Scale invariance and symmetries in inflation

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*Scale invariance in particle physics and cosmology*
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Outline

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2. Scale invariance during inflation
3. Symmetries during inflation
4. Consequences and beyond
   - Non-Gaussian consistency relation
   - Spectral tilt of power spectrum
   - Extension to multi-field case
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# Scale invariance and symmetries in inflation

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Why are universal relations important?

- Relations to underlying principles behind phenomenology
- Independent of model detail
- Symmetries behind

Strong discriminator for different classes of models
Examples

- Squeezed limit of scalar bispectrum & power spectrum
  \[ \lim_{q \to 0} \frac{B_R(q, k_1, k_2)}{P_R(q)} = (1 - n_R)P_R(k) \quad (k_1 \approx k_2 \equiv k) \]

- Similar relations also hold

Are such relations from symmetry?
Examples

- Squeezed limit of scalar bispectrum & power spectrum

\[ \lim_{q \to 0} \frac{B_R(q, k_1, k_2)}{P_R(q)} = (1 - n_R)P_R(k) \quad (k_1 \approx k_2 \equiv k) \]

- Similar relations also hold

Are such relations from symmetry scale invariance?
What am I going to discuss?

*Scale invariance in particle physics and cosmology*

- Scale invariance during inflation?
- Relation to symmetries during inflation?
- Consequences and further idea?
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Scale invariance with gravity

Inclusion of gravity requires a dimensionful coupling

\[ S = \int d^4 x \sqrt{-g} \frac{m_{\text{Pl}}^2}{2} R + S_{\text{matter}} \]

Scale invariance is broken from the beginning!
Quadratic action for cosmological perturbations

Spatial metric $g_{ij} = a^2(t)(e^h)_{ij}$ with $h_{ij} = 2H_L \delta_{ij} + \gamma_{ij} + \cdots$

$$S_2^{(s)} = \int d^4 x a^3 m_{pl}^2 \left[ \dot{\mathcal{R}}^2 - \frac{(\nabla \mathcal{R})^2}{a^2} \right] \left( \epsilon \equiv -\frac{\dot{H}}{H^2} \ll 1 \right)$$

Canonical form with $d\tau = a dt$, $u \equiv z \mathcal{R}$ and $z \equiv a\dot{\phi}_0 / H$

$$S_2^{(s)} = \int d\tau d^3 x \frac{1}{2} \left[ u'^2 - (\nabla u)^2 + \frac{z''}{z} u^2 \right] \equiv -m^2$$
Breaking of scale invariance

(Global) rescaling of coordinates $x^\mu$ and $u$ (N.B. $u$ has mass dim 1)

$$x^\mu \rightarrow e^\alpha x^\mu \quad \text{and} \quad u \rightarrow e^{-\alpha} u$$

Associated current is not conserved, but slightly broken

$$\partial_\mu j^\mu = \left( m^2 + \frac{1}{2} \frac{dm^2}{d\log \tau} \right) u^2 = O(\epsilon)$$
Current and Ward-Takahashi identities

- Action as a function of $\phi$ and $\partial_\mu \phi$: $S = \int d^4 x \mathcal{L}(\phi, \partial_\mu \phi)$
- Under $\phi \rightarrow \phi + \delta \phi$, $\delta S = \int d^4 x \Delta$ with $\partial_\mu j^\mu = \Delta$

WT identities i.t.o. connected Green's functions reads (e.g. Coleman 1985)

$$\frac{\partial}{\partial y^\mu} \left< T[j^\mu(y)\phi(x_1) \cdots \phi(x_n)] \right> = \left< T[\Delta(y)\phi(x_1) \cdots \phi(x_n)] \right>$$

$$- i\delta^{(4)}(y-x_1) \left< T[\delta \phi(x_1) \cdots \phi(x_n)] \right> - \cdots$$
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During inflation, space-time is almost de Sitter

1. Very special gravitational background
2. Symmetries we enjoy in dS:
   - GR: general coordinate invariance $x'^\mu \rightarrow x'^\mu + \xi^\mu$
   - Time-dependent: time translational symmetry is broken
     (EFT of inflation with Goldstone mode $\pi = -R/H$)
   - dS isometries: transformations that leave dS as dS
Isometries of dS

