Measurement of phase space density evolution in MICE Step IV

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Experimental apparatus at present (Step IV)

All the detectors are installed and working
- Three time-of-flight (TOF) detector stations
- Two Cherenkov counters and a downstream calorimetry module
- Two scintillating-fibre trackers

Part of the cooling channel (no RF yet)
- Two Spectrometer Solenoids (SS): 5 coils each
  - The two downstream match coils are currently not turned on
- An Absorber Focus Coil (AFC): 2 coils + absorber
Beam cooling setting optimization

Suitable optics have been found using two approaches to conjointly optimize transmission and cooling performance:

- Linear optics, scan in the parameter space of magnet currents;
- Genetic algorithm, best sets of optics bear the next generation, penalize transmission loss and encourage emittance reduction.

The **bottom lattice** setting is expected to have one of the best cooling performance – transmission trade-off and is **presented in the following**
Reproduction of optical functions in the simulation

Reliable reproduction of the data in the simulation for

- $\sim 6$ mm input beam
- 140 MeV/$c$ central momentum
- -0.68 central $\alpha_\perp$
- 787 mm central $\beta_\perp$

→ Excellent tool to systematically test novel density methods
Particle selection

Series of cuts applied to both data and simulation:

- Muon tagging using TOF01 (*Particle ID*)
- Upstream reference plane hit (*Reference*)
- Good track reconstruction quality (*Quality*)
- Track within the tracker fiducial (*Aperture*)
- Longitudinal momentum $\in [135, 145]$ MeV/$c$ (*Momentum*)

All
Transverse normalised RMS emittance

4D normalised RMS emittance:

\[ \epsilon_n = \frac{1}{m} |\Sigma|^{\frac{1}{4}}, \quad (1) \]

with \(|\Sigma|\) the determinant of the covariance matrix defined as

\[ \Sigma = \begin{pmatrix}
\sigma_{xx} & \sigma_{xp_x} & \sigma_{xy} & \sigma_{xp_y} \\
\sigma_{p_x x} & \sigma_{p_x p_x} & \sigma_{p_x y} & \sigma_{p_x p_y} \\
\sigma_{yx} & \sigma_{yp_x} & \sigma_{yy} & \sigma_{yp_y} \\
\sigma_{p_y x} & \sigma_{p_y p_x} & \sigma_{p_y y} & \sigma_{p_y p_y}
\end{pmatrix}. \quad (2) \]

The RMS emittance is directly related to the volume of the RMS ellipsoid through \( \epsilon_n = \sqrt{2V_{RMS}}/(m\pi) \) and as such is the most common probe of average phase space density:

\[ \rho_{RMS} = \frac{N}{V_{RMS}} = \frac{N}{\frac{1}{2} m^2 \pi^2 \epsilon_n^2} = \frac{N}{\frac{1}{2} \pi^2 |\Sigma|^{\frac{1}{2}}} \quad [\text{mm}^{-2} (\text{MeV}/c)^{-2}]. \quad (3) \]

→ It follows from Liouville’s theorem that the phase space volume should be conserved
Emittance evolution

Simulation [stat]
Relative change between reference planes: -20.83%

MICE [simulation]
ISIS Cycle 2016/04
Run setting 1.2_6mm
MAUS v2.9.1

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Power of density estimation

The emittance plot exhibits two obvious challenges:
- transmission losses yield apparent emittance reduction;
- filamentation in SSD yield apparent emittance growth.

The key to solving both problems lies in density estimation:
- Estimate density in the transverse 4D phase space (center figure);
- Select an identical fraction of the beam upstream and downstream from within the densest area of the space (right figure);
- Define cooling figure of merits on these subsamples;

→ The core, unlike the tails, is transmitted and linear.
Transverse single-particle amplitude

Single particle amplitude is defined as

\[ A_\perp = \epsilon_n u^T \Sigma^{-1} u \]  \hspace{1cm} (4)

with \( u = v - \mu \), the centered phase space vector, \( v = (x, p_x, y, p_y) \), of the particle.

