Three Scale Invariant Tales

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• Conformal/scale invariance and the cosmological constant/vacuum energy.
• Conformal/scale invariance and the cosmological constant/vacuum energy.

• On the road to removing the string scale.
Outline

• Conformal/scale invariance and the cosmological constant/vacuum energy.

• On the road to removing the string scale.

• Time translational invariance and SUSY breaking in a conformal setup.
• A Zel’dovich type argument for a problem:
Consider QED in the 40s. One has a proton, an electron and a photon. One understands physics up to the Rydberg scale \( R_Y \). Ergo in an effective field theory, valid up to \( R_Y \), the value of the cosmological constant would be at least \( R_Y^4 \).
This is a problem....
• Loophole: an effective low energy theory needs to include all symmetries, be they either explicit or spontaneously broken. This is true also if the underlying theory of nature has no scale.
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**Chiral Lagrangian**

\[ \mathcal{L}_4 = L_1 \left\{ \text{Tr}[D_\mu U(D^\mu U)^\dagger] \right\}^2 + L_2 \text{Tr} \left[ D_\mu U(D_\nu U)^\dagger \right] \text{Tr} \left[ D^\mu U(D^\nu U)^\dagger \right] + L_3 \text{Tr} \left[ D_\mu U(D^\mu U)^\dagger D_\nu U(D^\nu U)^\dagger \right] + L_4 \text{Tr} \left[ D_\mu U(D^\mu U)^\dagger \right] \text{Tr} \left( \chi U^\dagger + U \chi^\dagger \right) + L_5 \text{Tr} \left[ D_\mu U(D^\mu U)^\dagger (\chi U^\dagger + U \chi^\dagger) \right] + L_6 \left[ \text{Tr} \left( \chi U^\dagger + U \chi^\dagger \right) \right]^2 + L_7 \left[ \text{Tr} \left( \chi U^\dagger - U \chi^\dagger \right) \right]^2 + L_8 \text{Tr} \left( U \chi^\dagger U \chi^\dagger + \chi U^\dagger \chi U^\dagger \right) - i L_9 \text{Tr} \left[ f^R_{\mu\nu} D_\mu U(D^\nu U)^\dagger + f^L_{\mu\nu} (D^\mu U)^\dagger D^\nu U \right] + L_{10} \text{Tr} \left( U f^L_{\mu\nu} U^\dagger f^R_{\mu\nu} \right) + H_1 \text{Tr} \left( f^R_{\mu\nu} f^L_{\mu\nu} \right) + H_2 \text{Tr} \left( \chi \chi^\dagger \right) + \ldots \]

Where \[ U(x) = \frac{1}{F_0} \left[ \sigma(x) + i \vec{\tau} \cdot \vec{\pi}(x) \right] \]
• Claim: in a conformal/scale invariant theory the value of the vacuum energy does not depend on the theory being spontaneously broken or not.
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• The scale of the symmetry breaking does not affect the vacuum energy.
There is nothing like solving a concrete model
Illustrative Example in $d=3$: He(3)+ He(4)

- Diagram 1: $T$ vs $\mu$ for He I and He II, showing $\lambda$-line, TCP, PS-line, and $\Delta \mu_{TCP}$. 
- Diagram 2: $T$ vs $\mu$ for He I and He II, showing $\lambda$-line, CP, CEP, PS-line, and $\Delta \mu$. 
- Diagram 3: $g_2$ vs $\mu$ for He I and He II, showing $\lambda$-line, CP, $\langle \phi \rangle = 0$, and $\langle \phi \rangle > 0$. 

The diagrams illustrate the phase transitions and critical points in the system, with He I and He II phases and the corresponding parameters and lines.
Consider

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \vec{\phi} \partial_\nu \vec{\phi} + \frac{1}{6N^2} g_6 (\vec{\phi}^2)^3 \]

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\[ \sigma = \phi^2 \]

\[ V(\sigma) = f(g_6) |\sigma|^3 \]
Illustrative Example in d=3

- Let’s discuss the possible outcomes for a general $f(g_6)$.
  
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\[ f(g_6) = g_c - g_6 \quad \quad g_c = (4\pi)^2. \]
For $g_c = (4\pi)^2$ a flat direction develops in the $\sigma$ direction.
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Illustrative Example in $d=3$
• Though the qualitative results are correct, the actual derivation is more complicated.
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• The effective potential from which the full results are derived becomes non-local after integrating out the original degrees of freedom.
The Method

- Add a source term to the action allowing for symmetry breaking

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• The full overall \( N \)-dependence is now an explicit factor of \( N \), and the integral over \( t \) and \( m^2 \) is performed by the method of steepest descent.
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\]

where \(\tilde{\phi}, \tau, m^2\) are independent of \(x\).

