is doing its job!

# The $x$ dependence of $\Delta u(x) - \Delta d(x)$

Once we have the ME, we compute the qPDF:

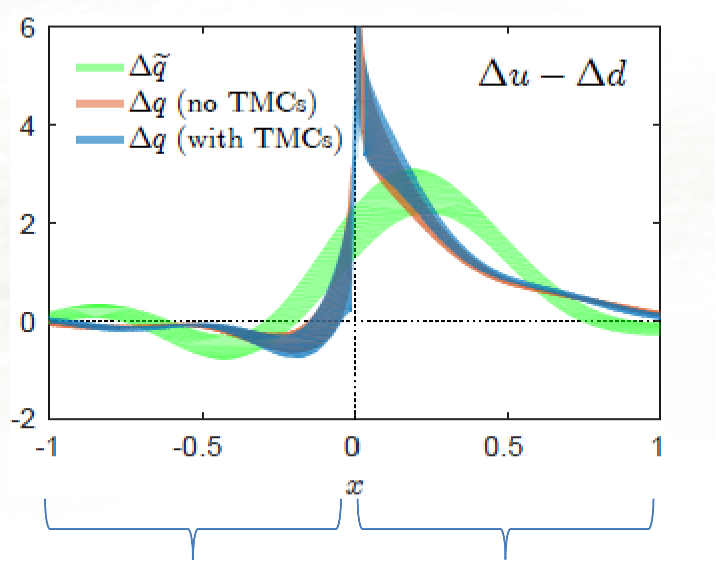
$$\Delta\tilde{q}(x,\mu^2,P_3) = \int \frac{dz}{4\pi} e^{-ixP_3 z} \langle P|\bar{\psi}(z)\gamma^3\gamma^5 W(z,0)\psi(0)|P\rangle$$

Continuum Euclidean qPDF = continuum Minkowski qPDF: Carlson, Freid, PRD 95 (2017) 094504
Briceño *et al.*, PRD 96 (2017) 014502

And then apply the matching plus target mass corrections to obtain the light-cone PDF:

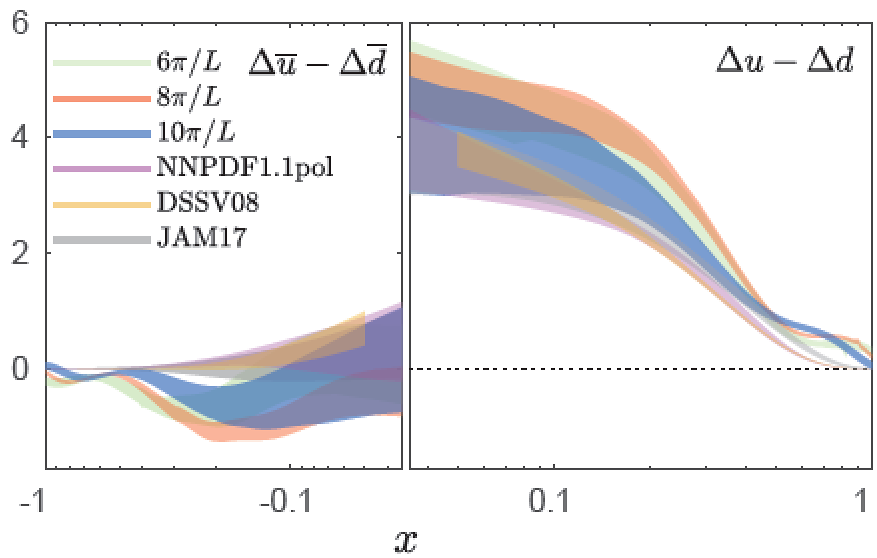
$$\Delta q(x,\mu) = \int_{-\infty}^{+\infty} \frac{d\xi}{\xi} C\left(\xi, \frac{\mu}{xP_3}\right) \Delta\tilde{q}\left(\frac{x}{\xi},\mu,P_3\right)$$

