Regions from the singularities of multi-loop multi-scale Feynman integrals

Ratan Sarkar

Centre for High Energy Physics,
Indian Institute of Science
Bangalore-560012, India

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With B. Ananthanarayan, Abhishek Pal, Sunethra Ramanan

Outlines

- Motivation
- The strategy of expansion by Regions
- Few key points
- Alpha Parametrization of Feynman Integrals
- Location of singularities in Alpha parameter space via Gröbner Basis
- Asymptotic solutions via Power Geometry (**Bruno, Bruno and Batkhin**)
- Algorithm : ASPIRE with examples
- Summary and future directions

Motivation

- Multi-scale multiloop Feynman integrals are very difficult to compute analytically.
- The difficulty lies in the presence of various scales of the problem.
- The strategy of expansion by Regions near threshold in terms of small parameter constructed out of the mass scales appear to be very promising in this context.
- One expands the integrand appropriately in various Regions before the explicit integration.
- Every integral that appear in the expansion has the form much simpler than the original one.

The Strategy of expansion by Regions

A given Feynman Integral F_Γ (corresponding to a graph Γ) can be written as,

$$F_\Gamma \sim \sum_\gamma M_\gamma F_\Gamma$$

STEPS :

- Identify the mass scales (large as well as small) in the given problem.
- Divide the loop integration domain into Regions, where each loop momentum can be taken to be the order of one of the scales in the given problem.
- In every region, a Taylor expansion is performed in the threshold parameter.
- After the expansion, integrate over the entire loop momentum in each region.
- The scaleless integrals are put to zero.
- The integrals need to be properly regularized when integrating over the entire domain. Since the expansions are only valid in a sub-domain, integrating over the entire domain requires one to introduce zero-bin subtractions.

M. Beneke, V.A. Smirnov, Nucl.Phys. B522 (1998) 321-344

Few Key points

- No mathematical proof of MoR yet!

- How one can get the complete set of the Regions for a given Feynman diagram for a given limit?

- Finding the set of Regions for a given Feynman integral can automatically be solved using geometric properties of certain polynomials. Automatic implementations in Mathematica have been done in the packages, Asy.m and Asy2.m.

  Alexey Pak, Alexander Smirnov, Eur.Phys.J. C71 (2011) 1626

  Bernd Jantzen, Alexander V. Smirnov, Vladimir A. Smirnov, Eur.Phys.J. C72 (2012) 2139

Few key points

- In scattering events, one deals with energy scales associated with momentum transfers, and considers also the particle productions, which necessarily implies the energy region at threshold.
- The pinched singularities of the integral correspond to the threshold of production of multi-particle states.
- We believe that the singularities of a given Feynman integral at the threshold for a given process can be utilized to extract the contributing Regions for that process.
- We present an approach, ASPIRE(Algebrogeometric analysis of Singular Polynomials for Identification of REgions) to find regions for a given integral using singularities, within the framework of Power geometry in order to make the existing formalism to be more robust.

Feynman Integral in Alpha parameter space

Consider a general Feynman integral,

$$I = \lim_{\epsilon \to 0+} \int \frac{d^d k_1 ......d^d k_l}{\sum_{r=1}^{n} [q_r^2 - m_r^2 + i\epsilon]}$$

$$= \lim_{\epsilon \to 0+} \int \frac{\prod_{j=1}^{l} d^d k_j \quad \prod_{i=1}^{n} d\alpha_i \quad \delta(\sum \alpha - 1)}{[\sum_{r=1}^{n} \alpha_r (q_r^2 - m_r^2) + i\epsilon]^n}$$

$$\propto \lim_{\epsilon \to 0+} \int \frac{\prod_{i=1}^{n} d\alpha_i \quad \delta(\sum \alpha - 1) \quad \mathcal{U}^{n-2l-2}}{[\mathcal{F} + i\epsilon]^{n-2l}}$$

Where $\mathcal{U}$ and $\mathcal{F}$ are the Symanzik polynomials of order l and (l+1) respectively.
Eden, Landshoff, Olive, Polkinghorne, The Analytic S-Matrix

LANDAU EQUATIONS :

$$\mathcal{F} = 0$$
$$\frac{\partial \mathcal{F}}{\partial \alpha_i} = 0$$

These equations can be combined and expressed using Gröbner Basis.