• The free energy is non-local.
The Method

\[ V_{\text{eff}} = |\sigma|^{d/(d-2)} \left( \frac{d}{|K_d|} \right)^{2/(d-2)} \left[ \left( \frac{d-2}{d} + \frac{1}{3} (-1)^{d/(d-2)} g \left( \frac{1}{2} |K_d| \right)^{2/(d-2)} \right) \right] \]

\[ K_d = \frac{\Gamma(1 - d/2)}{2^{d-1} \pi^{d/2}} < 0 \]

This potential has a flat direction for \( g = g_c \)
• The ground state is obtained by minimizing $\Gamma$ with respect to $\vec{\phi}, \tau, m^2$ leading to
The ground state is obtained by minimizing $\Gamma$ with respect to $\vec{\phi}, \tau, m^2$, leading to:

- $m^2 \vec{\phi} = 0$
- $2U'(\tau) - m^2 = 0$
- $\vec{\phi}^2 - \tau + \int \frac{dk}{k^2 + m^2} = 0$
• Diagonalizing the mass matrix, obtained from the effective potential, one redisCOVERS: one massless particle - the dilaton and \( N \) massive particles.
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• The $O(N)$ symmetry is not spontaneously broken.
The Results

• Diagonalizing the mass matrix, obtained from the effective potential, one redisCOVERs: one massless particle - the dilaton and $N$ massive particles.

• The $O(N)$ symmetry is not spontaneously broken.

• This example has only one scale.
• Let us consider an example with several possible scales: an $O(N) \times O(N)$ model in $d=3$ [Rabinovici, Saering, Bardeen 1987]
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$$S_E = \int d^3 x \left[ \frac{1}{2} (\nabla \phi_1)^2 + \frac{1}{2} (\nabla \phi_2)^2 + \frac{1}{2} \lambda_1 \phi_1^2 + \frac{1}{2} \lambda_2 \phi_2^2 + \frac{1}{4N} \left[ \mu_1 (\phi_1^2)^2 + \mu_2 \phi_1^2 \phi_2^2 + \mu_3 (\phi_2^2)^2 \right] + \frac{16\pi^2}{6N^2} \left[ h_1 (\phi_1^2)^3 + h_2 (\phi_1^2)^2 \phi_2^2 + h_3 \phi_1^2 (\phi_2^2)^2 + h_4 (\phi_2^3)^2 \right] \right]$$
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+ \frac{1}{4N} \left[ \mu_1 (\phi_1^2)^2 + \mu_2 \phi_1^2 \phi_2^2 + \mu_3 (\phi_2^2)^2 \right] \\
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• Maintaining the order of limits.
- The phase structure of this system is:
A Several-Scale System

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The vacuum energy is independent of both scales. The ratio between the scales is NOT fixed.
Flat potentials in $d=4$
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Flat potentials in d=4

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$$[\phi_i T^i, \phi_j T^j] = 0$$

- But here SUSY plays a role in maintaining quantum mechanically a classical flat direction.
Where is the Dilaton?

Breakdown of the Equivalence principle?

BEH like?
The Key Question Is:

Are there D=4 scale invariant NON SUSY systems With flat quantum directions?
Are there D=4 scale invariant NON SUSY systems With flat quantum directions? Or is there a No Go Theorem?
Are there D=4 scale invariant NON SUSY systems With flat quantum directions?

Or is there a No Go Theorem?

Caveat: In a super-selected sector with a non-vanishing value of the central charge (i.e. some dyonic charge) the ground state energy will reflect the energy of the dyonic distribution.
To include quantum gravity in a scale invariant theory one needs to know the structure of the theory at all available scales.

Topological phase, tensionless strings....
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\[ \alpha' \text{ scale delenda est} \]
Including Quantum Gravity

- To include quantum gravity in a scale invariant theory one needs to know the structure of the theory at all available scales.
- Topological phase, tensionless strings....

