Lattice QCD and Minimally Doubled Fermions

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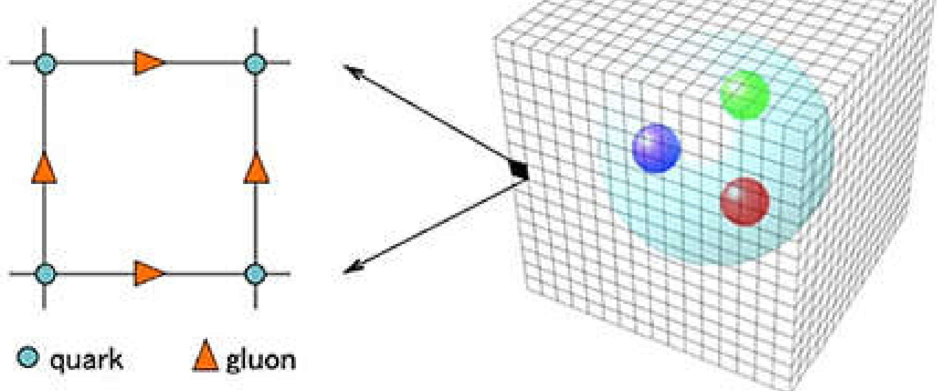
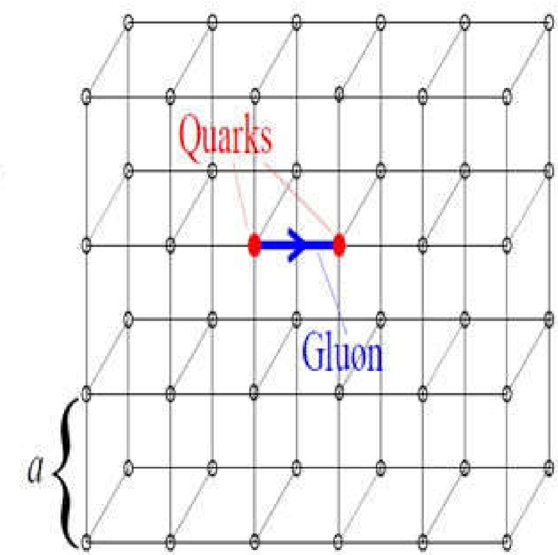
Lattice QCD

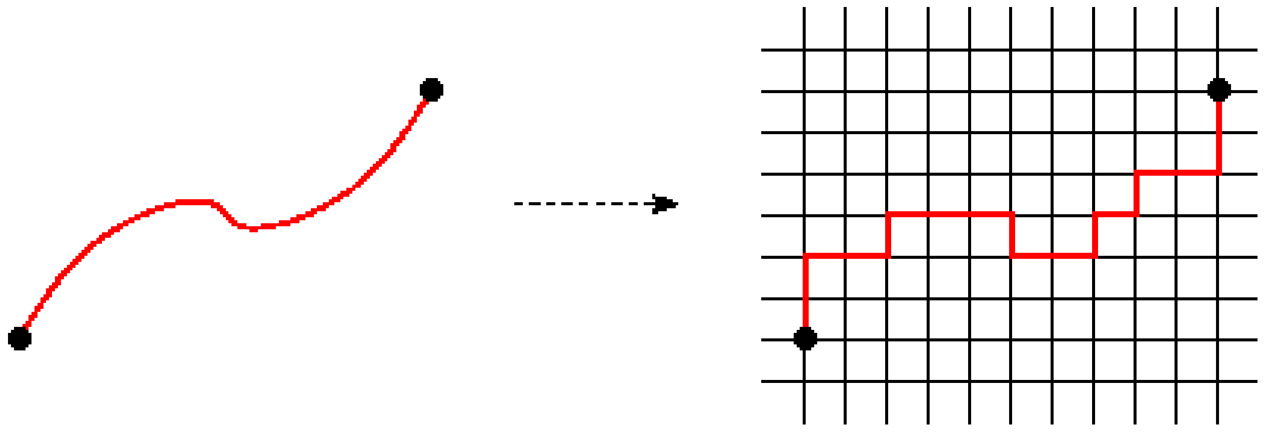
- Proposed by Wilson, 1974.
- Non - perturbative low - energy solution of QCD.
- The continuum spacetime discretized on 4D Euclidean space-time lattice.
- Quarks can only exist in lattice points and gluons are the links between them
- Solved by large scale numerical simulations on supercomputers.

QCD Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \sum_{q=u,d,s,c,b,t} \bar{q}\left[i\gamma^{\mu}(\partial_{\mu} - igA_{\mu}) - m_q\right]q$$







- From continuum to discretized lattice:

$$\int d^4x \rightarrow a^4 \sum_n$$

n - four-vector that labels the lattice site,

a - lattice constant

- Fermions and gauge action are discretized
- Take an appropriate continuum limit ($a \rightarrow 0$) to get back the continuum theory.

Fermions action discretization:

- Doubling problem
- If no doublers, chiral symmetry is broken

Nielsen – Ninomiya No – go theorem *(Nielsen, Ninomiya, 1981):*

"*A local, real, free fermion lattice action, having chiral and translational invariance, necessarily has fermion doubling* "

Doubling problem : obstacle to simulations

.