Definition: The Gröbner basis $\mathbb{G}$ of an ideal $\mathcal{I}$ over a polynomial ring $\mathcal{R}$ is the generating set of $\mathcal{I}$ with respect to some monomial ordering with the property that the leading term of any polynomial in $\mathcal{I}$ is divisible by the leading term of some element in $\mathbb{G}$.

Nice property of Gröbner Basis :
One of the most important properties of Gröbner basis of an ideal containing a set of algebraic varieties is that the zeros shared by the system of equations are also shared by the Gröbner basis elements.

Basic definitions :

Consider a finite sum $G = g(Q) = \sum g_R Q^R$
where $Q = (x, y, z) \in \mathbb{R}^3$ and $R = (r_1, r_2, r_3) \in \mathbb{R}^3, Q^R = x^{r_1} y^{r_2} z^{r_3}$ and $g_R = constant$.

- **Support of the Sum :**
  All the terms in the sum have the vector exponents forming a set $S(g) \in \mathbb{R}^3$.
- **Newton polytope :**
  The convex hull of $S(g)$ gives the Newton Polytope of the given sum $g(Q)$.
- **Truncated Polynomial :**
  Each surface of the Newton Polytope is made to correspond to a truncated poynomial $\hat{g}_j^d = \sum g_R Q^R$ over the generalized faces $\Gamma_j^{(d)}$ of different dimensions d and $R \in \Gamma_j^{(d)} \cap S(g)$. Here j labels the number of faces.
- **Normal Cone :**
  If $\mathbb{R}^3_*$ be the dual space to $\mathbb{R}^3$. The set of all points $P \in \mathbb{R}^3_*$ for which the scalar product$< R, P >$ achieves the maximum over all the points $R \in S(g)$ on the generalized face $\Gamma_j^{(d)}$ is called the normal cone $U_j^{(d)}$ of the generalized face $\Gamma_j^{(d)}$.

A. D. Bruno, A. B. Batkhin, Program Comput Soft (2012) 38: 57

Theorem

If curve

$$x = bt^{p_1}(1 + o(1))$$
$$y = ct^{p_2}(1 + o(1))$$
$$z = dt^{p_3}(1 + o(1))$$

where b,c,d, and p_i are constants, lie in the set g as $t \to \infty$ and vector $P = (p_1, p_2, p_3) \in U_j^{(d)}$, then the first approximation $x = bt^{p_1}, y = ct^{p_2}, z = dt^{p_3}$ of the curve satisfies the truncated equation $\hat{g}_j^d(Q) = 0$.

A. D. Bruno, A. B. Batkhin, Program Comput Soft (2012) 38: 57

Problem statement :

We want to study the set $g = \{Q : g(Q) = 0\}$ near a singular point Q^0, or a singular curve $\mathbb{S}$, or a singular surface $\mathcal{S}$.

The way to the asymptotic solution of a given polynomial :

- Make change of variables $Q(x, y, z) \to Q_1(x_1, y_1, z_1)$ such that the set of singularities gets mapped to the co-ordinate sub-space. By doing that, a singular point gets mapped to the origin, a singular curve to the co-ordinate axes, a singular surface to the coordinate planes.
- Compute $g(Q_1) = g_1$ and the support $S(g_1)$ of it. Find the Newton polytope Γ_{g_1}, and the two dimensional faces Γ_j^2 of the polytope.
- For each of the faces Γ_j^2, draw outward normal N_j.
- Use the above **Theorem** to obtain the desired asymptotic solutions, corresponding to each of the truncated faces.