- It is related to the volume of an ellipse, which is similar to the RMS ellipse, going through \( v \).
- Amplitude follows a \( \chi^2 \) distribution with \( d \) degrees of freedom

Particle amplitude provides a density estimate in every input point

\[ \rho(v_i) = \frac{1}{(2\pi)^2 |\Sigma|^{1/2}} \exp \left[ -u^T \Sigma^{-1} u / 2 \right] = \frac{1}{4\pi^2 m^2 \epsilon_n^2} \exp \left[ -\frac{A_\perp}{2\epsilon_n} \right]. \]  \hspace{1cm} (5)

\[ \rightarrow \text{Allows for the selection of a high density core} \]
Amplitude reconstruction

In the case of non-linear beams, special care must be taken in the reconstruction of amplitude as tails significantly bias the covariance matrix.

Optimal procedure for amplitude reconstruction:

- Compute $\Sigma$ and $\mu$ for the whole sample;
- 1. Calculate all the particle amplitudes $A^i_\perp$;
- 2. Register the highest amplitude in the distribution;
- 3. Update $\Sigma$ and $\mu$ by removing the highest amplitude point;
- 4. Iterate from 1.

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**Test Gaussian + outliers**

**Regular amplitudes (biased)**

**Corrected amplitudes**
Amplitudes at TKD station 5
Amplitude distribution evolution

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Subemittance definition and properties

The $\alpha$-subemittance, $e_\alpha$, is defined as the emittance of the core fraction $\alpha$ of the parent beam. For a truncated 4D Gaussian beam of covariance matrix $S$, it satisfies

$$\frac{e_\alpha}{\epsilon_n} = \frac{1}{|\Sigma|^{1/4}} = \frac{1}{2\alpha} \gamma \left( 3, \frac{R^2}{2} \right),$$

$$R^2 = Q_{\chi^2}^2(\alpha).$$

$$\rightarrow \frac{e_{\alpha}^{\text{out}} - e_{\alpha}^{\text{in}}}{e_{\alpha}^{\text{in}}} = \frac{\epsilon_{\text{out}}^{\text{in}} - \epsilon_{\text{out}}^{\text{in}}}{\epsilon_{\text{out}}^{\text{in}}}. \quad (7)$$

The statistical uncertainty carried by this measurement is identical to that of the emittance, scaled by the fraction $\alpha$ as

$$\frac{\sigma e_\alpha}{e_\alpha} = \frac{1}{\sqrt{\alpha}} \frac{\sigma \epsilon_n}{\epsilon_n} = \sqrt{\frac{2}{\alpha N d}}. \quad (8)$$
Subemittance evolution

Simulation [stat]
Relative change between reference planes: -7.54 %

MICE [simulation]
ISIS Cycle 2016/04
Run setting 1.2_6mm
MAUS v2.9.1

François Drielsma (UniGe)
The $\alpha$-fractional emittance, $\epsilon_\alpha$, is defined as the phase space volume occupied by the core fraction $\alpha$ of the parent beam. For a truncated 4D Gaussian beam of covariance matrix $S$, it satisfies

$$\epsilon_\alpha = \frac{1}{2} m^2 \pi^2 \epsilon^2_n R^4 = V_{\text{RMS}} R^4,$$

$$R^2 = Q \chi^2_4(\alpha).$$

\[ (9) \]

$$\epsilon_{\alpha}^{\text{out}} - \epsilon_{\alpha}^{\text{in}} \approx 2 \frac{\epsilon_{\alpha}^{\text{out}} - \epsilon_{\alpha}^{\text{in}}}{\epsilon_n^{\text{in}}}$$

\[ (10) \]

In 4D, a fraction $\alpha$ of 9% yields the volume of the RMS ellipsoid, $V_{\text{RMS}}$

The convex hull is a prime candidate for volume reconstruction. It computes the smallest volume that contains the core $\alpha N$ points.
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Toy analysis of fractional quantities

A toy analysis (Gaussian input beam, toy absorber) shows:

- The **same relative change** is seen in the RMS emittance and all of the fractional quantities, for any fraction.
- The change in fractional quantities exhibit the **same relation** with $\beta_\perp$ and the input emittance, $\epsilon_i$.
- The fractional quantities are **more robust** against losses and non-linearities as the tails do not influence their measurement.