- Wilson (*Wilson K, 1974)* ⇨ Broken chiral symmetry
- DW or Overlap

  (Kaplan D.B (1992); Shamir Y(1993) Neuberger H, 1999) ⇨ Numerical expensive (Non exact locality)
- Staggered *(Kogut J, Susskind L, 1975)* ⇨ Rooting procedure (4 tastes)

Motivations for minimal doubling

(Karsten, Wilczek, Creutz, Boriçi, Bedaque Buchoff Tiburzi Walker-Loud)

- Failure of rooting for staggered
- Lack of chiral symmetry for Wilson
- Computational demands of overlap, domain-wall approach
- Nature has two light flavors

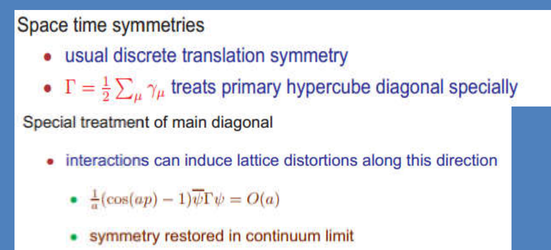
i) 2 flavors	←	4 in Staggered
ii) Exact chiral: *U(1)A⊂SU(2)*	←	Broken in Wilson
iii) Strict locality	←	Not strict in DW or Overlap

Symmetries

Translation, Gauge, $U(1)_B$, $\underline{U(1)_A} \subset SU(2)$

P.F.Bedaque, *et.al.*, PLB **662**, 449 (2008)

Common with all Minimal-doubling actions.



➤ <u>Discrete symmetries</u>

1. Karsten-Wilczek
2. Boriçi - Creutz
3. Twisted-Ordering

<u>*CT, P, Cubic,* Z_2</u>   <u>*CPT, S*$_4$*,* Z_2</u>   <u>*CPT, Z*$_4$*,* Z_2</u>

With gauge interaction, broken symmetry leads to anisotropy.

↓

Fine-tuning to cancel redundant operators is required.

Boriçi – Creutz action

Boriçi – Creutz fermionic action with the free Dirac operator (in the momentum space):

$$D(p) = \sum_{\mu} i\gamma_{\mu} \sin p_{\mu} + \sum_{\mu} i\gamma'_{\mu} \cos p_{\mu} - 2i\Gamma$$

This operator has zeros: $p_1 = (0,\ 0,\ 0,\ 0)$ dhe $p_2 = (\pi/2,\ \pi/2,\ \pi/2,\ \pi/2)$.

- There is always a special direction in euclidean space (given by the line that connects these two zeros: hypercubic diagonal)

↓

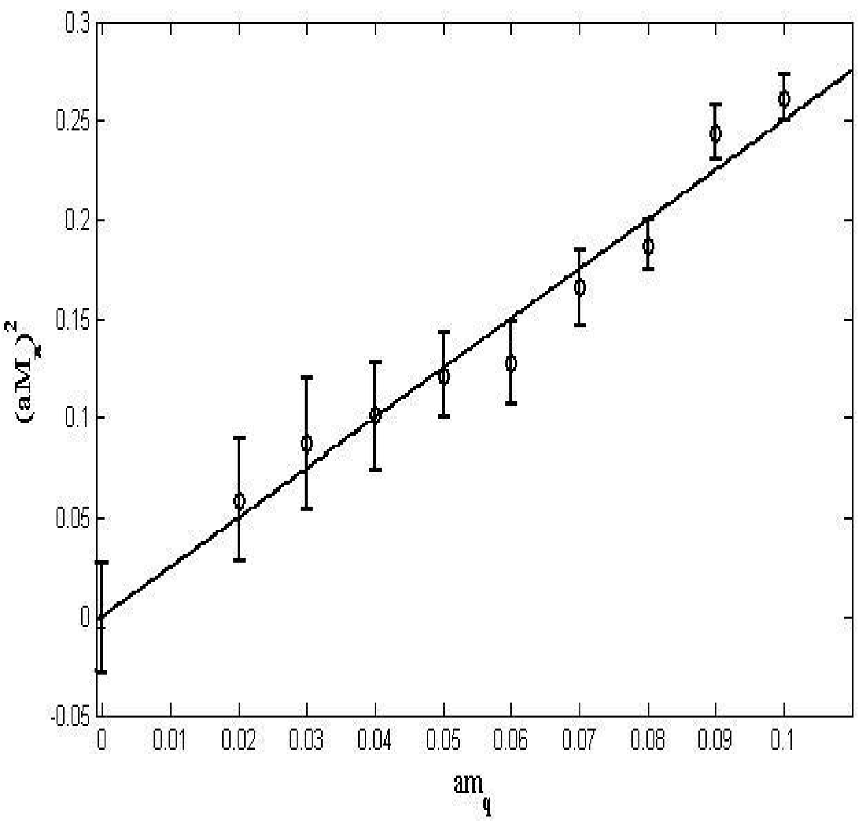
- Thus, these actions cannot maintain a full hypercubic symmetry *(P. F. Bedaque et al, 2008).*

