Low temperature condensation and scattering data

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C. Gattringer, M. Giuliani and O. Orasch, Phys. Rev. Lett. 120, 241601 (2018)

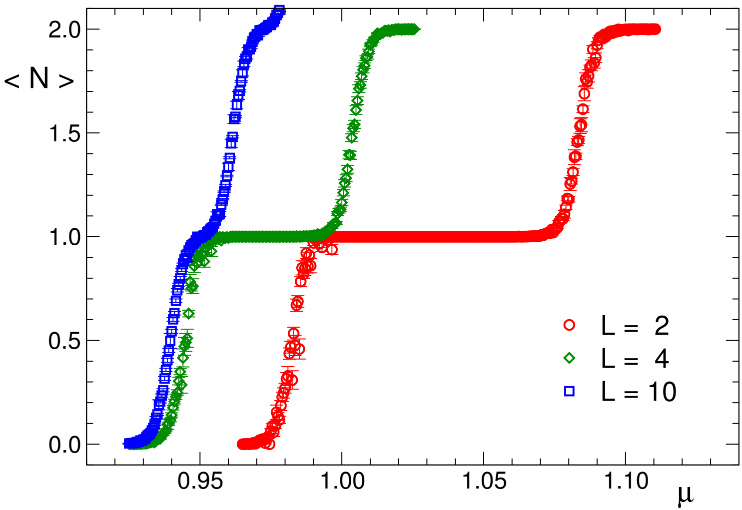
M. Giuliani, O. Orasch, C. Gattringer, EPJ Web of Confs. 175, 07007 (2018)

O. Orasch, C. Gattringer, M. Giuliani, PoS, Lattice 2018

Condensation at finite density in a QFT

Condensation thresholds

Expectation value $\langle N \rangle$ of the net particle number as a function of the chemical potential μ at very low temperature (charged scalar field):



- At critical values $\mu_n(L)$ one observes jumps from $\langle N \rangle = n-1$ to $\langle N \rangle = n$.
- The condensation thresholds $\mu_n(L)$ depend on the spatial extent L.

Connection of condensation thresholds and n-particle energies

- Grand canonical partition sum and grand potential:

$$Z \;=\; \text{Tr}\, e^{-\beta(\hat{H} - \mu \hat{N})} \;=\; e^{-\beta\,\Omega(\mu)}$$

- Low T: In each particle sector Z is governed by the minimal grand potential $\Omega(\mu)$

$$\Omega(\mu) \quad \xrightarrow{T\to 0} \quad \begin{cases} \Omega_{min}^{N=0} \;=\; 0\,, & \mu \;\in\; [0, \mu_1] \\ \Omega_{min}^{N=1} \;=\; m \,-\, 1\mu\,, & \mu \;\in\; [\mu_1, \mu_2] \\ \Omega_{min}^{N=2} \;=\; W_2 - 2\mu\,, & \mu \;\in\; [\mu_2, \mu_3] \\ \Omega_{min}^{N=3} \;=\; W_3 - 3\mu\,, & \mu \;\in\; [\mu_3, \mu_4] \\ \ldots & \end{cases}$$

- m: physical mass, W_2: minimal 2-particle energy, W_3: minimal 3-particle energy ...
- Use continuity of $\Omega(\mu)$ to relate the critical μ_n to m and the W_n.

F. Bruckmann, C. Gattringer, T. Kloiber, T. Sulejmanpasic, PRL 115, 231601 (2015)

Connection of condensation thresholds and n-particle energies

- Relations between the critical $\mu_n(L)$ and the minimal multi-particle energies:

$$m(L) = \mu_1(L) \,, \quad W_2(L) = \mu_1(L) + \mu_2(L) \,, \quad \ldots \quad W_n(L) = \sum_{k=1}^{n} \mu_k(L) \ldots$$

- The multi-particle energies are governed by low energy parameters.

- In particular their finite volume dependence can be related to scattering data.
  (K. Huang & C.N. Yang, M. Lüscher ....)

Goal: Describe condensation thresholds $\mu_n(L)$ with scattering data.

Challenge: Non-perturbative calculation at finite μ.

