

Master Majorana neutrino mass parametrization

Avelino Vicente
IFIC – CSIC / U. Valencia

In collaboration with
Isabel Cordero-Carrión and Martin Hirsch
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Portoroz 19



There are **MANY** Majorana neutrino mass models...

Tree-level

Radiative: 1-loop, 2-loop, 3-loop, ...

High scale

Low scale


Dimension-5: Weinberg operator

Higher dimensions: dim-7, dim-9, ...



The world of beers

↳ mass models



**See talks by Jacobo Lopez Pavon,
Andrea Romanino, Frank Deppisch
and Juan Herrero-Garcia**

The world of beers

mass models

This talk

Can one unify all models in a *master* neutrino mass formula?

Can one find a *master* solution to this formula?



Outline

Master formula

One formula to rule them all

Master parametrization

The most general solution to the master formula

An application

An illustrative example





*One formula to
rule them all*

Master formula

The master formula

$$m = f (y_1^T M y_2 + y_2^T M^T y_1)$$

The master formula

$$m = f (y_1^T M y_2 + y_2^T M^T y_1)$$

m : symmetric 3×3 Majorana neutrino mass matrix

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f : global factor [numerical factors, model parameters, mass ratios, ...]

The master formula

$$m = f \left(y_1^T M y_2 + y_2^T M^T y_1 \right)$$

m : symmetric 3×3 Majorana neutrino mass matrix

f : global factor [numerical factors, model parameters, mass ratios, ...]

y_1 : $n_1 \times 3$ Yukawa matrix

y_2 : $n_2 \times 3$ Yukawa matrix [$n_1 \geq n_2$]

The master formula

$$m = f \left(y_1^T M y_2 + y_2^T M^T y_1 \right)$$

m : symmetric 3×3 Majorana neutrino mass matrix

f : global factor [numerical factors, model parameters, mass ratios, ...]

y_1 : $n_1 \times 3$ Yukawa matrix

y_2 : $n_2 \times 3$ Yukawa matrix [$n_1 \geq n_2$]

M : $n_1 \times n_2$ matrix [dimensions of mass]

Some examples

To convince you that the **master formula** is indeed valid for all Majorana neutrino mass models

Type-I seesaw



everyone's
model

$$m = f (y_1^T M y_2 + y_2^T M^T y_1)$$

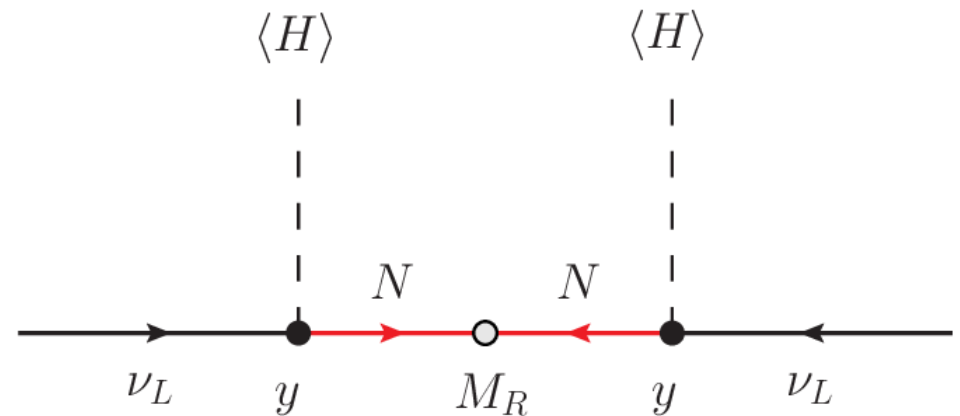
$$f = -1$$

$$n_1 = n_2 = 3$$

$$y_1 = y_2 = y/\sqrt{2}$$

$$M = \frac{v^2}{2} M_R^{-1}$$

$$m = -\frac{v^2}{2} y^T M_R^{-1} y$$



[Minkowski, 1977]

