QCD Masterclass 2019: Jets and their structure

Simone Marzani

June 17, 2019

Homework (with solutions)

Homework 1. *Which of the following observables are IRC safe (assuming the jet has been selected in an IRC safe fashion)? The jet invariant mass, the invariant mass of tracks in a jet, generalised angularities.*
Solution. The caveat has to do with the fact that, if we want to measure an internal property of a jet (e.g. its mass) we must make sure that the jet selection is not ambiguous in perturbation theory. For instance, the mass of the hardest jet is ambiguous because LO-type configurations are back-to-back. Instead the mass of the hardest jet in Z+jet events, is fine.

- The jet invariant mass is IRC safe. Let us consider a collection of massless particles in a jet

$$m^2 = (p_1 + p_2 + \dots p_i + p_{i+1} \dots)^2 \tag{1}$$

  If p_i is soft, its contribution to the mass vanishes. If p_i and p_{i+1} go collinear, we only have to prove that if $p_{i,i+1} = p_i + p_j$ then $p^2_{i,i+1} \to 0$. This is easy to show because

$$p^2_{i,i+1} = (p_i + p_j)^2 = 2p_i \cdot p_j = 2E_1 E_2 (1 - \cos\theta_{12}). \tag{2}$$

- Track observables are measured on electrically charged particles (the ones that leave a signature in the tracking system). Thus, when performing a calculation we have to impose different restrictions on real-emission and virtual diagrams, leading to a mis-cancellation of IR singularities and hence IRC unsafety.

- Let us define $z_i = \frac{p_{Ti}}{p_T}$, then

$$\lambda_{\kappa,\beta} = \sum_{i \in jet} z_i^{\kappa} \theta_i^{\beta}, \tag{3}$$

  where θ_i is measured with respect to the jet axis. The condition $\kappa > 0$ ensures that soft emissions, i.e. the ones with $z_i \to 0$ give vanishing contributions to the observable. Collinear safety is more subtle because the condition $\beta > 0$ is not sufficient. Let us consider the case where emissions i

and j become collinear. This means that $\theta_{ij} \to 0$, i.e. $\theta_i \simeq \theta_j$. The generalised angularity then becomes

$$\lambda_{\kappa,\beta} = z_1^\kappa \theta_1^\beta + \dots z_i^\kappa \theta_i^\beta + z_j^\kappa \theta_j^\beta + \dots z_n^\kappa \theta_n^\beta = z_1^\kappa \theta_1^\beta + \dots \left(z_i^\kappa + z_j^\kappa\right) \theta_i^\beta + \dots z_n^\kappa \theta_n^\beta \tag{4}$$

Thus, only if $\kappa = 1$ we can write

$$\lambda_{\kappa,\beta} = z_1 \theta_1^\beta + \dots (z_i + z_j)\, \theta_i^\beta + \dots z_n \theta_n^\beta, \tag{5}$$

which implies collinear safety. We conclude that generalised angularities are safe for $\kappa = 1$ and $\beta > 0$.

Homework 2. *Show that for an event made up of two particles all gen. kt algorithms recombine them is their azimuth-rapidity distance is less than R.*
Solution. Let us consider a jet made up for two particles with momenta p_1 and p_2. Then the generalised k_t distance is

$$\begin{aligned} d_{12} &= \min(p_{T1}^p, p_{T2}^p) \frac{\Delta R_{12}}{R} \\ d_1 &= p_{T1}^p, \quad d_2 = p_{T2}^p. \end{aligned} \tag{6}$$

The two particles are recombined if their mutual distance d_{12} is the minimum distance, i.e. $d_{12} < d_1$ and $d_{12} < d_2$. For Cambridge/Aachen ($p = 0$), this immediately implies $\Delta R_{12} < R$. For the k_t algorithm ($p > 0$) we have to consider two cases.

a) If $p_{T1} > p_{T2}$ then, 1 and 2 are recombined if

$$\begin{aligned} d_{12} &= p_{T2}^p \frac{\Delta R_{12}}{R} < p_{T1}^p \\ d_{12} &= p_{T2}^p \frac{\Delta R_{12}}{R} < p_{T2}^p \end{aligned} \tag{7}$$

and the second inequality implies $\Delta R_{12} < R$.

b) Analogously, when $p_{T2} > p_{T1}$ particles 1 and 2 are recombined if

$$\begin{aligned} d_{12} &= p_{T1}^p \frac{\Delta R_{12}}{R} < p_{T1}^p \\ d_{12} &= p_{T1}^p \frac{\Delta R_{12}}{R} < p_{T2}^p \end{aligned} \tag{8}$$

from which again we have $\Delta R_{12} < R$.