Special coordinate transformations that leave dS as dS

1. Translation: \( x^i \rightarrow x^i + a^i \)

2. Rotation: \( x^i \rightarrow x^i + \omega^i_j x^j \) with \( \omega^i_j = -\omega^j_i \)

3. Special conformal transformations:

\[
t \rightarrow t - 2H^{-1}(b \cdot x) \quad \text{and} \\
x^i \rightarrow x^i - b^i(-H^{-2} e^{-2Ht} + x^2) + 2(b \cdot x)x^i
\]

At the end of inflation (\( t \rightarrow \infty \)) time is not affected

✓ Dilatation: both time and spatial coord change as

\[
t \rightarrow t - H^{-1} \log(1 + \lambda) \quad \text{and} \quad x^i \rightarrow (1 + \lambda)x^i
\]

Again as \( t \rightarrow \infty \) only spatial coordinates matter
Scale invariance as a dS isometry

- Scale invariance is one of dS isometries, dilatation symmetry
- Departure from scale inv amounts to departure from perfect dS
- Scale inv manifests itself through appropriate perturbation

We need a good gauge choice
Different perturbations transform differently under $x^\mu \rightarrow x^\mu + \xi^\mu$, e.g. $\gamma_{ij} \rightarrow \gamma_{ij}$, so we can choose $\xi^\mu$ by setting specific variables zero.

Comoving gauge: $\delta \phi = 0$ and $\partial_j (h_{ij} - 2H_L \delta_{ij}) = 0$
Why dilatation is special in comoving gauge (or vice versa)

Under dilatation $x^i \to (1 + \lambda)x^i$, gauge conditions are invariant:

$$
\delta \phi \to \left(1 - \lambda x^k \partial_k\right)\delta \phi \\
h_{ij} - 2H_L \delta_{ij} \to \left(1 - \lambda x^k \partial_k\right) \left(h_{ij} - 2H_L \delta_{ij}\right)
$$

Form of metric remains intact under dilatation: residual symmetry, and e.g. $H_L = 0$ (flat gauge condition) is not preserved

$$
H_L \rightarrow \begin{cases} 
H_L - \lambda (1 + x^i \partial_i H_L) & \text{(dilatation)} \\
H_L + \left[x^2 b^i - 2(b \cdot x)x^i\right] \partial_i H_L - 2(b \cdot x) & \text{(SCTs)}
\end{cases}
$$
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WT identities for $\mathcal{R}$ with scale invariance

Integrating WT identity for $n$-point correlation function of $\mathcal{R}$

- Dilatation is an exact symmetry, so $\Delta = 0$
- Evaluated at the same time, $x_1^0 = x_2^0 = \cdots = x_n^0 \equiv t$

$$\langle [Q, \mathcal{R}(x_1) \cdots \mathcal{R}(x_n)] \rangle = -i \langle \delta \mathcal{R}(x_1) \cdots \mathcal{R}(x_n) \rangle \cdots - i \langle \mathcal{R}(x_1) \cdots \delta \mathcal{R}(x_n) \rangle$$

with $Q \equiv \int d^3 x f^0(t, x)$: dilatation charge
Why curvature perturbation is special

Q is the generator of transformation under dilatation: $\delta \phi = i[Q, \phi]$

$$\delta \mathcal{R} = -1 - x^i \partial_i \mathcal{R} \quad \delta \phi = -x^i \partial_i \phi \text{ for } \phi \in \{\gamma_{ij}\cdots\} \quad \rightarrow \langle Q\mathcal{R}(k) \rangle = \frac{i}{2} (2\pi)^3 \delta^{(3)}(k) + \text{real part}$$

1. $-1$ is the Goldstone nature of $\mathcal{R}$
2. Dilatation charge creates $\mathcal{R}$ out of vacuum with $k \approx 0$
Lowest-order relation

Specifying to $n = 2$

- **LHS**: inserting 1-ptl excited state $P_{\mathcal{R}}^{-1/2} \mathcal{R}(k) |\Omega\rangle$ gives

  $$-i \langle \Omega | [Q, \mathcal{R}(k_1) \mathcal{R}(k_2)] |\Omega\rangle = \lim_{q \to 0} \frac{B_{\mathcal{R}}(q, k_1, k_2)}{P_{\mathcal{R}}(q)}$$