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The Richness of CFT

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The Richness of CFT

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- 10d gravitons

![Diagram](image-url)
The Richness of CFT

- $\mathcal{N}=4$ SUSY $SU(N)$ in $d=4$ has:
  - 10d gravitons
  - Strings

![Diagram showing the behavior of entropy $S$ as a function of energy $E$ in AdS 5.](attachment:image.png)

(1) For energies $E \ll m_{10}$, the Hilbert space is the Fock space of supergravity particles and the spectrum is quantized in the unit of $R^{-1}$. For $E \ll m_{10} R^9$, the entropy is given by that of the gas of free supergravity particles in 10 dimensions:

$$S \sim (ER)^{9/10}.$$  \hspace{1cm} (3.73)

(2) For $m_{10} R^9 < E \ll m_8 P_{17}$, string excitations become important, and the entropy grows linearly in energy:

$$S \sim E l_{17}.$$  \hspace{1cm} (3.74)

(3) For $m_8 P_{17} \ll E \ll m_8 R^7$, the black holes start to show up in the Hilbert space. For $E \ll m_8 P_{17}$, the size of the black hole horizon is smaller than $R$, and the entropy is given by that of the 10-dimensional Schwarzschild solution:

$$S \sim (El_{P_{17}})^{8/7}.$$  \hspace{1cm} (3.75)
• $\mathcal{N}=4$ SUSY SU($N$) in d=4 has:
• 10d gravitons
• Strings
• Schwarzschild black holes
The Richness of CFT

- $\mathcal{N}=4$ SUSY $SU(N)$ in $d=4$ has:
  - 10d gravitons
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  - AdS black holes
• $\mathcal{N}=4$ SUSY $SU(N)$ in d=4 has:

• 10d gravitons

• Strings

• Schwarzschild black holes

• AdS black holes

• Quite a few phases of gravity…
BH-String Transition
The compact space, and such corrections are suppressed by factors of $l_s/R$ and $l_p/R$.

The leading $l_s/R$ corrections to (3.72) were studied in [290], and found to be of the order of $(l_s/R)^3$.

Summary

The above analysis gives the following picture about the structure of the Hilbert space of string theory on AdS when $l_s \ll R$ and $g_s \ll 1$.

1. For energies $E \ll m_s$, the Hilbert space is the Fock space of supergravity particles and the spectrum is quantized in the unit of $R^{-1}$. For $E \ll m_{10}$, the entropy is given by that of the gas of free supergravity particles in 10 dimensions:
   $$S \sim \left(\frac{ER}{R^{9/10}}\right)^{10/9}.$$  
   (3.73)

2. For $m_{10s} \ll E \ll m_{8p}$, string excitations become important, and the entropy grows linearly in energy:
   $$S \sim E l_s.$$  
   (3.74)

3. For $m_{8p} l_s \ll E \ll m_{8p} R^7$, the black hole starts to show up in the Hilbert space. For $E \ll m_{8p} R^{7/4}$, the size of the black hole horizon is smaller than $R$, and the entropy is given by that of the 10-dimensional Schwarzschild solution:
   $$S \sim \left(\frac{El_s}{R^{8/7}}\right)^{8/7}.$$  
   (3.75)
The leading $l_s/R$ corrections to (3.72) were studied in [290], and found to be of the order of $(l_s/R)^3$.

Summary
The above analysis gives the following picture about the structure of the Hilbert space of string theory on $\text{AdS}_5$ when $l_s \ll R$ and $g_s \ll 1$.

(1) For energies $E \ll m_{10}$, the Hilbert space is the Fock space of supergravity particles and the spectrum is quantized in the unit of $R^{-1}$. For $E \ll m_{10}$, the entropy is given by that of the gas of free supergravity particles in 10 dimensions:

$$S \sim \left(\frac{E}{R^9}\right)^{10/3}.$$ (3.73)

(2) For $m_{10} < E \ll m_8 R^{9/10}$, string excitations become important, and the entropy grows linearly in energy:

$$S \sim E l_s.$$ (3.74)

(3) For $m_8 l_s \ll E \ll m_8 R^{9/10}$, small black holes start to show up in the Hilbert space. For $E \ll m_8 R^{9/10}$, the size of the black hole horizon is smaller than $R$, and the entropy is given by that of the 10-dimensional Schwarzschild solution:

$$S \sim \left(\frac{E}{m_p l_s}\right)^{8/7}.$$ (3.75)

Figure 3.5: The behavior of the entropy $S$ as a function of the energy $E$ in $\text{AdS}_5$. (1) For energies $E \ll m_{10}$, the Hilbert space is the Fock space of supergravity particles and the spectrum is quantized in the unit of $R^{-1}$. For $E \ll m_{10}$, the entropy is given by that of the gas of free supergravity particles in 10 dimensions:

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(2) For $m_{10} < E \ll m_8 l_s$, string excitations become important, and the entropy grows linearly in energy:

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(3) For $m_8 l_s \ll E \ll m_8 R^{9/10}$, small black holes start to show up in the Hilbert space. For $E \ll m_8 R^{9/10}$, the size of the black hole horizon is smaller than $R$, and the entropy is given by that of the 10-dimensional Schwarzschild solution:

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String-BH transition

BH-String Transition
Summary

The above analysis gives the following picture about the structure of the Hilbert space of string theory on $\text{AdS}$ when $l_s \ll R$ and $g_s \ll 1$.