Inverse seesaw



$$m = f (y_1^T M y_2 + y_2^T M^T y_1)$$

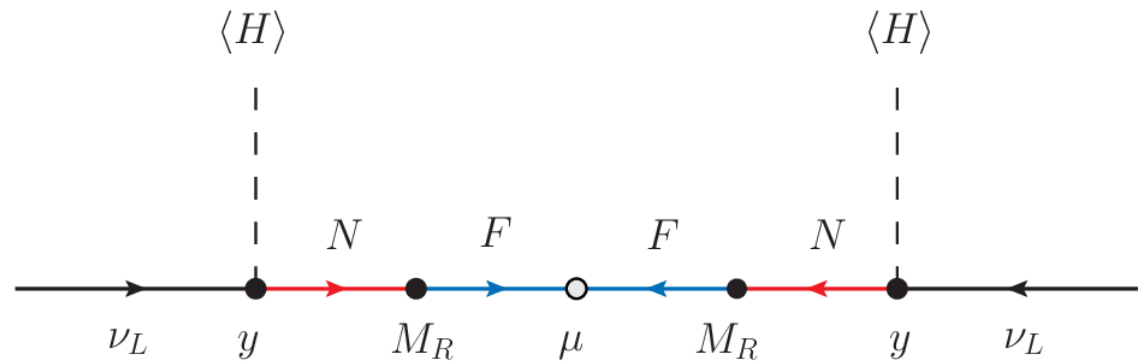
$$f = 1$$

$$n_1 = n_2 = 3$$

$$y_1 = y_2 = y/\sqrt{2}$$

$$M = \frac{v^2}{2} (M_R^T)^{-1} \mu M_R^{-1}$$

$$m = \frac{v^2}{2} y^T (M_R^T)^{-1} \mu M_R^{-1} y$$



[Mohapatra, Valle, 1986]

Scotogenic model



$$m = f (y_1^T M y_2 + y_2^T M^T y_1)$$

$$m = \frac{\lambda_5 v^2}{32\pi^2} y^T M_R^{-1} f_{\text{loop}} y$$

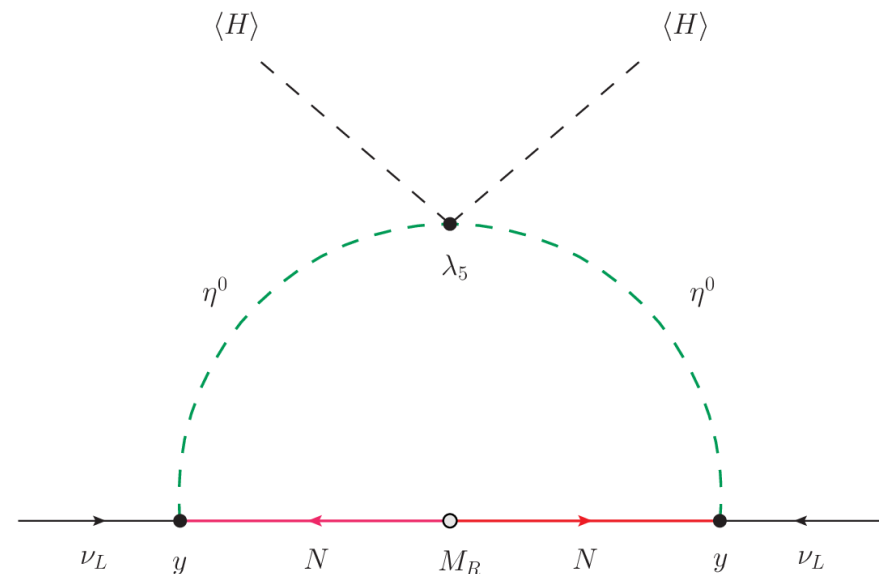
$$f = \frac{\lambda_5}{16\pi^2}$$

$$n_1 = n_2 = 3$$

$$y_1 = y_2 = y/\sqrt{2}$$

$$M = \frac{v^2}{2} M_R^{-1} f_{\text{loop}}$$

\Rightarrow



[Ma, 2006]