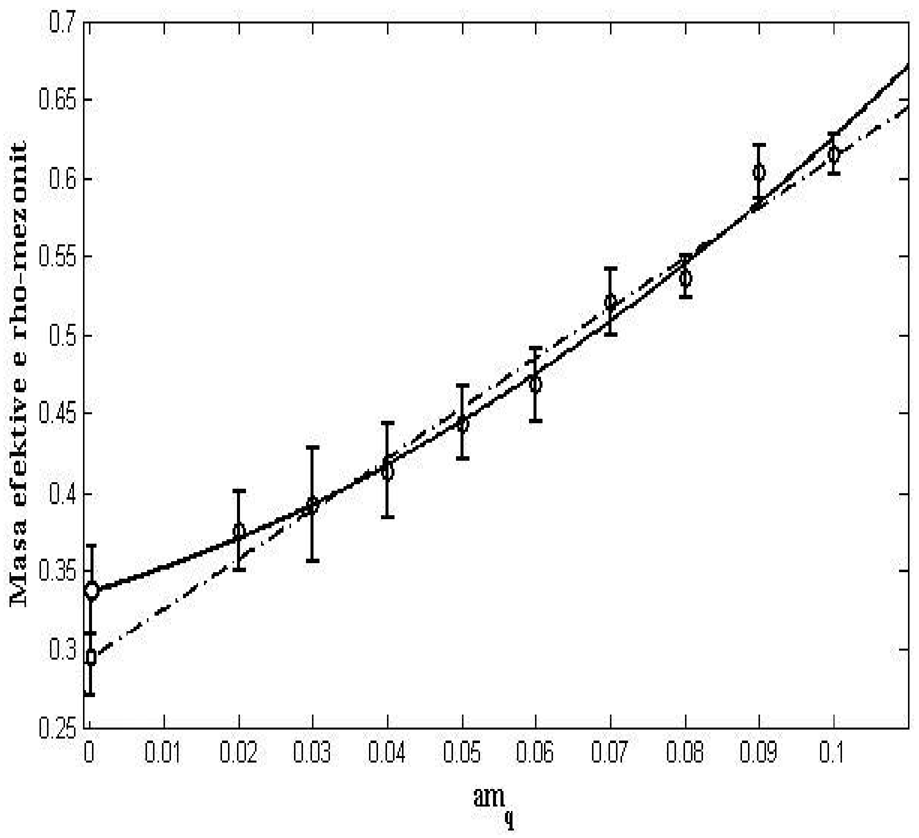
↓

Hypercubic symmetry has to be restored
(Perturbative calculations *Capitani et al, 2010*)

$$D_{BC}(p) = \sum_{\mu} [i\gamma_{\mu} \sin p_{\mu} + i(\Gamma - \gamma_{\mu}) \cos p_{\mu}] + i(c_3 - 2)\Gamma$$

Effects of the broken hypercubic symmetry





(0.33739 * 2 GeV) ± (0.0257 * 2 GeV) = 674.78 ± 51.4 MeV

M_eksp = 770 MeV

Evaluation of the broken hypercubic symmetry mass

$$\Delta(M^2_{\pi^+}) = (M^{(1,0,0,0)}_{\pi^+})^2 - (M^{(1,1,1,1)}_{\pi^+})^2$$

$$(M^{(1,0,0,0)}_{\pi^+})^2 = (M^{(1,1,1,1)}_{\pi^+})^2 = 0 \quad (M^2_{\pi^+} \xrightarrow[m_{q\to 0}]{} 0\ )$$

- Point splitting ($u(x)$, $d(x)$)
- Charged pion propogator calculation
- Calculation of the charged pion mass from two different directions

Non perturbative restaturation of broken hypercubic symmetry

Weber et al, 2013

1 –st Method :: $\Delta(M^2_{\pi^+}) = (M^{(1,0,0,0)}_{\pi^+})^2 - (M^{(1,1,1,1)}_{\pi^+})^2$