Lattice field theory and worldline techniques

Quantum field theory with path integrals

Lattice path integral for quantum field theory:

$$\langle \mathcal{O} \rangle \;=\; \frac{1}{Z} \int \mathcal{D}[\phi]\, e^{\,-\,S[\phi]}\, \mathcal{O}[\phi] \qquad , \qquad Z \;=\; \int \mathcal{D}[\phi]\, e^{\,-\,S[\phi]}$$

- Introduce a space-time lattice Λ, replace $\phi(x) \longrightarrow \phi_n,\; n \in \Lambda$ and discretize $S[\phi]$
- Integral over all field configurations is defined as $\int \mathcal{D}[\phi] \;\equiv\; \prod_n \int d\phi_n$

Monte Carlo simulations:

- Generate field configurations $\phi^{(j)}$, $j = 1, 2 ... N$ distributed with $P[\phi^{(j)}] \;\propto\; e^{-S[\phi^{(j)}]}$
- Quantum mechanical expectation values become averages

$$\langle \mathcal{O} \rangle \;=\; \frac{1}{N} \sum_{j=1}^{N} \mathcal{O}[\phi^{(j)}] \;+\; O\Big(1/\sqrt{N}\Big)$$

Sign problem (complex action problem)

Scalar field lattice action at finite chemical potential μ

$$S[\phi] = \sum_{n \in \Lambda} \left[(m_b^2 + 8) |\phi_n|^2 + \lambda |\phi_n|^4 - \sum_{\nu=1}^{3} \left[\phi_n^\star \phi_{n+\hat{\nu}} + \phi_{n+\hat{\nu}}^\star \phi_n \right] - \left[e^{-\mu} \phi_n^\star \phi_{n+\hat{4}} + e^{\mu} \phi_{n+\hat{4}}^\star \phi_n \right] \right]$$

The Boltzmann factor is complex and does not have a probability interpretation:

$$\frac{1}{Z} \, e^{-S[\phi]} \in \mathbb{C}$$

No direct Monte Carlo simulation! "Sign problem"

Idea: Solve the sign problem by transforming the path integral to new variables.

The idea of a worldline representation

- Lattice action: $\quad (\phi_n \in \mathbb{C} \, , \quad M^2 = m_b^2 + 8 \,)$

$$S = \sum_n \Big[M^2 \, |\phi_n|^2 \; + \lambda \, |\phi_n|^4 \, \Big] - \sum_{n,\nu} \Big[e^{-\mu \, \delta_{\nu,4}} \, \phi_n^\star \, \phi_{n+\widehat{\nu}} \; + \; e^{\mu \, \delta_{\nu,4}} \, \phi_x \, \phi_{n+\widehat{\nu}}^\star \Big]$$

- Expand the nearest neighbor terms of e^{-S}:

$$\prod_{n,\nu} \exp \left(e^{-\mu \, \delta_{\nu 4}} \, \phi_n^\star \, \phi_{n+\widehat{\nu}} \right) \; = \; \prod_{n,\nu} \sum_{j_{n,\nu}=0}^{\infty} \frac{\left(e^{-\mu \, \delta_{\nu 4}} \right)^{j_{n,\nu}}}{j_{n,\nu} \, !} \, \left(\phi_n^\star \; \phi_{n+\widehat{\nu}} \right)^{j_{n,\nu}}$$

- The integrals over the moments $\prod_{n,\nu} \left(\phi_n \, \phi_{n+\widehat{\nu}}^\star \right)^{j_{n,\nu}}$ can be solved.

- $j_{n,\nu}$ (and $\overline{j}_{n,\nu}$ for other NN-term) turn into the new worldline degrees of freedom.

Worldline representation - final form

- The original partition function is mapped exactly to a sum over configurations of the worldline variables $j_{n,\nu}, \overline{j}_{n,\nu} \in \mathbb{N}_0$ with $d_{n,\nu} \equiv j_{n,\nu} - \overline{j}_{n,\nu}$.

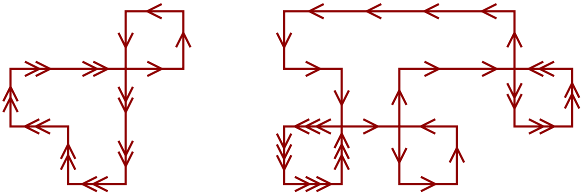
$$Z \;=\; \sum_{\{j,\overline{j}\}} \mathcal{W}[\,j,\overline{j}\,]\,\mathcal{C}[\,d\,]\,e^{\,\mu\,\beta\,\omega[d]}$$

- $\mathcal{W}[\,j,\overline{j}\,]$: Real and positive weight from radial d.o.f. and combinatorics.

- Constraints from integrating over the symmetry group ($d_{n,\nu} = j_{n,\nu} - \overline{j}_{n,\nu}$):

$$\mathcal{C}[\,d\,] \;=\; \prod_n \delta\left(\vec{\nabla}\vec{d}_n\right)$$

- Particle number N $\Leftrightarrow$ temporal winding number $\omega[d]$ of $d_{n,\nu}$-flux:



Condensation thresholds and scattering data

Determination of the critical values μ_n

- Using worldlines we compute $\langle N \rangle$ as function of μ.