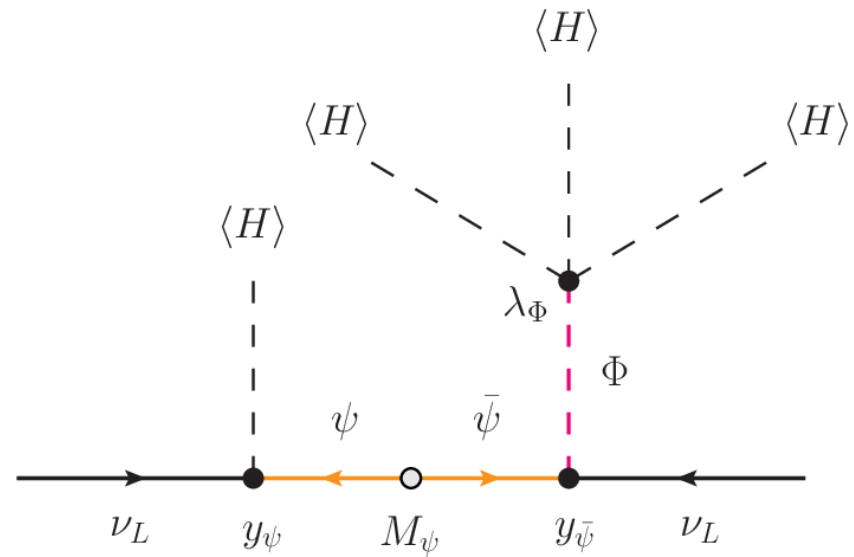
BNT model



$$m = f (y_1^T M y_2 + y_2^T M^T y_1)$$

$$m = \frac{\lambda_\Phi v^4}{4 M_\Phi^2} \left[y_\psi^T M_\psi^{-1} y_{\bar{\psi}} + y_{\bar{\psi}}^T (M_\psi^{-1})^T y_\psi \right]$$

$$\left. \begin{aligned} f &= \frac{\lambda_\Phi v^2}{2 M_\Phi^2} \\ n_1 &= n_2 = 3 \\ y_1 &= y_\psi \neq y_2 = y_{\bar{\psi}} \\ M &= \frac{v^2}{2} M_\psi^{-1} \end{aligned} \right\} \Rightarrow$$



[Babu, Nandi, Tavartkiladze, 2009]

Towards a master parametrization

$$m = f \left(\underbrace{y_1^T M y_2 + y_2^T M^T y_1}_{\text{Model parameters}} \right)$$

↑
Neutrino oscillation experiments

↑
Model parameters

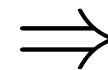
Goal:

To establish a general, complete and programmable parametrization of the y_1 and y_2 Yukawa matrices

Particular case: Type-I seesaw

Casas-Ibarra parametrization

[Casas, Ibarra, 2001]



Master parametrization

The master parametrization



$$V = G \cdot h = B \cdot e^2 \cdot r$$

The master parametrization

$$y_1 = \frac{1}{\sqrt{2f}} V_1^\dagger \begin{pmatrix} \Sigma^{-1/2} W A \\ X_1 \\ X_2 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

$$y_2 = \frac{1}{\sqrt{2f}} V_2^\dagger \begin{pmatrix} \Sigma^{-1/2} \widehat{W}^* \widehat{B} \\ X_3 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

The master parametrization

$$y_1 = \frac{1}{\sqrt{2f}} V_1^\dagger \begin{pmatrix} \Sigma^{-1/2} W A \\ X_1 \\ X_2 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

$$y_2 = \frac{1}{\sqrt{2f}} V_2^\dagger \begin{pmatrix} \Sigma^{-1/2} \widehat{W}^* \widehat{B} \\ X_3 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

$$D_m = \text{diag}(m_1, m_2, m_3) = U^T m U$$

$$r_m = \text{rank}(m) \begin{cases} \bar{D}_{\sqrt{m}} = \text{diag}(\sqrt{m_1}, \sqrt{m_2}, \sqrt{m_3}) \\ \bar{D}_{\sqrt{m}} = \text{diag}(\sqrt{m_1}, \sqrt{m_2}, \sqrt{v}) \end{cases}$$

2 or 3

U : 3×3 unitary matrix

$$U^\dagger U = U U^\dagger = \mathbb{I}_3$$

Leptonic mixing matrix

The master parametrization

$$y_1 = \frac{1}{\sqrt{2f}} V_1^\dagger \begin{pmatrix} \Sigma^{-1/2} W A \\ X_1 \\ X_2 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

$$y_2 = \frac{1}{\sqrt{2f}} V_2^\dagger \begin{pmatrix} \Sigma^{-1/2} \widehat{W}^* \widehat{B} \\ X_3 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

$$M = V_1^T \widehat{\Sigma} V_2$$

Singular-value decomposition

$$\widehat{\Sigma} = \begin{pmatrix} \Sigma & 0 \\ 0 & 0_{n_2-n} \\ \hline 0_{n_1-n_2} \end{pmatrix}$$

$$\widehat{\Sigma} : n_1 \times n_2$$

$$V_1 : n_1 \times n_1$$

$$V_2 : n_2 \times n_2$$

Unitary matrices

$$\Sigma : n \times n$$

$$\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n) \quad (\sigma_i > 0)$$