$$M^2_{\pi^+} \xrightarrow[m_{q\to 0}]{} 0$$

Minimisation of the anisotropy

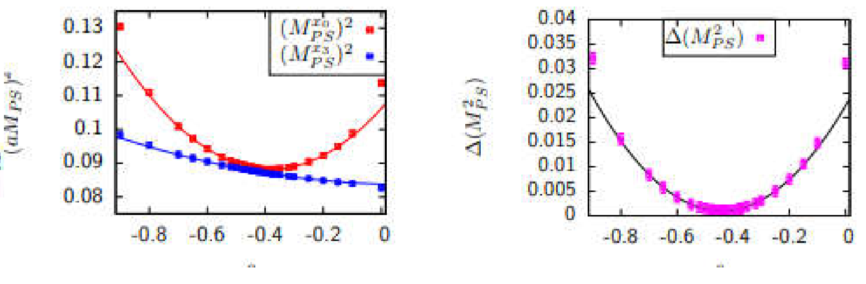
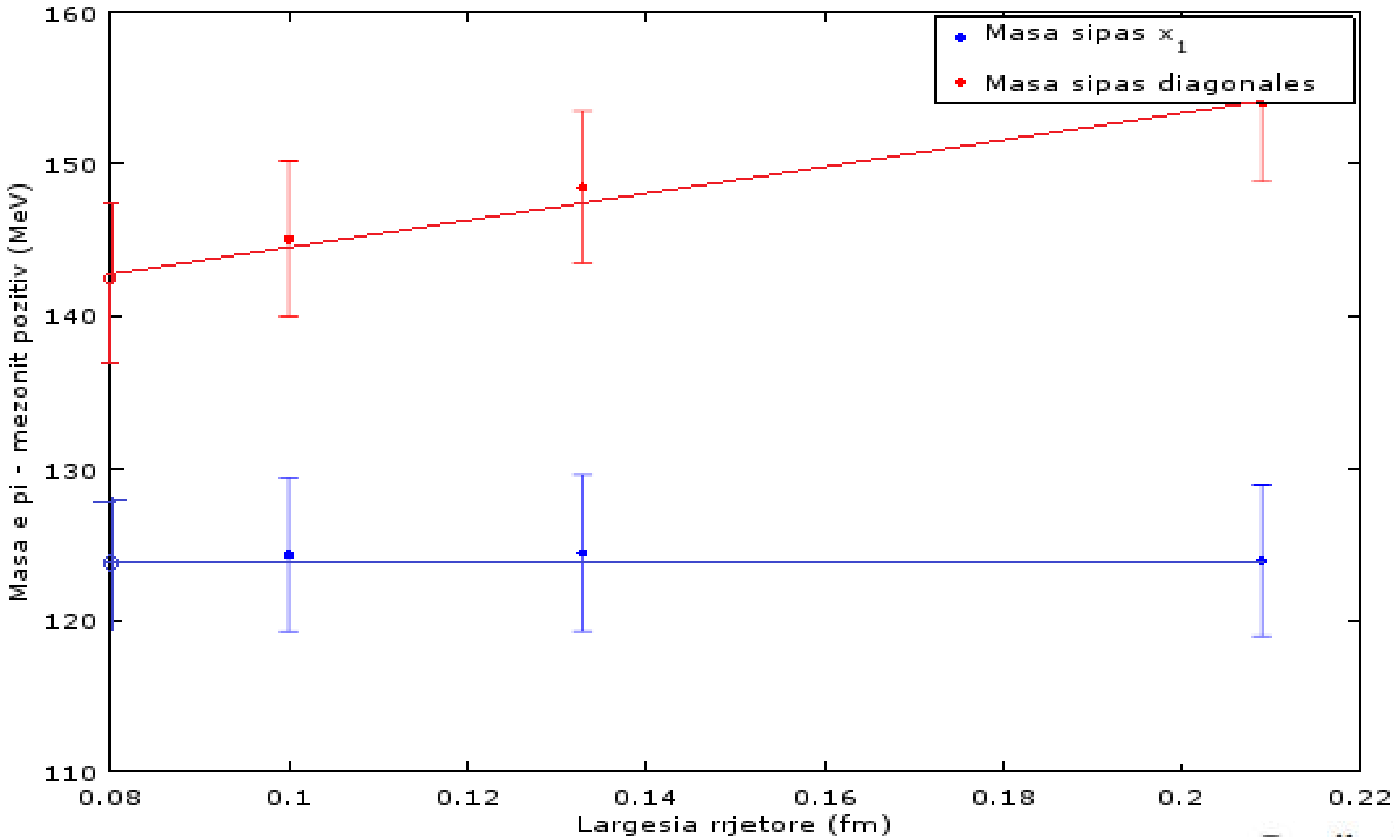


Figure : Fit masses ($\beta = 6.0$, $L = 32$, $m_0 = 0.02$, $d = 0.0$) are interpolated in $c \in [-0.65, -0.25]$. The minimum of $\Delta(M^2_{PS})$ as a function of c (right plot) is shallow with respect to statistical errors.

2-nd Method: Exploring the phase – structure of QCD in T=0K

(Banks- Casher relation and Lanczos quadrature)

Evaluation of the broken hyper - cubic symmetry mass



$$m_{\pi^+(diag)} = 141.54 \pm 5 MeV$$

$$m_{\pi^+(x_1)} = 123.89 \pm 5 MeV$$

Details of the simulations:

- Lattices 8^4, 12^4 dhe 16^4
- Quenched approximation
- Wilson gauge action
- Boriçi – Creutz action
- CGNE inverter
- Five different quark masses

Banks-Casher relation

- Banks – Casher relation *(T. Banks, A. Casher, 1980)* provides a link between the chiral condensate and the spectral density $\rho(\lambda,m)$

$$\lim_{\lambda\to 0}\lim_{m\to 0}\lim_{V\to\infty}\rho(\lambda,m)=\frac{\Sigma}{\pi} \qquad \Sigma=-\lim_{m\to 0}\lim_{V\to\infty}\langle\bar{\psi}\psi\rangle$$

(If chiral symmetry is spontaneously broken by a non-zero value of the condensate the density of the quark modes in infinite volume does not vanish at the origin. A non-zero density conversely implies that the symmetry is broken)

- Instead of the spectral density, the average number ν(M,m) of eigenmodes of the Dirac operator with eigenvalues α ≤ M2 turns out to be a more convenient quantity to consider. Since

$$\nu(\Lambda)=V\int_{-\Lambda}^{\Lambda}d\lambda\rho(\lambda,m)$$

the mode number ultimately carries the same information as the spectral density.