Helicity iso-vector quark distribution




$$P_3 = \frac{10\pi}{L} \approx 1.38 \text{ GeV}$$

# Helicity iso-vector quark distribution



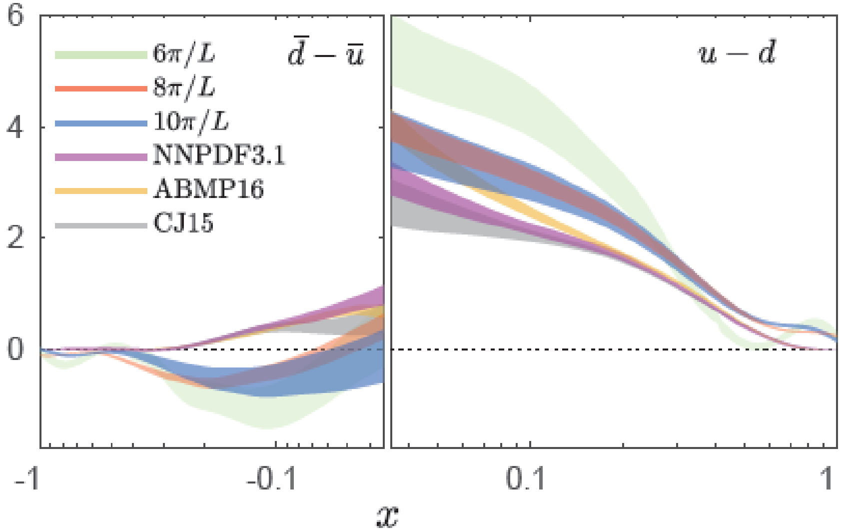

C. Alexandrou et al., 1803.02685,
PRL 121, 112001 (2018)

Remarkable qualitative agreement

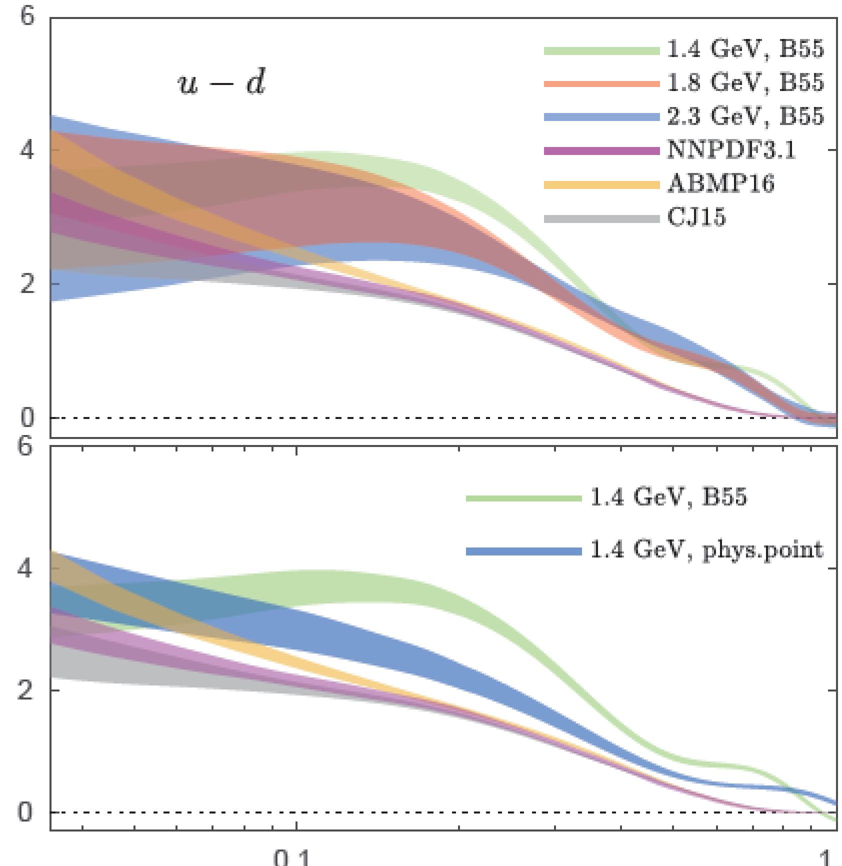
For the values of $P_3$ used here, the ME do not decay fast enough, that is, before $e^{-ixP_3z}$ becomes negative