The master parametrization

$$y_1 = \frac{1}{\sqrt{2f}} V_1^\dagger \begin{pmatrix} \Sigma^{-1/2} W A \\ X_1 \\ X_2 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

$$y_2 = \frac{1}{\sqrt{2f}} V_2^\dagger \begin{pmatrix} \Sigma^{-1/2} \widehat{W}^* \widehat{B} \\ X_3 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

$$X_1 : (n_2 - n) \times 3$$

Arbitrary matrices

$$X_2 : (n_1 - n_2) \times 3$$

[Note: absent if $n_1 = n_2 = n$]

$$X_3 : (n_2 - n) \times 3$$

The master parametrization

$$y_1 = \frac{1}{\sqrt{2f}} V_1^\dagger \begin{pmatrix} \Sigma^{-1/2} W A \\ X_1 \\ X_2 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

$$y_2 = \frac{1}{\sqrt{2f}} V_2^\dagger \begin{pmatrix} \Sigma^{-1/2} \widehat{W}^* \widehat{B} \\ X_3 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

$$\widehat{W} = \begin{pmatrix} W & \bar{W} \end{pmatrix}$$

$$r = \text{rank}(W) \\ \leq \min(n, 3)$$

\widehat{W} : $n \times n$ unitary matrix

$$W : n \times r$$

$$\bar{W} : n \times (n - r) \longrightarrow \begin{matrix} \text{Absent if} \\ n = r \end{matrix}$$

$$\widehat{W}^\dagger \widehat{W} = \widehat{W} \widehat{W}^\dagger = \mathbb{I}_n$$

The master parametrization

$$y_1 = \frac{1}{\sqrt{2f}} V_1^\dagger \begin{pmatrix} \Sigma^{-1/2} W A \\ X_1 \\ X_2 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$
$$y_2 = \frac{1}{\sqrt{2f}} V_2^\dagger \begin{pmatrix} \Sigma^{-1/2} \widehat{W}^* \widehat{B} \\ X_3 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

$$A : r \times 3$$

$$A = T C_1$$

$$T : r \times r$$

Invertible upper triangular matrix

$$(T)_{ii} \in \mathbb{R}, (T)_{ii} > 0$$

$$C_1 : r \times 3$$

Numerical matrix whose form depends on r_m and r

The master parametrization

$$y_1 = \frac{1}{\sqrt{2f}} V_1^\dagger \begin{pmatrix} \Sigma^{-1/2} W A \\ X_1 \\ X_2 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

$$y_2 = \frac{1}{\sqrt{2f}} V_2^\dagger \begin{pmatrix} \Sigma^{-1/2} \widehat{W}^* \widehat{B} \\ X_3 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

$$\widehat{B} = \begin{pmatrix} B \\ \bar{B} \end{pmatrix}$$

$$B : r \times 3$$

$$B = (T^T)^{-1} [C_1 C_2 + K C_1]$$

$$C_2 : 3 \times 3$$

Matrix whose form depends on r_m and r

$$\widehat{B} : n \times 3$$

$$\bar{B} : (n - r) \times 3 \longrightarrow \text{Absent if } n = r$$

$K : r \times r$
antisymmetric

The master parametrization

$$y_1 = \frac{1}{\sqrt{2f}} V_1^\dagger \begin{pmatrix} \Sigma^{-1/2} W \mathbf{A} \\ X_1 \\ X_2 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$
$$y_2 = \frac{1}{\sqrt{2f}} V_2^\dagger \begin{pmatrix} \Sigma^{-1/2} \widehat{W}^* \widehat{\mathbf{B}} \\ X_3 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

For $r_m = r = 3$:

$$C_1 = \mathbb{I}_3 \quad C_2 = \mathbb{I}_3 + K_{12} \frac{T_{13}}{T_{11}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

[Different matrix forms for other values of (r_m, r) . Ask me if you want to see them]

The master parametrization

$$y_1 = \frac{1}{\sqrt{2f}} V_1^\dagger \begin{pmatrix} \Sigma^{-1/2} W A \\ X_1 \\ X_2 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

$$y_2 = \frac{1}{\sqrt{2f}} V_2^\dagger \begin{pmatrix} \Sigma^{-1/2} \widehat{W}^* \widehat{B} \\ X_3 \end{pmatrix} \bar{D}_{\sqrt{m}} U^\dagger$$

Summary:

	Experimental input:	Model input:	Free Yukawa parameters:	
v-data	$\left\{ \begin{array}{l} \bar{D}_{\sqrt{m}} \\ U \end{array} \right\}$	$M \left\{ \begin{array}{l} f \\ \Sigma \\ V_{1,2} \end{array} \right\}$	$\left\{ \begin{array}{l} \widehat{W} \\ X_{1,2,3} \\ T \\ K \\ (C_2) \end{array} \right\}$	$\left\{ \begin{array}{l} A \\ \widehat{B} \end{array} \right\}$

The definitions are over!