- So we can use: $$\Sigma_{eff}=\frac{\pi}{2}\frac{\nu(\Lambda)}{\Lambda V} \qquad \Lambda=\sqrt{M^2-m^2}$$

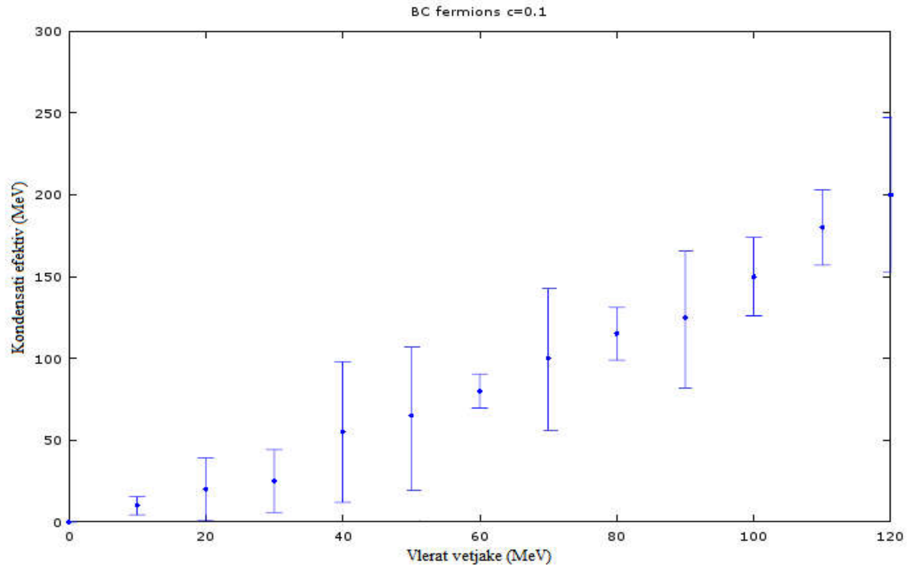
The algorithm for evaluating chiral condensate

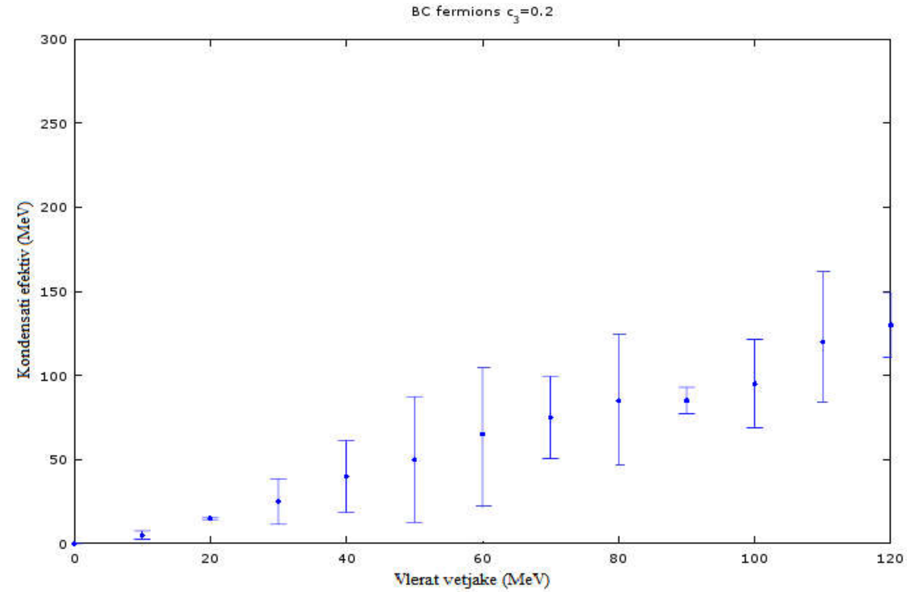
Algorithm Lanczos algorithm for solving $Ax = b$
Let be $A = D^*D$ and $b = z_2$
Set $\beta_0 = 0$, $\rho_1 = 1/\|b\|_2$, $q_0 = o$, $q_1 = \rho_1 b$
for $i = 1, \ldots$ **do**
$v = Aq_i$
$\alpha_i = q_i^\dagger v$
$v := v - q_i\alpha_i - q_{i-1}\beta_{i-1}$
$\beta_i = \|v\|_2$
$q_{i+1} = v/\beta_i$
$\rho_{i+1} = -\frac{(\rho_i\alpha_i + \rho_{i-1}\beta_{i-1})}{\beta_i}$
if $\left|\frac{\rho_1}{\rho_{i+1}}\right| < tol$ **then**
$n = i$
stop
end if
end for