1. Find $\mathcal{U}$ and $\mathcal{F}$ polynomials from a given Feynman integral
2. Find the Gröbner Basis Elements (GBE) of Landau Equations
3. Map each of the GBE to the coordinate axes or to origin, resulting certain transformations
4. Find $\mathcal{G} = \mathcal{U} + \mathcal{F}$ with each of the transformations
5. Find the support of $\mathcal{G}$ in each sector
6. Find the convex hull of these supports
7. Look for the Normal vector corresponding to each of the surfaces
8. Get the unique Regions

Two point function

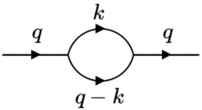

Figure: A two point one loop diagram

$$I(q^2, m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)((k-q)^2 - m^2)}$$

Where $x = m^2 - \frac{q^2}{4}$ is the small expansion parameter.

$$\textit{Hard region:} \ (k_0 \sim q, k \sim q)$$

$$\textit{Potential region:} \ \left(k_0 \sim \frac{x}{q}, k \sim \sqrt{x}\right), \left(k_0 \sim \sqrt{x}, k \sim \frac{x}{q}\right)$$

The Symanzik polynomials are,

$$\mathcal{U} = x_1 + x_2$$

$$\mathcal{F} = \frac{q^2}{4}(x_1 - x_2)^2 + x(x_1 + x_2)^2$$

Mathematica function $\mathcal{UF}$ has been used for computing $\mathcal{U}, \mathcal{F}$.

The Gröbner basis elements of $\left\{\mathcal{F}, \frac{\partial \mathcal{F}}{\partial x_1}, \frac{\partial \mathcal{F}}{\partial x_2}\right\}$ is

$$\{q^2 x x_2, x(x_1 + x_2), q^2(x_1 - x_2)\}$$

By mapping Gröbner basis elements to the origin, and/or to the co-ordinate axes, the obtained transformations are,

$$T_1 \equiv \{x_1 \to x_1, x_2 \to x_2\}$$
$$T_2 \equiv \left\{x_1 \to \frac{x_1}{2}, x_2 \to x_2 + \frac{x_1}{2}\right\}$$
$$T_3 \equiv \left\{x_1 \to x_1 + \frac{x_2}{2}, x_2 \to \frac{x_2}{2}\right\}$$

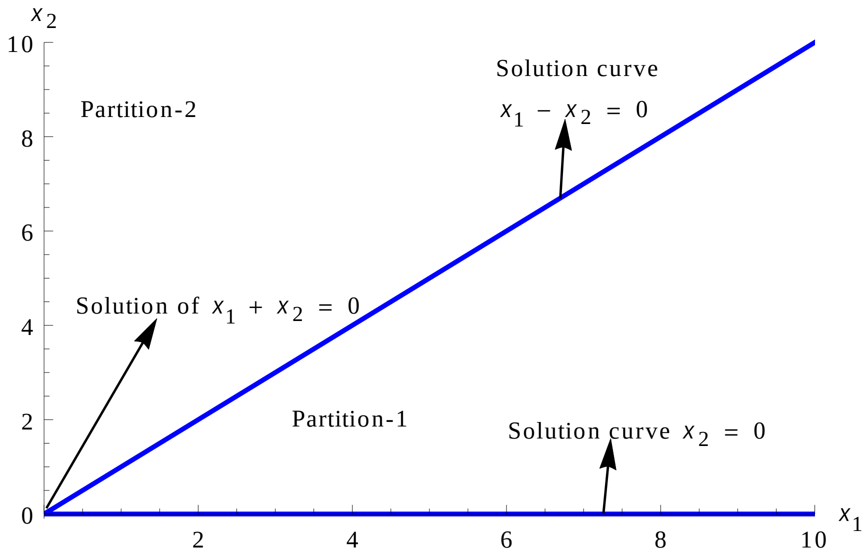

Figure: Partitioning of Alpha parameter space by solution curves of the Gröbner basis elements

With the above transformations, $\mathcal{G} = \mathcal{U} + \mathcal{F}$ are obtained,

$$\mathcal{G}_1 \equiv \frac{1}{4}q^2\,x_1^2 - \frac{1}{2}q^2\,x_1\,x_2 + \frac{1}{4}q^2\,x_2^2 + x\,x_1^2 + 2\,x\,x_1\,x_2 + x_1 + x\,x_2^2 + x_2,$$

$$\mathcal{G}_2 \equiv \frac{1}{4}q^2\,x_1^2 + x\,x_1^2 + 2\,x\,x_1\,x_2 + x_1 + x\,x_2^2 + x_2,$$

$$\mathcal{G}_3 \equiv \frac{1}{4}q^2\,x_2^2 + x\,x_1^2 + 2\,x\,x_1\,x_2 + x_1 + x\,x_2^2 + x_2$$

We have written Mathematica function **getMul** to get the support of a polynomial.