The Casas-Ibarra limit

Particular case: Type-I seesaw

$$m = -\frac{v^2}{2} y^T M_R^{-1} y$$

The Casas-Ibarra limit

Particular case: Type-I seesaw

$$m = -\frac{v^2}{2} y^T M_R^{-1} y$$



$$\left. \begin{aligned} f &= -1 \\ n_1 = n_2 = n &= 3 \\ r_m = r &= 3 \\ y_1 = y_2 &= y/\sqrt{2} \\ M &= \frac{v^2}{2} M_R^{-1} \end{aligned} \right\}$$

The Casas-Ibarra limit

Particular case: Type-I seesaw

$$m = -\frac{v^2}{2} y^T M_R^{-1} y$$



$$\left. \begin{aligned} f &= -1 \\ n_1 = n_2 = n &= 3 \\ r_m = r &= 3 \\ y_1 = y_2 &= y/\sqrt{2} \\ M &= \frac{v^2}{2} M_R^{-1} \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} V_1 = V_2 = V & \text{ (= } \mathbb{I} \text{ in mass basis)} \\ X_{1,2,3}, \bar{W} \text{ and } \bar{B} & \text{ are absent} \\ W^T W A = B & \Rightarrow B = (A^T)^{-1} \\ & \Rightarrow R = W A \\ & \text{orthogonal } 3 \times 3 \\ R^T R = R R^T &= \mathbb{I} \end{aligned} \right.$$

The Casas-Ibarra limit

Particular case: Type-I seesaw

$$m = -\frac{v^2}{2} y^T M_R^{-1} y$$

⇓

$$\left. \begin{aligned} f &= -1 \\ n_1 &= n_2 = n = 3 \\ r_m &= r = 3 \\ y_1 &= y_2 = y/\sqrt{2} \\ M &= \frac{v^2}{2} M_R^{-1} \end{aligned} \right\}$$

⇒

$$\left\{ \begin{aligned} V_1 &= V_2 = V \quad (= \mathbb{I} \text{ in mass basis}) \\ X_{1,2,3}, \bar{W} \text{ and } \bar{B} &\text{ are absent} \\ W^T W A &= B \Rightarrow B = (A^T)^{-1} \\ &\Rightarrow R = W A \\ &\text{orthogonal } 3 \times 3 \\ R^T R &= R R^T = \mathbb{I} \end{aligned} \right.$$

Casas-Ibarra parametrization [Casas, Ibarra, 2001]

$$y = i \Sigma^{-1/2} R D_{\sqrt{m}} U^\dagger$$

⇑



An application

BNT model



	generations	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
Φ	1	1	4	3/2
$\psi_{L,R}$	3	1	3	-1

$$\Phi = \begin{pmatrix} \Phi^{+++} \\ \Phi^{++} \\ \Phi^{+} \\ \Phi^0 \end{pmatrix}$$

$$-\mathcal{L} \supset y_\psi \bar{L} H \psi_R + y_{\bar{\psi}} \bar{L}^c \Phi \psi_L + M_\psi \bar{\psi} \psi$$

An example of $y_1 \neq y_2$

$$\psi_{L,R} = \begin{pmatrix} \psi^0 \\ \psi^- \\ \psi^{--} \end{pmatrix}_{L,R}$$

$$\mathcal{V} \supset \lambda_\Phi H^3 \Phi$$

 \Rightarrow

$$\langle \Phi^0 \rangle = \frac{\lambda_\Phi v^3}{2\sqrt{2}M_\Phi^2}$$

 \Rightarrow

$$\Delta L = 2$$

Lepton number violation

[Babu, Nandi, Tavartkiladze, 2009]