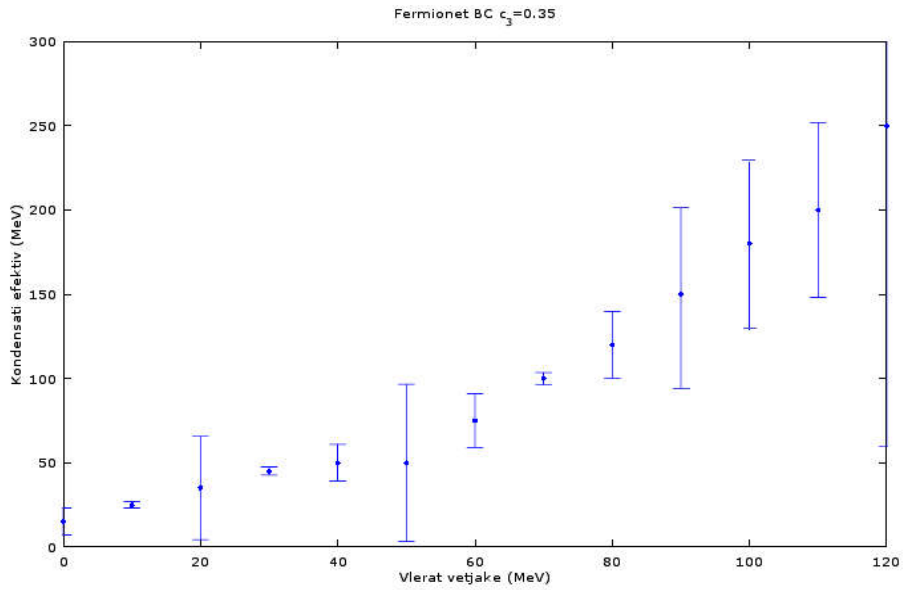
Let $A \in \mathbb{C}^{N \times N}$ be a hermitian matrix and $b \in \mathbb{R}^N$ a starting vector. Then the following algorithm computes the Gauss - Lanczos quadrature [17, 16]

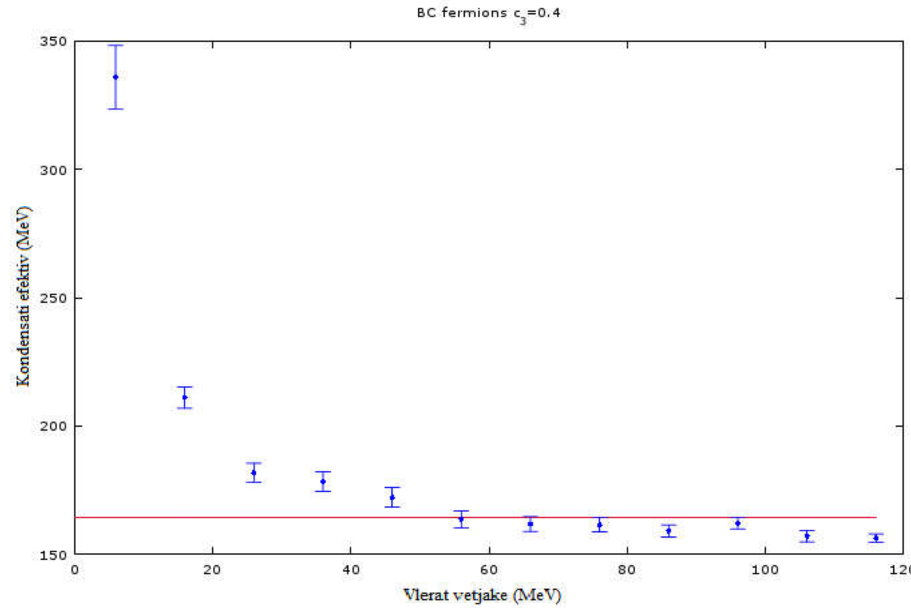
Algorithm 1 Algorithm for the Gauss - Lanczos quadrature
Compute α_i and β_i using Lanczos algorithm for $Ax = b$
Set $(T_n)_{i,i} = \alpha_i$, $(T_n)_{i+1,i} = (T_n)_{i,i+1} = \beta_i$ otherwise $(T_n)_{i,j} = 0$
Compute eigenvalues λ_i and eigenvectors v_i of T_n, where $i = 1 \ldots n$
Sort eigenvalues and eigenvectors in the increasing order of eigenvalues
Set k as the maximum index which correspond to the cut-off eigenvalue
Set θ_i to the positive square root of the original eigenvalues
Set z_i the first element of eigenvectors v_i where $i = 1 \ldots n$
Set $\omega_i = z_i^2$
Compute the mode number $\nu_k = \sum_{i=1}^{k} \omega_i$

Results

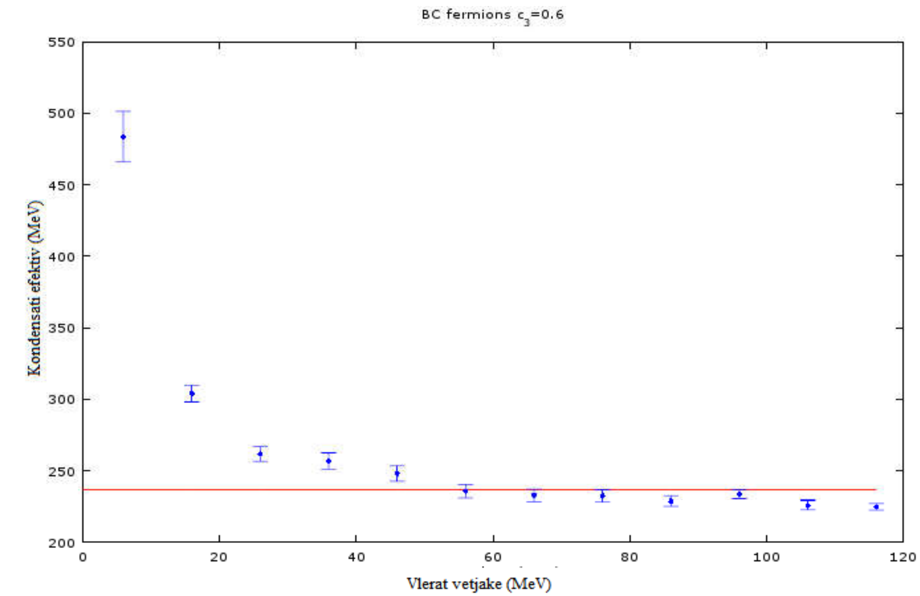

BC fermions c=0.1
Kondensati efektiv (MeV)
Vlerat vetjake (MeV)