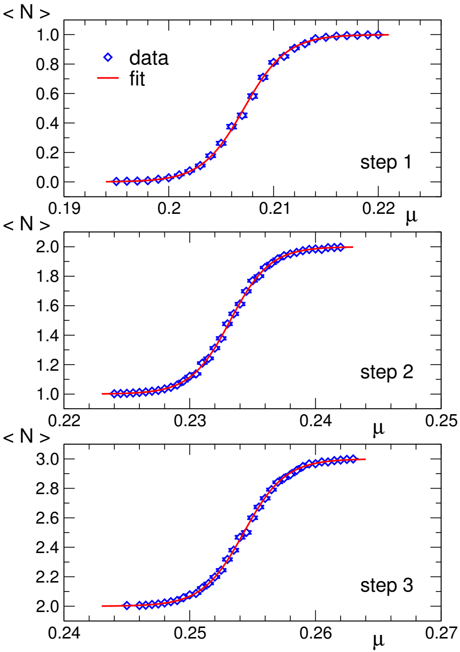
- Near the steps we fit $\langle N \rangle$ with a logistic function:

$$\langle N \rangle \sim \frac{1}{1 + e^{-a_n(\mu - \mu_n)}} + n - 1$$

- The μ_n are obtained as fit parameters.

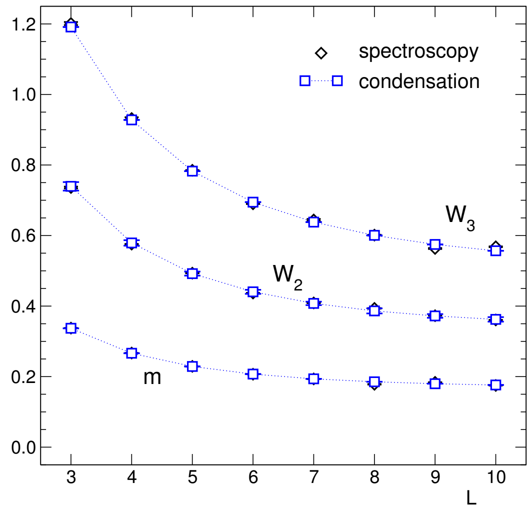
- From the $\mu_n(L)$ at different L we obtain

$$\begin{aligned} m(L) &= \mu_1(L) \\ W_2(L) &= \mu_1(L) + \mu_2(L) \\ W_3(L) &= \mu_1(L) + \mu_2(L) + \mu_3(L) \end{aligned}$$



Cross-check for the interpretation of the thresholds

We compute $m(L), W_2(L)$ and $W_3(L)$ also in a conventional spectroscopy calculation from 2-, 4-, and 6-point functions.



The good agreement confirms the interpretation of the $\mu_n(L)$ in terms of multi-particle energies, and their determination from the fits.

Finite volume analysis in 4d

Finite volume relations: $(\mathcal{I} = -8.914, \mathcal{J} = 16.532)$

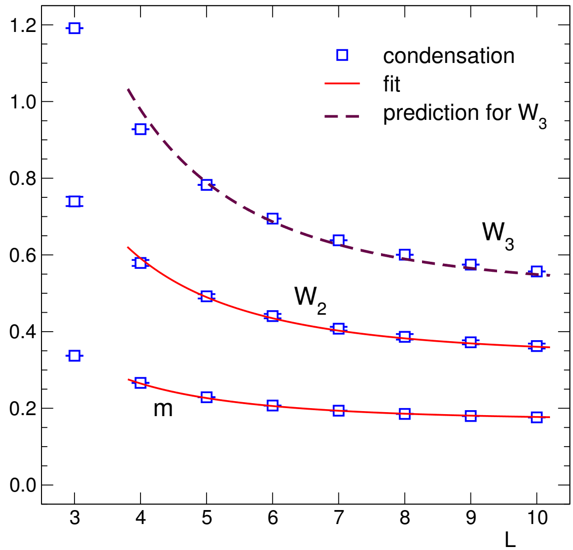
(K. Huang, C.N. Yang, M. Lüscher, S.R. Beane, W. Detmold, M.J. Savage, S.R. Sharpe, M.T. Hansen)

$$
\begin{aligned}
m(L) &= m_\infty + \frac{A}{L^{\frac{3}{2}}}\, e^{-L\, m_\infty} \,, \\
W_2(L) &= 2m + \frac{4\pi a}{mL^3}\left[1 - \frac{a}{L}\frac{\mathcal{I}}{\pi} + \left(\frac{a}{L}\right)^2 \frac{\mathcal{I}^2 - \mathcal{J}}{\pi^2} + \mathcal{O}\left(\frac{a}{L}\right)^3\right], \\
W_3(L) &= 3m + \frac{12\pi a}{mL^3}\left[1 - \frac{a}{L}\frac{\mathcal{I}}{\pi} + \left(\frac{a}{L}\right)^2 \frac{\mathcal{I}^2 + \mathcal{J}}{\pi^2} + \mathcal{O}\left(\frac{a}{L}\right)^3\right].
\end{aligned}
$$

- From fitting the mass data we obtain m_∞.
- In $W_2(L)$ we use $m(L)$ for m on the rhs. and obtain a as a fit parameter.
- In $W_3(L)$ we use $m(L)$ and a from $W_2(L)$ to "predict" the 3-particle data.