- **RHS**: using $\delta \mathcal{R} = -1 - x^i \partial_i \mathcal{R}$ gives (with $|k_1| \approx |-k_2| \equiv k$)

  $$-(3 + k_2 \cdot \nabla_{k_2}) \langle \mathcal{R}(k_1) \mathcal{R}(k_2) \rangle = (1 - n_{\mathcal{R}}) P_{\mathcal{R}}(k)$$

Equating LHS and RHS gives lowest-order nG consistency relation
Dilatation charge is not conserved so now \( \Delta \) term is included:

\[
\left\langle \left[ i \int_{\tau_0}^{\tau} d\tau' \int d^3 y \left( m^2 + \frac{1}{2} \frac{dm^2}{d\log \tau'} \right) u^2(\tau', y), u(\tau, x_1) u(\tau, x_2) \right] \right\rangle
\]

This gives

\[
n_R = 3 + \frac{d \log |u_k|^2}{d \log \tau} + 4 \Im \left[ \frac{u_k^*}{|u_k|^2} \int_{\tau_0}^{\tau} d\tau' \left( m^2 + \frac{1}{2} \frac{dm^2}{d\log \tau'} \right) u_k^2(\tau') \right]
\]

definition of \( n_R \)

\[
= 2 \left[ 2 - \log 2 - \gamma - \log(-k\tau) \right] \epsilon \eta \text{ with } \eta \equiv \dot{\epsilon}/(H \epsilon)
\]
Comoving gauge in multi-field inflation

1. **Spatial condition:** \( \partial_j (h_{ij} - 2H_L \delta_{ij}) = 0 \), the same

2. **Temporal condition:** \( \dot{\phi}_0 a \delta \phi^a = 0 \) (equiv to \( T^0_i = 0 \))
   - Decompose \( \delta \phi^a \) along and orthogonal to time evolution

\[
\delta \phi^a = \delta \phi^a_{\perp} + \dot{\phi}_0^a \pi \\
\text{with} \quad \dot{\phi}_0 a \delta \phi^a_{\perp} = 0
\]

- \( \dot{\phi}_0 a \delta \phi^a = \dot{\phi}_0^2 \pi = 0 \) means simply \( \pi = 0 \), so MS variables becomes

\[
\delta \phi^a - \frac{\dot{\phi}_0^a}{H} \left( H_L - \frac{\Delta}{3} H_T \right) = \delta \phi^a_{\perp} - \frac{\dot{\phi}_0^a}{H} \left( H_L - \frac{\Delta}{3} H_T - H \pi \right) \\
\equiv \mathcal{R} \text{ with } H_T = 0 \text{ & } \pi = 0
\]

1 d.o.f. in gravity, and \( n - 1 \) d.o.f. in matter
While $\delta \phi_{\perp}$ is orthogonal to time evolution, it can interact with $\mathcal{R}$

$$
\lim_{q \to 0} \frac{B_{\mathcal{R}}(k_1, k_2, q)}{P_{\mathcal{R}}(q)} = (1 - n_{\mathcal{R}})P_{\mathcal{R}}(k)
$$

$$
+ \lim_{q \to 0} \left[ P_a(q) - \frac{P_{a\mathcal{R}}(q)}{P_{\mathcal{R}}(q)} - \sum_{b \neq a} \frac{P_{a b}(q)}{P_b(q)} \right]^{-1} \frac{P_{a\mathcal{R}}(q)}{P_{\mathcal{R}}(q)}
\times \left[ B_{\mathcal{R}a}(k_1, k_2, q) - \frac{P_{a\mathcal{R}}(q)}{P_{\mathcal{R}}(q)} B_{\mathcal{R}}(k_1, k_2, q) \sum_{b \neq a} \frac{P_{a b}(q)}{P_b(q)} B_{\mathcal{R}b}(k_1, k_2, q) \right]
$$

- Existence of interactions imposes normalization factor
- No assumption except for weak interactions between $\delta \phi_{\perp}$'s
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Conclusions

- Scale invariance during inflation
  - Scale invariance is broken
  - But the breaking is not strong

- Symmetries during inflation
  - dS isometries that leave dS as dS
  - Scale invariance as one of dS isometries
  - Good gauge choice is necessary: comoving gauge

- Consequences and beyond
  - WT identities based on symmetries
  - Various consequences follow