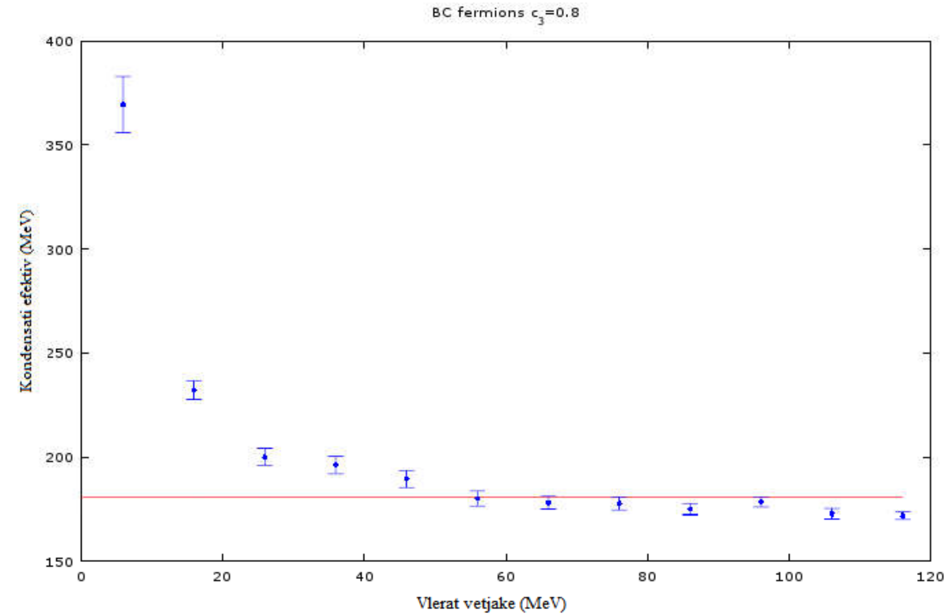

BC fermions c_3=0.2
Kondensati efektiv (MeV)
Vlerat vetjake (MeV)

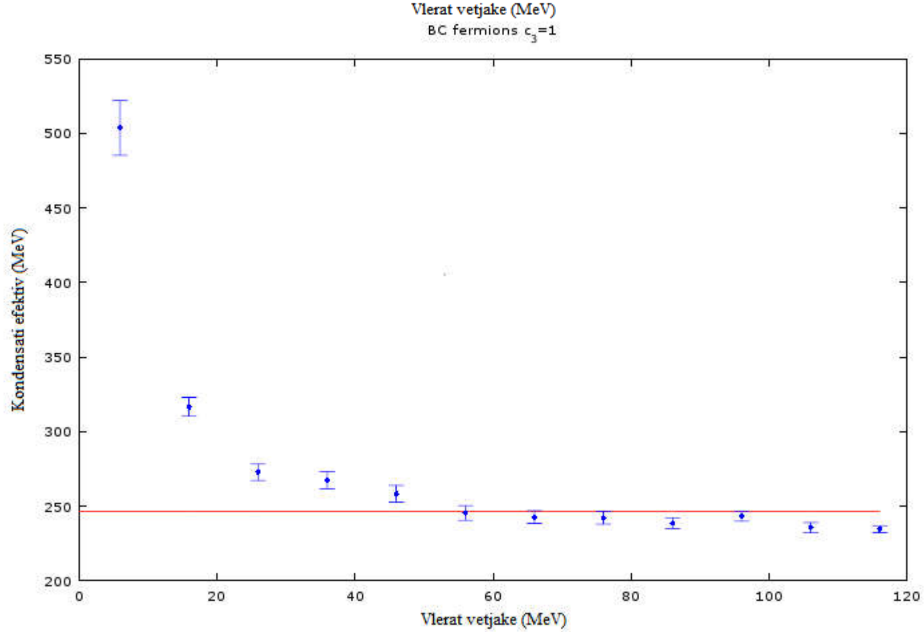

Fermionet BC c_3=0.35
Kondensati efektiv (MeV)
Vlerat vetjake (MeV)


BC fermions c_3=0.4
Kondensati efektiv (MeV)
Vlerat vetjake (MeV)