→ Plots produced for a core 9% selection, i.e. size of the RMS ellipse.
Non-parametric density estimation: \(k\)NN

For a given point \(x\), find the \(k\) closest points in the input cloud. Find the distance \(R_k\) to the \(k^{th}\) point and compute the 4D local density estimate as

\[
\rho(x) = \frac{k}{V_k} = \frac{2k}{\pi^2 R_k^4},
\]

(11)

with \(V_k\) the volume of the 4-ball centred in \(x\) of radius \(R_k\).

- Rule of thumb choice of \(k = \sqrt{N}\) yields quasi-optimal results for a broad array of distributions.
- Right plot shows great agreement between Cauchy distribution (red) and estimation (blue).
Density estimation at TKD station 5

**MICE [simulation]**
ISIS Cycle 2016/04
Run setting 1.2_6mm
MAUS v2.9.1
Transverse phase space evolution

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Conclusions

Status of the amplitude-based analysis:

- Selecting the low amplitude core gets rid of apparent emittance reduction due to scraping and apparent emittance growth due to beam filamentation in the downstream section;
- A toy MC shows that the exact same behaviour is observed for the subemittance and fractional emittance as for the RMS definition;
- Method shows a clean cooling signal in a realistic MC.

Status of the non-parametric analysis:

- Systematic study well advanced, $k$NN robust in 4D, low error and bias for large samples with the rule-of-thumb $k$ selection;
- Method applied to the toy MC to study its behaviour, identical trend as with the amplitude-based fractional emittance;
- Method also shows cooling signal in a realistic MC.
Back-up slides
Convex hull volume uncertainty

Computing the volume of the convex hull of a core fraction $\alpha$ of a set of $N$ i.i.d. random $d$-Gaussian points yields

$$\frac{E[\hat{V} - V]}{V} \sim -(1 + C^{-\frac{d+1}{2}} \alpha N)^{-\frac{2}{d+1}},$$

$$\frac{\sigma_{\hat{V}}}{\hat{V}} \sim \sqrt{\frac{1 - \alpha}{\alpha N}} \exp \left[ Q_{\chi^2_d}(\alpha)/4 \right]$$

(12)

The factor $C$ is purely deterministic, albeit complex in nature...

$$C = \frac{d + 1}{2(d + 3)(d - 1)!} \Gamma \left( \frac{d + 3}{d + 1} + d \right) \left( \frac{2\pi}{B \left( \frac{1}{2}, \frac{d}{2} + 1 \right)} \right)^{\frac{2}{d+1}}$$

(13)
Density estimator Mean Integrated Squared Error

A critical characteristic of estimators is **consistency**. For large $N$, the estimator must converge to the true value that is estimated,

$$\lim_{N \to +\infty} \hat{\theta}_N = \theta.$$  \hspace{1cm} (14)

Deviation from the true estimated distribution can be quantified by computing the Mean Integrated Squared Error (MISE).
Contour volume reconstruction

In 4D, the most efficient way to estimate an arbitrary volume is to generate **random points** inside a 4-box that bounds the contour and to count the amount of points that are above the contour level.

This is a Binomial process which yields an uncertainty of

\[
\frac{\sigma_{\hat{V}}}{\hat{V}} = \sqrt{1 - \frac{\xi}{\xi N}}, \quad \xi = \frac{N_{\text{in}}}{N}.
\]

(15)
In addition to the counting error, which can be tamed by adding random points, there is a certain level of uncertainty related to the density estimation. The bias and variance satisfy

\[
\frac{E[\hat{V} - V]}{V} \propto (\alpha N)^{-1/\lambda},
\]

\[
\frac{\sigma_{\hat{V}}}{\hat{V}} \sim \sqrt{\frac{1 - \alpha \rho_0}{\alpha N \rho_\alpha}},
\]

which are distribution dependent.