Finally, the proof for anti-k_t (when $p < 0$) proceeds in the same was as for k_t but with cases a) and b) swapped.

Homework 3. *Consider the emission of a soft gluon with momentum k off a fermion line of momentum p mass m and derive the corresponding eikonal factor.*

Solution. Using standard notation for the QCD Feynman rules, we have

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_r \frac{i(\not{p} - \not{k} + m)}{(p-k)^2 - m^2 + i\epsilon} \left(-ig_s t^a_{ij}\gamma^\mu\right) u(p)\varepsilon^*_\mu(k) \\
&= \mathcal{M}_r \frac{i[\gamma^\mu(-\not{p} + \not{k} + m) + 2(p^\mu - k^\mu)]}{-2p\cdot k + i\epsilon} \left(-ig_s t^a_{ij}\right) u(p)\varepsilon^*_\mu(k) \\
&= \mathcal{M}_r \frac{i[\gamma^\mu\not{k} + 2(p^\mu - k^\mu)]}{-2p\cdot k + i\epsilon} \left(-ig_s t^a_{ij}\right) u(p)\varepsilon^*_\mu(k),
\end{aligned} \tag{9}$$

where in the last step we have used the Dirac equation $\not{p}u(p) = mu(p)$. We now take the soft limit $k^\mu \ll p^\mu$ and obtain

$$\mathcal{M} = \mathcal{M}_r u(p) \frac{p^\mu}{-p\cdot k + i\epsilon} g_s t^a_{ij}\varepsilon^*_\mu(k), \tag{10}$$

Homework 4. The bulk of the $O(R^2)$ contribution to the jet mass spectrum arises from the initial-state radiation . Calculate the contribution to the jet mass from the dipole which is formed by the initial-state partons [hint: it's easier to work with rapidity and azimuth].
Solution. Following the hint, we describe the kinematics with

$$\begin{aligned}
&p_1 = \frac{\sqrt{s}}{2}x_1\,(1,0,0,1)\,, \quad p_2 = \frac{\sqrt{s}}{2}x_2\,(1,0,0,-1)\,, \\
&p_3 = p_t\,(\cosh y, 1, 0, \sinh y)\,, \quad k = k_t\,(\cosh\eta, \cos\phi, \sin\phi, \sinh\eta)\,,
\end{aligned} \tag{11}$$

where p_1 and p_2 denote the four-momenta of the incoming hard partons, p_3 the momentum of the jet, and k of the soft gluon. It is understood that the jet must recoil against a system with momentum p_4 (not specified above), over which we are inclusive.

Provided the soft gluon is clustered with the jet, its contribution to the jet mass is

$$m^2 = (p_3 + k)^2 = 2p_3\cdot k = 2p_t k_t\,(\cosh(\eta - y) - \cos\phi)\,. \tag{12}$$

We can now add together real and virtual pieces and write the contribution to the cumulative distribution from the 12 as

$$\begin{aligned}
\alpha_s\Sigma^{(1)}_{12} = -C_{12}\int k_t dk_t d\eta \frac{d\phi}{2\pi}\frac{\alpha_s\,(\kappa_{12})}{2\pi}\frac{(p_1.p_2)}{(p_1.k)(p_2.k)}\Theta\left((\eta - y)^2 + \phi^2 < R^2\right) \\
\cdot\,\Theta\left(\frac{2k_t}{p_t R^2}\,(\cosh(\eta - y) - \cos\phi) > \rho\right),
\end{aligned} \tag{13}$$

where the first Θ function is the jet clustering condition and we have introduced $\rho = \frac{m^2}{p_t^2 R^2}$. We next note that