For our couplings: $m_\infty = 0.168(1)\ (l.u.),\ \ a = -0.078(7)\ (l.u.),\ \ am_\infty = -0.013(1)$

Comparison of threshold data with the finite volume relations



The good agreement shows that one can describe condensation thresholds with scattering data.

Finite volume analysis in 2d

- 2-particle wave function and energy at zero total momentum: $(p_1 = -p_2 \equiv p)$

$$\psi(r) \;=\; e^{-ipr} \;\;, \qquad W_2(L) \;=\; 2\sqrt{m(L)^2 + p(L)^2}$$

- Quantization condition from boundary condition:

$$2\,\delta(p) \;=\; -\,p\,L \qquad \Rightarrow \qquad \delta(p(L)) \;=\; -\,p(L)\,L/2 \;\equiv\; \delta(L)$$

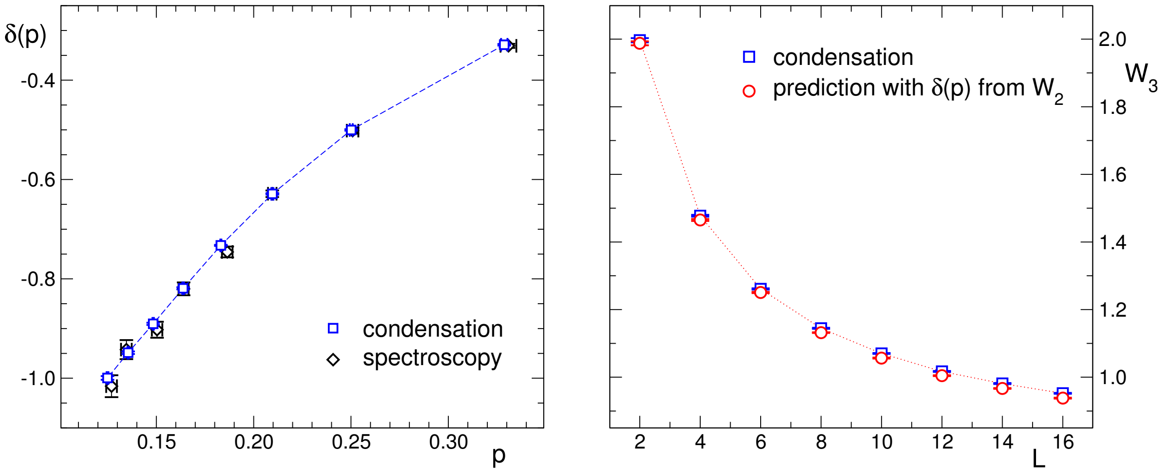
- 3-particle wave function and energy at zero total momentum: $(p_3 = -p_1 - p_2)$

$$\psi(r) \;=\; e^{-ip_1 r_1}\, e^{-ip_2 r_2} \;\;, \qquad W_3(L) \;=\; \sum_{j=1}^{3} \sqrt{m(L)^2 + p_j(L)^2}$$

- Two quantization conditions from boundary conditions:

$$2\,\delta(p_j) \;=\; -\,p_j\,L \qquad \Rightarrow \qquad p_j(L) \;=\; -\,2\,\delta(L)/L$$

Comparison of threshold data with the finite volume relations



The good agreement shows that one can describe condensation thresholds with scattering data.

Summary

- At low temperature the particle number develops condensation steps as function of μ.
- The critical values $\mu_n(L)$ are related to multi-particle energies.
  (cross-checked with spectroscopy)
- The multi-particle energies and thus the $\mu_n(L)$ depend on scattering parameters.
- We explored the relation between condensation thresholds and scattering data using worldline simulations of the ϕ^4 field in 2 and 4 dimensions.
- In 4d we computed the scattering length from $\mu_1(L)$ and $\mu_2(L)$ and used it in the relations for $W_3(L)$ to "predict" the 3-particle energy and thus $\mu_3(L)$.
- In 2d we computed the phase shift from $\mu_1(L)$ and $\mu_2(L)$ and used it in the relations for $W_3(L)$ to "predict" the 3-particle energy and thus $\mu_3(L)$.
- In both 2d and 4d we obtained a satisfactory understanding of the $\mu_n(L)$ from scattering data.
- Possible future work: Study QC_2D and QCD with isospin chemical potential.