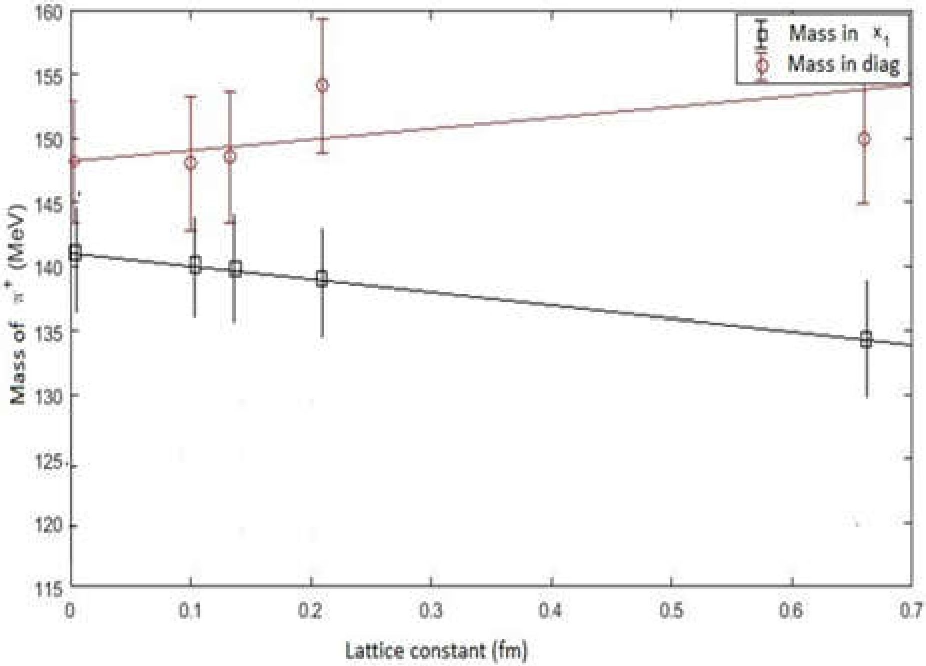
Results





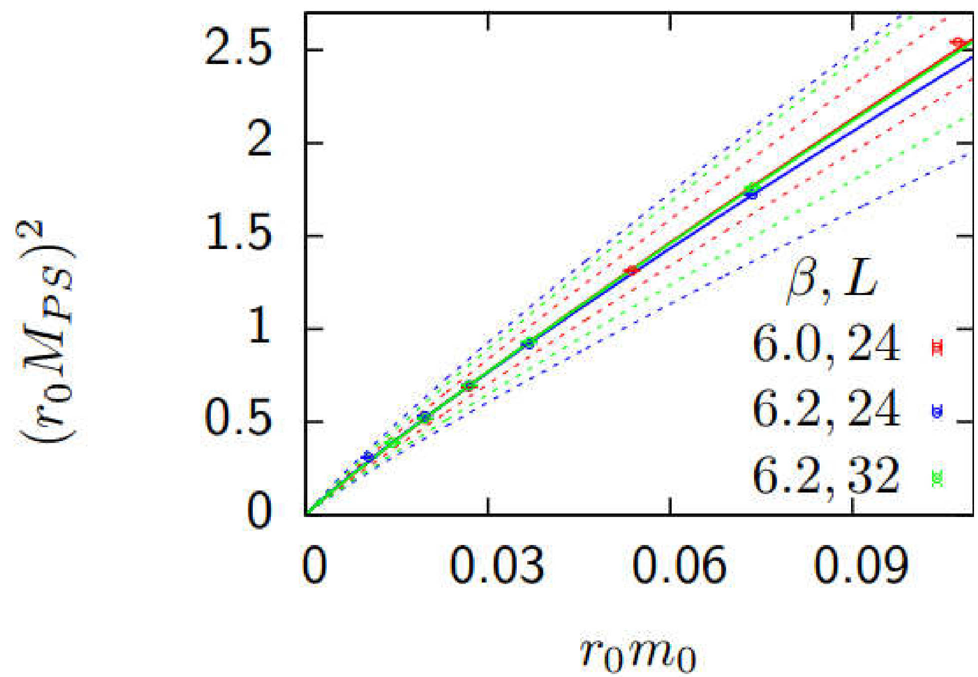


- Lattice 12^4
- Quenched approximation
- Wilson gauge action ($\beta = 6$)
- Boriçi – Creutz action
- Lanczos inverter
- Zero quark mass (BC fermions are chiral fermions)
- Seven different counterterms c3 (0.1, 0.2 0.35, 0.4, 0.6,0.8,1)



Pion masses in two different directions, in the case of the modified BC action.

Weber et al, 2013



The pseudoscalar mass at $\beta = 6.0$ and $\beta = 6.2$ agrees well

Conclusions

- Minimally doubled fermions present an alternative for the Lattice QCD simulations.
- They preserve an exact chiral symmetry for a degenerate doublet of quarks (chiral symmetry protects mass renormalization).
- This kind of fermions remains at the same time also strictly local, which means that are fast for simulations.
- MDF fermions have a preferred direction in euclidean spacetime → breaking of the hypercubic symmetry
- The calculations of charged pions masses in two different directions show clearly the broken hypercubic symmetry.
- Chiral symmetry and spontanous chiral symmetry breaking is very important in QCD.
- The chiral condensate can be used as an order parameter for BC fermions, and help us to find the proper counterterms that restore partially the broken hypercubic symmetry.
- Using Lanczos quadrature and Banks – Casher relation we can explore the chiral symmetry breaking and find the counterterms for which we have spontanous chiral symmetry breaking
- This value depends on the lattice and the coupling constant we use on simulations, which means that the procedure should be repeated in every different situation.
- Also, the total computational cost (for defining the value of the counterterm in each case and for the calculation of the hadrons spectrum) has to be defined, in order to understand if this kind of fermions are suitable for the hadrons spectrum calculation.

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Thank you!