$$\kappa^2_{12} = 2\frac{(p_1.k)(p_2.k)}{(p_1.p_2)} = k_t^2. \tag{14}$$

Eq. (13) therefore exhibits a logarithmic enhancement at small k_t as expected. To isolate the leading (NLL) contribution, we can as usual just retain the dependence of the jet mass on k_t in the second line of (13), and neglect the dependence on y, η and ϕ which produces terms beyond NLL accuracy. We can then carry out the integration over η and ϕ which simply measures the jet area πR^2 and obtain

$$\alpha_s \Sigma_{12}^{(1)} = -C_{12} R^2 \int_{\rho p_t}^{p_t} \frac{\alpha_s(k_t)}{2\pi} \frac{dk_t}{k_t}, \tag{15}$$

where the lower limit of integration stems from the constraint on the jet mass.

Homework 5. *The QCD splitting $g \to b\bar{b}$ is an important background for $H \to b\bar{b}$. What's the average mass of the QCD splitting ? (Assume $m_b = 0$).*

Solution. If we assume the b-quark to be massless, then the collinear branching $g \to b\bar{b}$ is described by the splitting function

$$P_{qg}(z) = T_R \left[z^2 + (1-z)^2\right]. \tag{16}$$

Thus, following what was done in the lecture,

$$\begin{aligned}
\langle m^2 \rangle &= \frac{\alpha_s T_R}{2\pi} p_T^2 \int_0^{R^2} \frac{d\theta^2}{\theta^2} \int_0^1 dz \, \left[z^2 + (1-z)^2\right] z\,(1-z)\,\theta^2 \\
&= \frac{\alpha_s T_R}{2\pi} p_T^2 R^2 \int_0^1 dz \, \left[z^2 + (1-z)^2\right] z\,(1-z) = \frac{1}{20} \frac{\alpha_s T_R}{\pi} p_T^2 R^2.
\end{aligned} \tag{17}$$

Homework 6.1. *Compute the jet mass distribution at $\mathcal{O}(\alpha_s)$ for a jet tagged by mMDT. You can work, as in the lecture, in the collinear limit, neglecting power corrections in the jet radius. You can also work in the limit of small z_{cut}.* What is the role of the mass-drop parameter μ at this order?

Solution.

In a leading-order configuration the jet consists of just two partons. When the jet is declustered, each of the prongs is massless, so that the mass-drop condition is automatically satisfied, rendering the μ parameter irrelevant. There are then two possibilities: if the asymmetry condition is satisfied the jet is tagged, with the tagged mass equal to the original jet mass. Otherwise the jet does not contribute to the tagged jet mass distribution. We may write the differential cross section for the jet to have a given tagged mass as

$$\begin{aligned}
\frac{1}{\sigma}\frac{d\sigma}{dm^2}^{(\text{mMDT, LO})} &= \frac{\alpha_s}{2\pi} \int dz P_{gq}(z) \frac{d\theta^2}{\theta^2}\, \delta\big(m^2 - z(1-z) p_T^2 \theta^2\big) \times \\
&\qquad \times \Theta\left(z - z_{\text{cut}}\right) \Theta\left(1 - z - z_{\text{cut}}\right) \Theta\left(R^2 - \theta^2\right).
\end{aligned} \tag{18}$$

Proceeding as the previous LO calculations, we obtain

$$\frac{\rho}{\sigma}\frac{d\sigma}{d\rho}^{(\text{mMDT, LO})} = \frac{\alpha_s C_F}{\pi} \left[\Theta(\rho - z_{\text{cut}}) \ln\frac{1}{\rho} + \Theta(z_{\text{cut}} - \rho) \ln\frac{1}{z_{\text{cut}}} - \frac{3}{4} + \mathcal{O}(\rho, z_{\text{cut}})\right]. \tag{19}$$

The result is characterised by two regimes: it is linear in $\ln\frac{1}{\rho}$ when $\rho > z_{\rm cut}$ (i.e. "double" logs), and saturates at a constant value ($\ln\frac{1}{z_{\rm cut}} - \frac{3}{4}$) for $\rho < z_{\rm cut}$ (i.e. "single" logs).

Homework 6.2. *Compute the pruned mass distribution at $\mathcal{O}(\alpha_s)$. You can work, as in the lecture, in the collinear limit, neglecting power corrections in the jet radius. You can also work in the limit of small z_{cut}. How does the result compare to mMDT at this order?*
Solution.