1. For energies $E \ll m_{10}s$, the Hilbert space is the Fock space of supergravity particles and the spectrum is quantized in the unit of $R^{-1}$. For $E \ll m_{10}s R^9$, the entropy is given by that of the gas of free supergravity particles in 10 dimensions:
   \[ S \sim \left( \frac{E R}{m_{10}s} \right)^{9/10}. \]
   (3.73)

2. For $m_{10}s R^9 < E \ll m_8P l_7s$, string excitations become important, and the entropy grows linearly in energy:
   \[ S \sim E l_s. \]
   (3.74)

3. For $m_8P l_7s \ll E \ll m_8P R^7$, the black holes start to show up in the Hilbert space. For $E \ll m_8P R^7$, the size of the black hole horizon is smaller than $R$, and the entropy is given by that of the 10-dimensional Schwarzschild solution:
   \[ S \sim \left( \frac{E m_8P}{R^7} \right)^{8/7}. \]
   (3.75)

- “Quark-Hadron transition”
“Quark-Hadron transition”

- One can also engineer a “transition as a number of flavors” in string theory.
On The Road to Tensionless Strings

• In some string theory systems there is a phase transition as a function of a negative cosmological constant. It occurs when the cosmological constant is string scale [Giveon, Kutasov, Sever and Rabinovici].
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• Long strings are free in one phase and bind to BHs in another.
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• At high energies it is observed by entropy considerations.
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The IR manifestation is having two discrete infinity trajectories of massless particles which carry all spins. [Gaberdiel-Gopakumar-Hull]
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• An island on the way of having a theory with no scale [Giribet-Kleban-Hull-Porrati-Rabinovici] at $k=1$
• The IR manifestation is having two discrete infinity trajectories of massless particles which carry all spins. [Gaberdiel-Gopakumar-Hull]

• An island on the way of having a theory with no scale [Giribet-Kleban-Hull-Porrati-Rabinovici] at k=1

• This is a NEW type of phase transition with a discrete infinity of massless particles. More massless particles than at the BEH and Goldstone transitions…
Conformal and super-conformal quantum mechanics

• The Hamiltonian
Conformal and super-conformal quantum mechanics

- The Hamiltonian

\[ H = \frac{1}{2}(p^2 + gx^{-2}) \]
Conformal and super-conformal quantum mechanics

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is also special since \( g \) has a meaning that it does more than determine an energy scale.
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• \( H \) is part of the following algebra
Conformal and super-conformal quantum mechanics

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\[ [H, D] = iH, \quad [K, D] = iK, \quad [H, K] = 2iD \]
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Conformal and super-conformal quantum mechanics

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- This forms an SO(2,1) algebra where

  \[ D = -\frac{1}{4}(xp + px), \quad K = \frac{1}{2}x^2 \]

- The Casimir is given by

  \[ \frac{1}{2}(HK + KH) - D^2 = \frac{g}{4} - \frac{3}{16} \]
Why is \[ H = \frac{p^2}{2m} + \frac{g}{2x^2} \] special?

\[ H = \frac{p^2}{2m} + \frac{gx^n}{2} \]
Why is $H = \frac{p^2}{2m} + \frac{gx^2}{2}$ special?

- No perturbation possible.
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\[ \tilde{p} = a \, p, \quad \tilde{x} = a^{-1} x \]
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- $m$ and $g$ only determine the energy scale.

\[ \tilde{p} = ap, \quad \tilde{x} = a^{-1}x \]

\[ H = \frac{\tilde{p}^2}{2ma^2} + g\frac{\tilde{x}^n a^n}{2} \]
Why is \( H = \frac{p^2}{2m} + \frac{g}{2x^2} \) special?