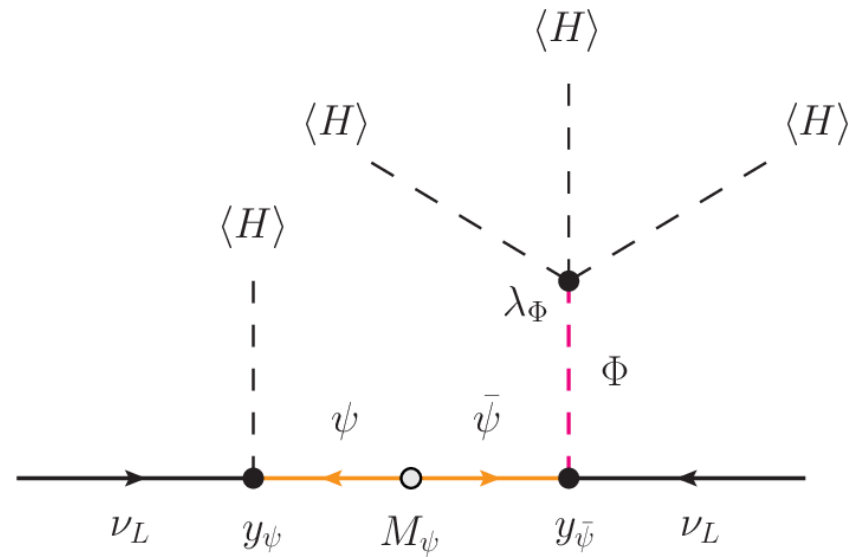
BNT model



$$m = f (y_1^T M y_2 + y_2^T M^T y_1)$$

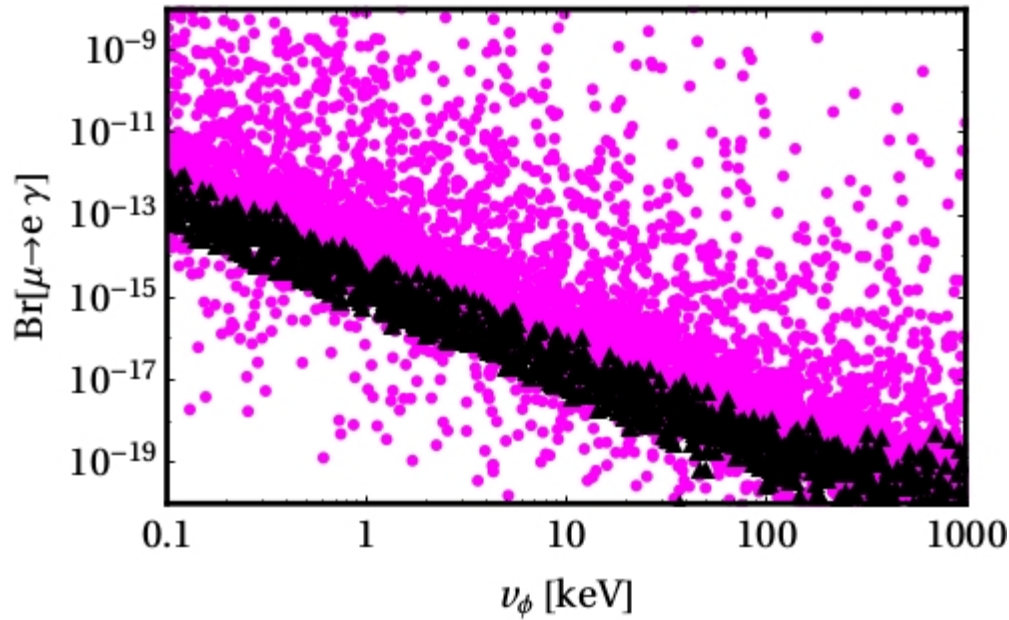
$$m = \frac{\lambda_\Phi v^4}{4 M_\Phi^2} \left[y_\psi^T M_\psi^{-1} y_{\bar{\psi}} + y_{\bar{\psi}}^T (M_\psi^{-1})^T y_\psi \right]$$

$$\left. \begin{aligned} f &= \frac{\lambda_\Phi v^2}{2 M_\Phi^2} \\ n_1 &= n_2 = 3 \\ y_1 &= y_\psi \neq y_2 = y_{\bar{\psi}} \\ M &= \frac{v^2}{2} M_\psi^{-1} \end{aligned} \right\} \Rightarrow$$



[Babu, Nandi, Tavartkiladze, 2009]