The Supports of $\mathcal{G}'$s are,

$$\mathbb{S}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 2 \end{pmatrix}, \mathbb{S}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \mathbb{S}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

We use the Mathematica function **CHNQuickHull** to find the convex hull of the supports (Newton polytope). (**Loren Petrichs implementation downloadable at http://lpetrich.org/Science/)**

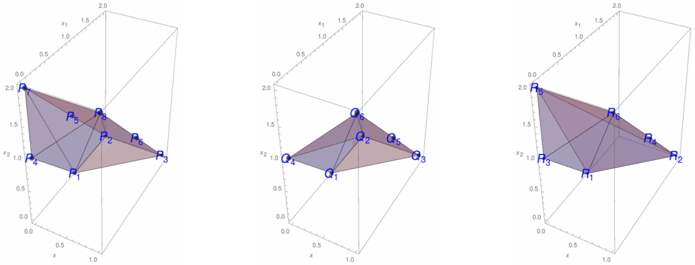

Figure: Newton polytopes of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$

We have written Mathematica functions, **getNormal** and **getCoordinatesNormal,** to find the components of the normal vector of the facets of the Newton polytope.

The normal vectors of Newton polytope of $\mathcal{G}_1$,

$$\left\{\begin{array}{c} \{v(1) \to 0, v(2) \to 0, c \to 0, \text{surf} \to -1\}, \\ \text{Null}, \\ \{v(1) \to -1, v(2) \to -1, c \to -1, \text{surf} \to 1\} \\ \{v(1) \to 0, v(2) \to 0, c \to 0, \text{surf} \to -1\} \\ \{v(1) \to -1, v(2) \to -1, c \to -1, \text{surf} \to 1\} \\ \text{Null} \\ \text{Null} \\ \text{Null} \end{array}\right\}$$

$\{0, 0\} \to$ **Hard region!**

The normal vectors of Newton polytope of $\mathcal{G}_2$,

$$\left\{\begin{array}{c} \{v(1)\to 0, v(2)\to 0, c\to 0, \text{surf}\to -1\} \\ \text{Null} \\ \{v(1)\to -1, v(2)\to -1, c\to -1, \text{surf}\to 1\} \\ \{v(1)\to -1, v(2)\to -1, c\to -1, \text{surf}\to 1\} \\ \text{Null} \\ \{v(1)\to -\frac{1}{2}, v(2)\to -1, c\to -1, \text{surf}\to -1\} \end{array}\right\}$$

$\{0,0\} \to$ **Hard region!**

$\{-\frac{1}{2}, -1\} \to$ **Potential region!**

The normal vectors of Newton polytope of $\mathcal{G}_3$,

$$\left\{\begin{array}{c} \{v(1)\to -1, v(2)\to -\frac{1}{2}, c\to -1, \text{surf}\to -1\} \\ \{v(1)\to -1, v(2)\to -1, c\to -1, \text{surf}\to 1\} \\ \{v(1)\to 0, v(2)\to 0, c\to 0, \text{surf}\to -1\} \\ \{v(1)\to -1, v(2)\to -1, c\to -1, \text{surf}\to 1\} \\ \text{Null} \\ \text{Null} \end{array}\right\}$$

$\{0, 0\} \to$ **Hard region!**

$\{-1, -\frac{1}{2}\} \to$ **Potential region!**

We have written mathematica function **UniqueRegion** for obtaining the unique normal vectors.

The contributing Regions for the given integral :

$$\begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & -1 \\ -1 & -\frac{1}{2} \end{pmatrix}$$

Summary and future directions

- ASPIRE presents an effective algorithm, which identifies all the Regions, needed for the asymptotic expansion of a given Feynman integral in a given limit, by exploiting the singularities of the integral via Power Geometry.
- Potential and Glauber Regions have been isolated within our formalism using transformations.
- The issue of “top facet” scaling, as we pointed out in for the Newton polytope, is a subject of future investigation.
- The analysis of non-planer diagrams shows a rich family of very complicated Gröbner basis elements, which need to be treated carefully. We take this treatment as a work of future direction.

Thanks a lot!