When doing the Fourier transform, unphysical oscillations appear, remarkably for $x > 0.5$, and an unphysical minimum at $x \approx -0.2$

# Unpolarized iso-vector quark distribution

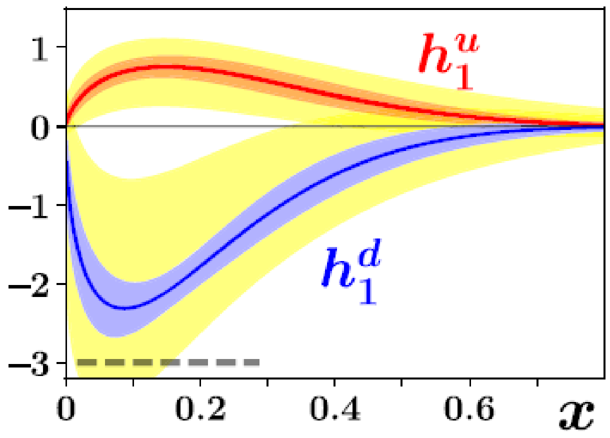


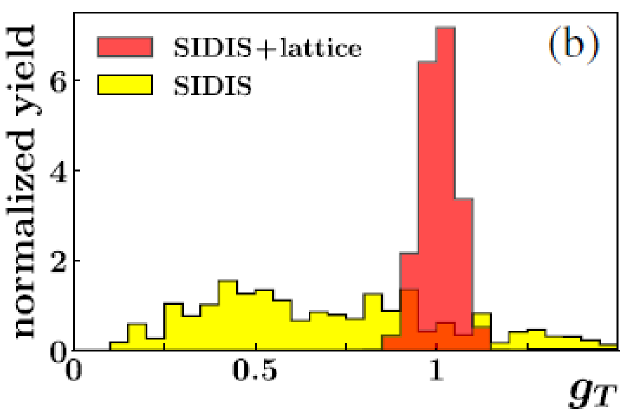

**Unphysical pion mass**
$m_\pi \approx 372$ **MeV**




**Unphysical x Physical** $m_\pi$

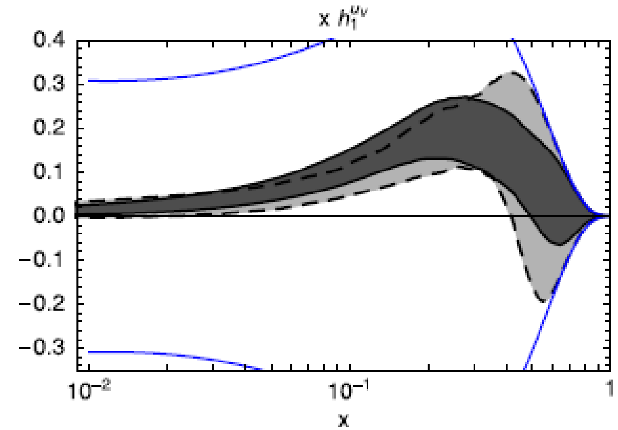
# Iso-vector transversity distributions

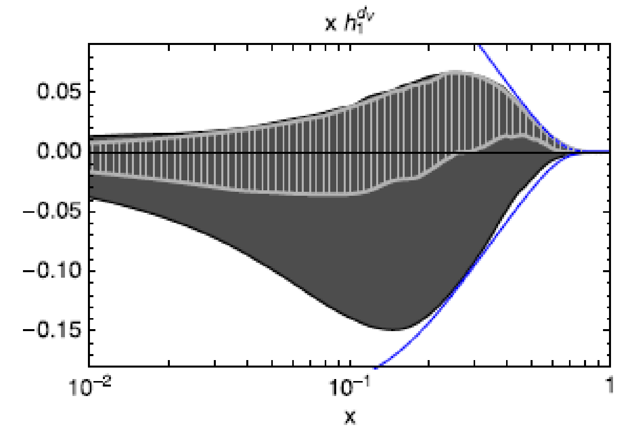



$$g_T = \int_0^1 dx \left( h_1^u(x) - h_1^d(x) \right) = 1.0(1)$$

H.-W. Lin PRL 120, 152502 (2018)