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H = \frac{p^2}{2m} + \frac{g x^n}{2}
\]

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- \( m \) and \( g \) only determine the energy scale.

\[
\tilde{p} = a p, \quad \tilde{x} = a^{-1} x
\]

\[
\tilde{H} = \frac{\tilde{p}^2}{2m a^2} + g \frac{\tilde{x}^n a^n}{2}
\]

\[
\frac{1}{ma^2} = ga^n \implies a = (mg)^{\frac{1}{n+2}}
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\frac{1}{ma^2} = ga^n \implies a = (mg)^{\frac{1}{n+2}}
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\[
H = E(m, g) \frac{1}{2} \left( \tilde{p}^2 + \tilde{x}^n \right)
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\[
H = E(m, g) \frac{1}{2} (\tilde{p}^2 + \tilde{x}^n)
\]

Note that for \( n=-2 \) the coupling \( g \) has a meaning
The spectrum of the Hamiltonian 126 is the open set $(0, \infty)$, the spectrum is therefore continuous and bounded from below. The wave functions are given by:

$$\psi_E(x) = \sqrt{x}J_{\frac{1}{4}}(\sqrt{2E}x) \quad E \neq 0.$$  
(136)

We will now attempt to find the zero energy state. Take the ansatz $\phi(x) = x^\alpha$:

$$Hx^\alpha = 0.$$  
(137)

This implies

$$g = -\alpha(\alpha - 1) \quad \text{(138)}$$

solving this equation gives

$$\alpha = -\frac{1}{2} \pm \sqrt{1 + 4g^2}.$$  
(139)

This gives two independent solutions, and by completeness all other solutions.

$\alpha + > 0$ does not lead to a normalizable solution since the function diverges at infinity.

$\alpha - < 0$, is not normalizable either since the function diverges at the origin (a result of the scale symmetry).

Thus, there is no normalizable $E=0$ solution (not even plane wave nor normalizable)!

Most of the analysis in field theory proceeds by identifying a ground state and the fluctuations around it. How do we deal with a system in the absence of a ground state?
The spectrum of the Hamiltonian $126$ is the open set $(0, \infty)$, the spectrum is therefore continuous and bounded from below. The wave functions are given by:

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We will now attempt to find the zero energy state. Take the ansatz $\phi(x) = x^\alpha$:

$$
H = \left(-\frac{d^2}{dx^2} + g x^2\right) x^\alpha = 0.
$$

(137)

This implies

$$
g = -\alpha (\alpha - 1) \quad (138)
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Figure 3: The absence of a normalizable ground state for this potential

Most of the analysis in field theory proceeds by identifying a ground state and the fluctuations around it. How do we deal with a system in the absence of a ground state?

Breaking of time translation invariance and SUSY
In a sense - time translational invariance is spontaneously broken.

The spectrum of the Hamiltonian 126 is the open set \((0, \infty)\), the spectrum is therefore continuous and bounded from below. The wave functions are given by:

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\psi_{E}(x) = \sqrt{xJ} \sqrt{g + \frac{1}{4}(\sqrt{2E})} \quad E \neq 0.
\] (136)

We will now attempt to find the zero energy state. Take the ansatz \(\phi(x) = x^\alpha\):

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H = \left( -\frac{d^2}{dx^2} + gx^2 \right) x^\alpha = 0.
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Most of the analysis in field theory proceeds by identifying a ground state and the fluctuations around it. How do we deal with a system in the absence of a ground state?
In a sense - time translational invariance is spontaneously broken.

The system has an open positive energy spectrum bounded from below but the potential ground state at energy 0 is not plane wave normalizable.

Figure 3: The absence of a normalisable ground state for this potential
In a sense - time translational invariance is spontaneously broken.

The system has an open positive energy spectrum bounded from below but the potential ground state at energy 0 is not plane wave normalizable.

In the SUSY case, SUSY can’t be broken, BUT SUSY will be spontaneously broken by this new mechanism. Also an example of $\mathcal{N}=2$ SUSY breaking down to $\mathcal{N}=1$ SUSY.
Summary

Are there D=4 scale invariant NON SUSY systems With flat quantum directions?

Are all scales Spontaneously Generated?

Does Nature have runaway potentials of some kind?
The spectrum of the Hamiltonian $126$ is the open set $(0, \infty)$, the spectrum is therefore continuous and bounded from below. The wave functions are given by:

$$\psi_E(x) = \sqrt{x} J_{\frac{1}{4}} \left( \sqrt{2E} x \right)$$

$(136)$

We will now attempt to find the zero energy state. Take the ansatz $\phi(x) = x^\alpha$:

$$H = \left( -\frac{d^2}{dx^2} + g x^2 \right) x^\alpha = 0$$

$(137)$

This implies $g = -\alpha (\alpha - 1)$ $(138)$

Solving this equation gives

$$\alpha = -\frac{1}{2} \pm \sqrt{1 + 4g^2}$$

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This gives two independent solutions and by completeness all these solutions.

$\alpha + > 0$, does not lead to a normalizable solution since the function diverges at infinity.

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Most of the analysis in field theory proceeds by identifying a ground state and the fluctuations around it. How do we deal with a system in the absence of a ground state?