Quantum mechanics with magnetic backgrounds with manifest symmetry and locality

Joseph Tooby-Smith

University of Cambridge

Based on: /hep-th/1905.11999 with Ben Gripaios and Joe Davighi

"As you progress in this PhD you're solving increasingly simple problems using increasingly complicated techniques"

Current project

$$\int \frac{1}{2}\dot{x}^2 dt$$

Current project

$$\int \frac{1}{2}\dot{x}^2 dt$$

Spoiler alert, answer:

$$e^{ikx}, \quad E_k = \frac{1}{2}k^2$$

This talk

$$\int (\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - By\dot{x})dt$$

This talk

$$\int (\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - By\dot{x})dt$$

Want to solve this in a generalisable way!

This talk

$$\int (\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - By\dot{x})dt$$

Want to solve this in a generalisable way!
Symmetry under $(x, y) \to (x + x', y + y')$

$$\int (\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - By\dot{x} - By'\dot{x})dt$$

How to solve a really simple QM problem using an overly complicated technique

Joseph Tooby-Smith

University of Cambridge

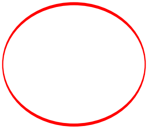
Based on: /hep-th/1905.11999 with Ben Gripaios and Joe Davighi

Step 1: Add redundant degree of freedom

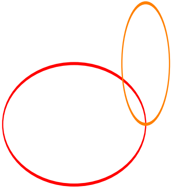
$$\int (\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 \underbrace{-\dot{s} - By\dot{x}}_{A})dt$$

Magnetic field (P, A) we consider the dynamics on P.

Mathematical interlude: Principal bundle



1) Take your manifold (e.g. a S^1).



2) At each point add a copy of $U(1)$.

3) Join to form a manifold.

Step 2: A new symmetry

$$\int (\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \dot{s} - By\dot{x})dt$$

Is strictly invariant under the Heisenberg group[1]

$$[(x', y', s')] \cdot [(x, y, s)] = [(x + x', y + y', s + s' - By'x)]. \qquad (1)$$

a $U(1)$-central extension of the group of translations, $\mathbb{R}^2$.

[1] Tuynman, G.M., Wiegerinck, W.A.J.J., 1987. Central extensions and physics. Journal of Geometry and Physics 4, 207–258. https://doi.org/10.1016/0393-0440(87)90027-1

Mathematical interlude: Central Extensions

$$0 \longrightarrow U(1) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 0$$

where $U(1)$ is in the centre of $\tilde{G}$.

Step 3: Formulating QM on this new system

$$\int (\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \dot{s} - By\dot{x})dt$$

has the (total) Hamiltonian

$$\hat{H} = \frac{1}{2}\left(-i\frac{\partial}{\partial x} + By\right)^2 - \frac{1}{2}\frac{\partial}{\partial y^2} + v(t)\left(-i\frac{\partial}{\partial s} + 1\right)$$

Step 3: Formulating QM on this new system

$$\int (\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \dot{s} - By\dot{x})dt$$

has the (total) Hamiltonian

$$\hat{H} = \frac{1}{2}\left(-i\frac{\partial}{\partial x} + By\right)^2 - \frac{1}{2}\frac{\partial}{\partial y^2} + v(t)\left(-i\frac{\partial}{\partial s} + 1\right)$$

acting on the Hilbert space

$$\mathcal{H} = \{\psi \in L^2(\mathrm{Hb}) \mid (-i\partial_s + 1)\,\psi = 0\}$$

Step 4: Decompose wavefunction into irreps

$\mathcal{H}$ has a Left-regular rep

$$\rho(g) \circ \psi(p) = \psi(g^{-1} p)$$

Step 4: Decompose wavefunction into irreps

$\mathcal{H}$ has a Left-regular rep

$$\rho(g) \circ \psi(p) = \psi(g^{-1}p)$$

Decomposition of a rep $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$. In our case

$$\psi(x, y, s) = \int drdt \frac{|B|}{2\pi} \left\{ e^{iB(xr - s/B)} \delta(r + y - t) \right\} f(r, t)$$

Step 5: Apply to the Schrödinger equation

Since

$$\hat{H}\rho(g) = \rho(g)\hat{H}$$

we get the simplification of the SE

$$\left(\frac{1}{2}B^2t^2 - \frac{1}{2}\frac{\partial^2}{\partial t^2} - E\right)f(r,t) = 0.$$

Step 6: Solve and remove redundancy

$$\Psi_n(x, y, s) = \frac{|B|}{2\pi} \int drdt \; H_n\left(\sqrt{|B|}t\right) e^{-|B|t^2/2} g(r) e^{i(Bxr-s)} \delta(r + y - t)$$

with energies

$$E_n = \frac{|B|}{2}(2n + 1)$$

Advantages

1. Gauge independent
2. Generalizable
3. Solves two problems:
    3.1 Lagrangian shifting by a total derivative under the symmetry group
    3.2 Not being able to write a lagrangian down

Disadvantages

1. Need to know the group representations
2. Need to know how to do decomposition

Other Examples

1. Rigid body
2. Dirac Monopole
3. Dyon
4. Landau levels using Euclidian group
5. Heisenberg group
6. Trapped particle

Thanks for Listening!