ν -data NH within 3σ $M_\psi \in [0.5, 2]$ TeV
 $W = \mathbb{I}$

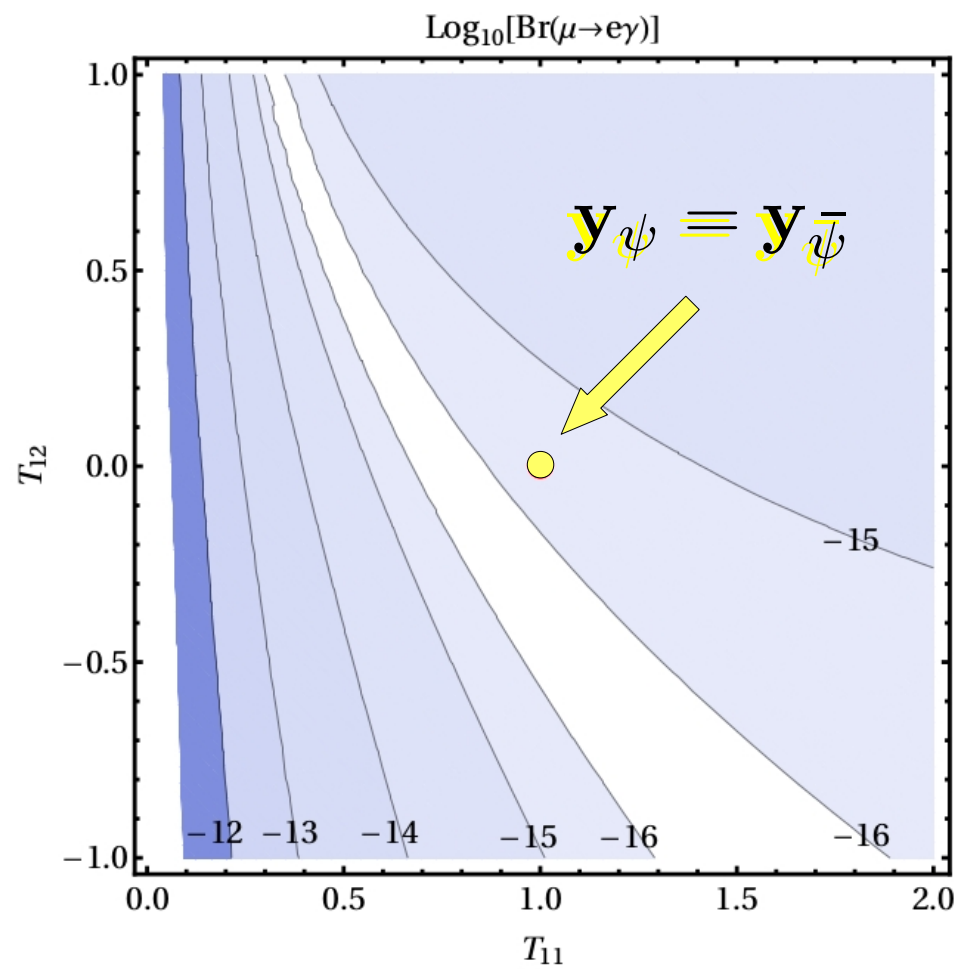


Black: Trivial scan [$T = \mathbb{I}$ & $K = 0$]

Purple: General scan

A large parameter space that can only be covered with the **master parametrization**

ν -data NH BFP $v_\Phi = 10^{-5}$ GeV
 $M_\psi = 0.5$ TeV



Model in **SARAH** and scans with **SSP** [Staub]
 BR computed with **FlavorKit** [Porod, Staub, AV]

Final discussion

Final discussion

The master parametrization allows one to explore the parameter space of any Majorana neutrino mass model **in a complete way**

Potential **limitation**: models with Yukawas that have additional restrictions

Easy to **program**: parameter space exploration easier than ever

The master parametrization may also provide **analytical insight** on some scenarios

To the organizers
of Portoroz 19!

Cheers!



Apologies to
those whose
favorite beer was
not mentioned

No, Heineken
does not qualify
as "beer"

Backup slides

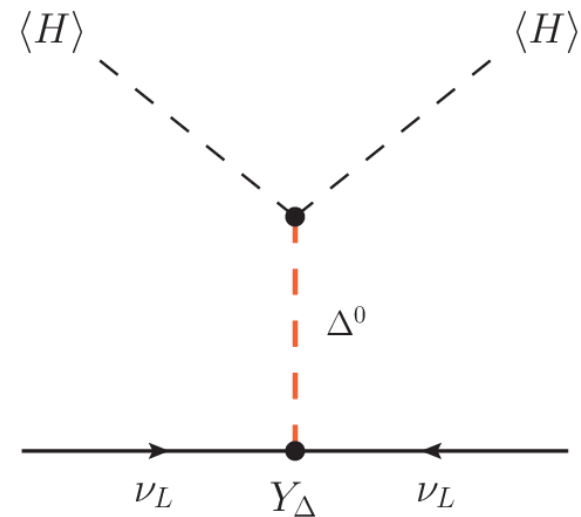
Type-II seesaw



$$m = f (y_1^T M y_2 + y_2^T M^T y_1)$$

$$\left. \begin{aligned} f &= 1 \\ n_1 &= n_2 = 3 \\ y_1 &= y_2 = \mathbb{I}/\sqrt{2} \\ M &= Y_\Delta v_\Delta \end{aligned} \right\} \Rightarrow$$

$$m = Y_\Delta v_\Delta$$



[Schechter, Valle, 1982]

Cases (I)

$$(r_m, r) = (3, 3) \quad C_1 = \mathbb{I}_3 \quad C_2 = \mathbb{I}_3 + K_{12} \frac{T_{13}}{T_{11}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$(r_m, r) = (3, 2)$
 1st & 2nd columns
 of $WA \dots$