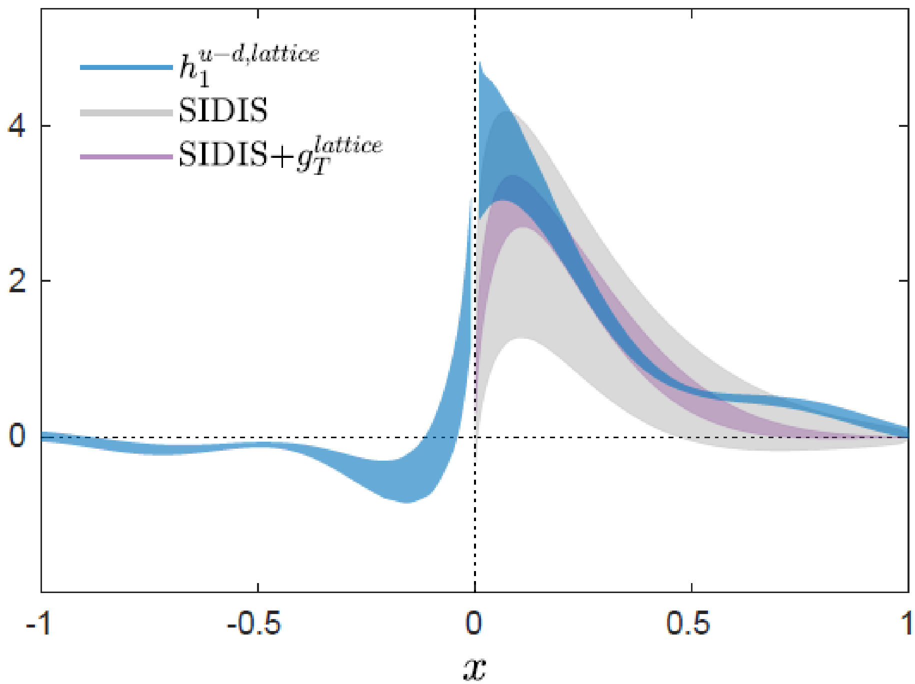




$$g_T = 0.53(25)$$

Radici and Bacchetta PRL 120, 192001 (2018)

$$\left\langle P \middle| \bar{\psi}(z) \frac{\sigma_{31} + \sigma_{32}}{2} \, W\,(z, 0) \tau^3 \psi(0) \middle| P \right\rangle$$

# Tranvesveristy iso-vector quark distribution




C. Alexandrou et all.,
1807.00232

$$g_T = \int_{-1}^{+1} dx h_1^{u-d} = 1.09(11)$$

This should be compared to: $g_T = 1.06(1)$ from dedicated lattice QCD calculation

C. Alexandrou et al., PRD95, 114514 (2017)

$g_T = 1.0(1)$ from Monte Carlo global analysis

H.-W. Lin PRL 120, 152502 (2018)

$g_T = 0.53(25)$ from global analysis of $ep$ and $pp$ data

Radici and Bacchetta PRL 120, 192001 (2018)

# Summary

We have also shown an *ab initio* computation of the $x$ dependence of the iso-vector unpolarized, helicity, and transversity PDFs directly at the physical point;

No input nor any assumption on their functional dependence, this was unthinkable of just few years ago;

Enormous progress over the last couple of years:

- a complete non-perturbative prescription for the ME has emerged
- the matching equations relating the qPDFs to the light-cone PDFs have been improved

Still, many challenges remain:

- How to go to higher values of $P_3$?
- Unphysical oscillations
- Discretization and volume effects
- Higher twist

Physical point computation also presented in Huey-Wen Lin et all., 1803.04393, 1807.07431

Quasi-PDFs are intrinsically related to pseudo-PDFs, see Radyushkin, PRD 96 (2017) 034025