At leading order, i.e. a jet involving a single $1 \to 2$ splitting, $R_{\rm prune} = \frac{m}{p_T} = d_{ij}\sqrt{z(1-z)}$, which guarantees that $d_{ij} > R_{\rm prune}$. To establish the pruned jet mass, one then needs to examine the second part of the pruning condition: if $\min(z, 1-z) > z_{\rm cut}$ then the clustering is accepted and the pruned jet has a finite mass. Otherwise the pruned jet mass is zero. This pattern is true independently of the angle between the two prongs. This leads to the following result for the mass distribution:

$$\begin{aligned}
\frac{1}{\sigma}\frac{d\sigma}{dm^2}^{\text{(prune, LO)}} &= \frac{\alpha_s}{2\pi}\int dz\, P_{gq}(z)\,\frac{d\theta^2}{\theta^2}\,\delta\big(m^2 - z(1-z)p_T^2\theta^2\big)\times \qquad (20)\\
&\qquad\qquad \times\Theta\left(z - z_{\rm cut}\right)\Theta\left((1-z) - z_{\rm cut}\right)\Theta\left(R^2-\theta^2\right),\\
&= \frac{\alpha_s}{2\pi}\int dz\, P_{gq}(z)\,\frac{1}{m^2}\,\Theta\left(z - z_{\rm cut}\right)\Theta\left(z - \frac{m^2}{p_T^2 R^2}\right), \qquad (21)
\end{aligned}$$

where to obtain the last line we have made use of the fact that $z_{\rm cut}$ is small and that the integral is dominated by the region $z \ll 1$. The final z-integration is straightforward to perform and gives

$$\frac{\rho}{\sigma}\frac{d\sigma}{d\rho}^{\text{(prune, LO)}} = \frac{\alpha_s C_F}{\pi}\left[\Theta(\rho - z_{\rm cut})\ln\frac{1}{\rho} + \Theta(z_{\rm cut}-\rho)\ln\frac{1}{z_{\rm cut}} - \frac{3}{4} + \mathcal{O}(\rho, z_{\rm cut})\right]. \qquad (22)$$

This result is identical to what we found for mMDT, i.e. a rise linear in $\ln\rho$ for ρ down to $z_{\rm cut}$, and then it is constant below. However, this similarity only holds at leading order. Beyond that, the pruned mass distribution develops a more complicated structure.

Homework 7. *Compute the LL (fixed-coupling) expression for the Iterated Soft Drop ν for $\beta < 0$ and for $\beta > 0$, with an additional cutoff on the angular separation θ_{min}. Show that the ISD multiplicity exhibits the same scaling as the track multiplicity (while being IRC safe).*
Solution.

$$\nu = \frac{\alpha_s C_i}{\pi}\int_0^1 \frac{d\theta^2}{\theta^2}\int_0^1\frac{dz}{z}\Theta(z > z_{\rm cut}\theta^\beta) = \frac{\alpha_s C_i}{\pi}\int_0^1\frac{d\theta^2}{\theta^2}\left[\log\frac{1}{z_{\rm cut}} + \beta\log\frac{1}{\theta}\right]\Theta(z_{\rm cut}\theta^\beta < 1). \qquad (23)$$

If $\beta < 0$, the above expression is finite and we have

$$\nu = \frac{2\alpha_s C_i}{\pi}\int_{z_{\rm cut}^{\frac{1}{|\beta|}}}^1 \frac{d\theta}{\theta}\left[\log\frac{1}{z_{\rm cut}} - |\beta|\log\frac{1}{\theta}\right] = \frac{\alpha_s C_i}{\pi|\beta|}\left[\log^2\frac{1}{z_{\rm cut}}\right]. \qquad (24)$$

If instead $\beta > 0$, we have to introduce an angular cut-off:

$$\nu = \frac{2\alpha_s C_i}{\pi}\int_{\theta_{\rm cut}}^1\frac{d\theta}{\theta}\left[\log\frac{1}{z_{\rm cut}} + \beta\log\frac{1}{\theta}\right] = \frac{2\alpha_s C_i}{\pi}\left[\log\frac{1}{z_{\rm cut}}\log\frac{1}{\theta_{\rm cut}} + \frac{\beta}{2}\log^2\frac{1}{\theta_{\rm cut}}\right]. \qquad (25)$$

Given that the ISD multiplicity is Poisson-distributed, its mean value is given by ν. Hence at LL, we have

$$\frac{\langle n_{ISD}\rangle_g}{\langle n_{ISD}\rangle_q} = \frac{\nu_g}{\nu_q} = \frac{C_A}{C_F}, \tag{26}$$

which is the same relation we have found for the track multiplicities.