↗ l.i. $C_1 = \begin{pmatrix} 1 & 0 & z_1 \\ 0 & 1 & z_2 \end{pmatrix} \quad C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
 with $1 + z_1^2 + z_2^2 = 0$

↘ l.d. $C_1 = \begin{pmatrix} 1 & \pm i & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$(r_m, r) = (2, 3) \quad C_1 = \mathbb{I}_3 \quad C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + K_{12} \frac{T_{13}}{T_{11}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Cases (II)

$(r_m, r) = (2, 2)$
1st & 2nd columns
of $WA \dots$

l.i.

$$C_1 = \begin{pmatrix} 1 & 0 & z_1 \\ 0 & 1 & z_2 \end{pmatrix} \quad C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with $z_1^2 + z_2^2 = 0$

l.d.

$$C_1 = \begin{pmatrix} 1 & \pm i & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$(r_m, r) = (2, 1)$

$$C_1 = \begin{pmatrix} 1 & \pm i & 0 \end{pmatrix} \quad C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Parameter counting

$$\#_{\text{free}} = \#_{y_1} + \#_{y_2} - \#_{\text{eqs}} - \#_{\text{extra}}$$

$$\#_{y_1} = 6n_1 \quad \#_{\text{eqs}} = \begin{cases} 12 & \text{for } r = 3 \text{ or } 2 \\ 10 & \text{for } r = 1 \end{cases}$$

$$\#_{y_2} = 6n_2 \quad \#_{\text{extra}} : \text{any extra condition on } y_{1,2}$$

$$\#_{\text{free}} = \#_{X_1} + \#_{X_2} + \#_{X_3} + \#_T + \#_W + \#_K + \#_{\bar{B}} + \#_{C_1}$$

$$\#_{X_1} = 6(n_2 - n) \quad \#_K = r(r - 1)$$

$$\#_{X_2} = 6(n_1 - n_2) \quad \#_{\bar{B}} = 6(n - r)$$

$$\#_{X_3} = 6(n_2 - n) \quad \#_W = r(2n - r)$$

$$\#_T = r^2 \quad \#_{C_1} = 0 \text{ or } 2$$

Parameter counting

Scenario	n_1	n_2	n	case	$\#_{\text{eqs}}$	$\#_{\text{extra}}$	$\#_{\text{free}}$	$\#_{X_1}$	$\#_{X_2}$	$\#_{X_3}$	$\#_T$	$\#_W$	$\#_K$	$\#_{\bar{B}}$	$\#_{C_1}$
1	3	3	3	(3, 3)	12	0	24	-	-	-	9	9	6	-	-
2	4	3	2	(3, 3)	12	0	42	6	6	6	9	9	6	-	-
3	3	3	3	(3, 2) _a	12	2	22	-	-	-	4	8	2	6	2
4	2	2	2	(3, 2) _a	12	0	12	-	-	-	4	4	2	-	2
5	3	3	3	(3, 2) _b	12	4	20	-	-	-	4	8	2	6	-
6	2	2	2	(2, 2) _a	12	0	12	-	-	-	4	4	2	-	2
7	2	2	2	(2, 2) _b	12	2	10	-	-	-	4	4	2	-	-
8	2	2	2	(2, 1)	10	4	10	-	-	-	1	3	-	6	-

↑
 (r_m, r)

↑
Absent if
 $n_2 = n$

↑
Absent if
 $n_1 = n_2$

↑
Absent if
 $n = r$

The general parameter counting can be easily adapted to any model

The master parametrization

Matrix	Dimensions	Property	Real parameters
X_1	$(n_2 - n) \times 3$	Absent if $n = n_2$	$6(n_2 - n)$
X_2	$(n_1 - n_2) \times 3$	Absent if $n_1 = n_2$	$6(n_1 - n_2)$
X_3	$(n_2 - n) \times 3$	Absent if $n = n_2$	$6(n_2 - n)$
W	$n \times r$		$r(2n - r)$
T	$r \times r$	Upper triangular with $(T)_{ii} > 0$	r^2
K	$r \times r$	Antisymmetric	$r(r - 1)$
\bar{B}	$(n - r) \times 3$	Absent if $n = r$	$6(n - r)$
C_1	$r \times 3$	Case-dependent	0 or 2
C_2	3×3	Case-dependent	-

$$\#_{\text{free}} = \#_{X_1} + \#_{X_2} + \#_{X_3} + \#_T + \#_W + \#_K + \#\bar{B} + \#_{C_1}$$

A philosophical moment

Occam's razor:

The simplest explanation is the correct one

Occam's laser:

The most awesome explanation is the correct one

Occam's hammer:

My explanation is the correct one

All credit goes to
Alberto Aparici