Homework 8. *Consider the momentum fraction z_g between the two prongs that pass soft drop. Compute the Sudakov-safe integral for $\beta > 0$and show that you obtain a non-analytic expansion in the square-root of α_s. Repeat the exercise for the IRC safe case $\beta < 0$ and show that its expansion is now analytic in α_s and it reproduce the first-order result discussed during the lecture.).*

Solution. Following the definition of Sudakov safety discussed in the lecture, we have to evaluate

$$\frac{1}{\sigma_0}\frac{d\sigma}{dz_g} = P_i(z_g)\frac{\alpha_s C_i}{\pi}\int_0^1 \frac{d\theta_g}{\theta_g}\exp\left[-\frac{\alpha_s C_i}{\pi\beta}\Big(\log^2(z_{\rm cut}\theta_g^\beta) - \log^2(z_{\rm cut})\Big)\right]\Theta(z_{\rm cut}\theta_g^\beta < z_g), \tag{27}$$

where the exponential is the Sudakov form factor for the companion variable θ_g, evaluated at leading-logarithmic accuracy, in the fixed-coupling approximation.

We first consider $\beta > 0$, the evaluation of Eq. (27) gives

$$\frac{1}{\sigma_0}\frac{d\sigma}{dz_g} \approx \sqrt{\frac{\alpha_s}{4\beta C_i}}e^{\frac{\alpha_s C_i}{\pi\beta}\log^2(z_{\rm cut})}\left[1 - \mathrm{erf}\left(\sqrt{\frac{\alpha_s C_i}{\pi\beta}}\log\left(\frac{1}{\min(z_g, z_{\rm cut})}\right)\right)\right]P_i(z_g). \tag{28}$$

Although at first sight this looks similar to what was previously obtained, Eq. (28) (for positive β) shows a significantly different behaviour compared to Eq. (30) for negative β, as a direct consequence of the fact that z_g is only Sudakov safe for $\beta > 0$. Indeed, for $\beta > 0$, the distribution has the expansion

$$\beta > 0: \quad \frac{1}{\sigma_0}\frac{d\sigma}{dz_g} = \sqrt{\frac{\alpha_s}{4\beta C_i}}\, P_i(z_g) + \mathcal{O}\left(\alpha_s\right), \tag{29}$$

and the presence of $\sqrt{\alpha_s}$ implies non-analytic dependence on α_s.

Let us consider the case $\beta < 0$ for which $z_g > z_{\rm cut}$ and we get

$$\begin{aligned}\frac{1}{\sigma_0}\frac{d\sigma}{dz_g} \approx & \sqrt{\frac{\alpha_s}{4|\beta| C_i}}e^{-\frac{\alpha_s C_i}{\pi|\beta|}\log^2(z_{\rm cut})}\\ & \left[\mathrm{erfi}\left(\sqrt{\frac{\alpha_s C_i}{\pi|\beta|}}\log\left(\frac{1}{z_{\rm cut}}\right)\right) - \mathrm{erfi}\left(\sqrt{\frac{\alpha_s C_i}{\pi|\beta|}}\log\left(\frac{1}{z_g}\right)\right)\right]P_i(z_g),\end{aligned} \tag{30}$$

where $\mathrm{erfi}(x) = -i\,\mathrm{erf}(ix)$ is the imaginary error function. For $\beta < 0$, z_g is an IRC-safe observable and, accordingly, the above result admits an expansion in powers of the strong coupling:

$$\beta < 0: \quad \frac{1}{\sigma_0}\frac{d\sigma}{dz_g} = \frac{\alpha_s}{\pi|\beta|}\, P_i(z_g)\log\left(\frac{z_g}{z_{\rm cut}}\right)\Theta(z_g - z_{\rm cut}) + \mathcal{O}(\alpha_s^2). \tag{31}$$